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ON STABILITY IN RISK AVERSE
STOCHASTIC BILEVEL PROGRAMMING

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Abstract. From a conceptual point of view, two-stage stochastic programs and bilevel problems under stochastic uncertainty are closely related. However, the step from the first to the latter mirrors the step from optimal values to optimal solutions and entails a loss of desirable analytical properties. This work focuses on mean risk formulations of stochastic bilevel programs. Weak continuity of the objective function with respect to perturbations of the underlying measure is derived based on a growth condition. Implications regarding stability for a comprehensive class of risk averse models are pointed out.

Key words. Stochastic bilevel programming, mean risk models, stability

AMS subject classifications. 90C15

1. Introduction.

Bilevel problems arise from the interplay between two decision makers on different levels of a hierarchy. The *leader* decides first and passes the *upper level decision* on to the *follower*. Incorporating the leaders decision as a parameter, the follower then solves the *lower level problem* that reflects his or her own goals and returns an optimal solution back to the leader. The leaders own objective function depends on both his or her decision and the solution that is fed back from the lower level. In bilevel optimization, it is assumed that the leader has full information about the influence of his or her decision on the lower level problem. As the latter may have more than one solution, one typically assumes that the follower returns either the best (optimistic approach) or the worst (pessimistic approach) solution with respect to the leaders objective. The bilevel optimizaion problem is to find an optimal upper level decision. Such problems have first been discussed in economics ([27]).

The present work is on stochastic bilevel problems. In this setting, the realization of some random vector whose distribution does not depend on the upper level decision enters the lower level problem as an additional parameter. It is assumed that the leader has to make his or her decision without knowing the random parameter, while the follower decides under full information. Stochastic bilevel problems can be seen as an extension of classical two-stage stochastic programs, where upper and lower level mirror first and second stage, respectively. As in those problems, the upper level objective function gives rise to a random variable. However, this random variable now depends on an optimal solution rather than just on the optimal value of the lower level (or second stage) problem. This is a crucial difference that results in weaker analytical properties and a less stable behavior.

Nevertheless, stochastic bilevel problems are of great relevance for practical applications and have been discussed in the context of transportation ([1], [21]), the pricing of electricity swing options ([18]), economics ([7]), supply chain planning ([25]), telecommunications ([28]), structural optimization ([8]) and general agency problems ([11]). Other works focus on solution methods ([4]), stochastic bilevel problems with Knapsack constraints ([17]), nonlinear bilevel programming under uncertainty ([22]) or stochastic equilibrium problems ([19]) and their stability ([20], [23]).

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The recent work [12] has addressed stability of stochastic bilevel problems based on a so called quantile criterion. Using the optimistic approach and assumptions on the linearity of the upper and lower level problems, stability with respect to perturbations of the underlying probability measure have been examined. The focus in the present work is on stability of mean risk formulations of stochastic bilevel problems. While the goals are similar to the case of quantile criteria, the model and the methodology to be employed differ greatly.

Mean risk formulations arise if the random variables in question are ranked by applying a weighted sum of the expectation and some quantification of risk. In a recent work on stability of two-stage stochastic programs, it has been shown that for a comprehensive class of risk measures including all coherent ones, continuity of the resulting objective function can be derived from a growth condition imposed on the underlying models (see [9]). This paper extends the approach to stochastic bilevel problems.

The paper is organized as follows: After introducing the general setting of stochastic bilevel programming and basic assumptions on the models (Section 2), the relevant risk functionals are discussed (Section 3). In the resulting setting, stability of the mean risk bilevel program is entirely governed by properties of the underlying deterministic problem. The crucial growth condition is verified for uniquely solvable quadratic lower level problems in Section 4. Section 5 is devoted to problems where the uncertainty only affects the upper level objective function and the feasible set of the lower level. The implications of the resulting continuity of the objective function for stability are pointed out in Section 6.

2. Setting. Consider the parametric bilevel optimization problem

$$\text{"}\min_x\text{"}\{c(x,z) + q(y) \mid x \in X, y \in C(x,z)\}, \tag{1}$$

where the leader variable x is to be chosen from a fixed nonempty set $X \subseteq \mathbb{R}^n$, the upper level objective function is given as the sum of the mappings $c : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$ and $q : \mathbb{R}^m \to \mathbb{R}$ and the lower level problem is described by the multifunction $C : \mathbb{R}^n \times \mathbb{R}^s \to 2^{\mathbb{R}^m}$,

$$C(x,z) = \operatorname{argmin}_y\{y^\top Dy + j(x,z)^\top y \mid Ay \le h(x,z)\}, \tag{2}$$

involving the parameter $z \in \mathbb{R}^s$, matrices $A \in \mathbb{R}^{k\times m}$ and $D \in \mathbb{R}^{m\times m}$ and mappings $j : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^m$ and $h : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^k$. Without loss of generality, D is assumed to be symmetric. Furthermore, assume that

- **(B1)** there is a constant $\gamma_c > 0$ and a locally bounded mapping $\kappa : \mathbb{R}^n \to \mathbb{R}$, i.e. for every converging sequence $\{x_n\}$, the sequence $\{\kappa(x_n)\}$ is bounded, such that $|c(x,z)| \le \kappa(x)(\|z\|^{\gamma_c} + 1)$ for all $(x,z) \in \mathbb{R}^n \times \mathbb{R}^s$ and
- **(B2)** there exist an exponent $\gamma_q > 0$ and constants $C, q, L_q > 0$ such that $|q(y) - q(y')| \le L_q\|y - y'\|^{\gamma_q} + C_q$ for all $y, y' \in \mathbb{R}^m$.

Remark 1. *(B1) especially holds true if c is locally bounded and does not depend on z.*

Impose an additional information constraint on the problem by assuming that only the follower has full information, while the leader's decision has to be made without knowledge of the parameter z. Furthermore, assume that $z = z(\omega)$ is the realization of some random vector whose distribution does not depend on the upper level decision x. This leads to the following pattern of decision:

$$\text{decide } x \quad \to \quad \text{observe } z = z(\omega) \quad \to \quad \text{decide } y.$$

In an optimistic setting, the problem can be reformulated as

$$\min_x \{ \underbrace{c(x,z) + \min_y \{q(y) \mid y \in C(x, z(\omega))\}}_{=: f_{opt}(x, z(\omega))} \mid x \in X\},$$

while a pessimistic approach leads to

$$\min_x \{ \underbrace{c(x,z) + \max_y \{q(y) \mid y \in C(x, z(\omega))\}}_{=: f_{pes}(x, z(\omega))} \mid x \in X\}.$$

Remark 2. *This is exactly the setting of two-stage stochastic programming if an optimistic approach is taken and the upper and lower level objective functions coincide.*

Throughout the paper, assume $f = f_{opt}$ or $f = f_{pes}$, depending on which approach is considered. Mean risk models are obtained by ranking the random variables in the family $\{f(x, z(\omega)) \mid x \in X\}$ via a weighted sum of the expectation and some quantification of risk.

The resulting objective can be seen as a function of both the leader's variable x and the Borel probability measure $\mu = \mathbb{P} \circ z^{-1}$ induced by the entering random vector z. This allows to investigate the behavior of the objective under perturbations of the underlying probability measure and is motivated by the fact that in applications, the exact distribution of z might not be known. Let $\mathcal{P}(\mathbb{R}^s)$ denote the space of Borel probability measures on $\mathbb{R}^s$. Equipping this space with the topology induced by weak convergence of probability measures has proven to be instrumental in the context of stability analysis for stochastic programming models and was first done in [13]. A sequence $\{\nu_n\} \subset \mathcal{P}(\mathbb{R}^s)$ converges weakly to $\nu \in \mathcal{P}(\mathbb{R}^s)$ if $\lim_{n\to\infty} \int_{\mathbb{R}^s} l(t)\nu_n(dt) = \int_{\mathbb{R}^s} l(t)\nu(dt)$ for any bounded continuous function $l : \mathbb{R}^s \to \mathbb{R}$ (see [3]).

The present paper investigates under which conditions the objective function is continuous with respect to the product topology of the Euclidean topology on $\mathbb{R}^n$ and the topology of weak convergence on $\mathcal{P}(\mathbb{R}^s)$.

3. Mean risk objective functions. Following the approach provided in [9], consider the class of risk quantifiers defined in the following way: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space, i.e. $A \in \mathcal{F}$ and $\mathbb{P}(A) > 0$ imply that there exists $B \in \mathcal{F}$ such that $B \subsetneq A$ and $\mathbb{P}(B) > 0$. Such a probability space supports a random variable U that is uniformly distributed on the open unit interval. Fix $p \geq 1$ and consider a mapping $\rho : L^p(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ that is convex, nondecreasing with respect to the $\mathbb{P}$-almost sure partial order and law-invariant, i.e. $\rho(X) = \rho(Y)$ whenever X and Y have the same law under $\mathbb{P}$.

Note that every law-invariant convex or coherent risk measure in the sense of [2] has the desired properties, so the approach especially covers the Expected Exceedance of a given target level, the Conditional Value-at-Risk and the Upper Semideviation. In view of the mean risk models it is important to mention that the expectation and every weighted sum of covered risk measures with nonnegative weights also yield a feasible choice for ρ.

ρ gives rise to a function $\mathcal{R}_\rho : \mathcal{M}^p_1 := \{\nu \in \mathcal{P}(\mathbb{R}) \mid \int_{\mathbb{R}} |t|^p \nu(dt) < \infty\} \to \mathbb{R}$ via $\mathcal{R}_\rho(\nu) := \rho(F^{\leftarrow}_\mu(U))$, where $F^{\leftarrow}_\nu$ denotes the left-continuous quantile function of the distribution function of ν.

Assuming that the mapping $f : \mathbb{R}^n \times \mathbb{R}^s \to \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ is real-valued and Borel measurable, the mean risk objective function in the above setting can be written as

$$Q(x, \nu) := \mathcal{R}_\rho\left((\delta_x \otimes \nu) \circ f^{-1}\right),$$

where $(\delta_x \otimes \mu) \circ f^{-1}$ denotes the image measure of the product probability measure of the Dirac measure at $x \in \mathbb{R}^n$ and $\nu \in \mathcal{P}(\mathbb{R}^s)$ under f. Q is well-defined under conditions guaranteeing that the image measure is in $\mathcal{M}_1^p$.

By Theorem 2.2 in [9], continuity of Q with respect to the product topology of the standard topology on $\mathbb{R}^n$ and the relative topology of weak convergence on $\mathcal{M}$ can be derived whenever the following growth condition is fulfilled:

- **(G)** There is a constant $\gamma > 0$ and a locally bounded mapping $\bar{\sigma} : \mathbb{R}^n \to \mathbb{R}$ such that $|f(x,z)| \leq \bar{\sigma}(x)(\|z\|^\gamma + 1)$ for all $(x,z) \in \mathbb{R}^n \times \mathbb{R}^s$.

Note that the growth condition only depends on the underlying deterministic bilevel program and does not involve the quantification of risk.

4. Quadratic lower level problem with unique solutions. Assume that

- **(U1)** the problem $\min_y \{y^\top D y + t_1^\top y \mid Ay \leq t_2\}$ has a unique optimal solution whenever $(t_1, t_2) \in \operatorname{conv}\{(j(x,z), h(x,z)) \mid (x,z) \in X \times \mathbb{R}^s\}$ and
- **(U2)** there is a constant $\gamma_{j,h} > 0$ and a locally bounded mapping $\sigma : \mathbb{R}^n \to \mathbb{R}$ such that $\|j(x,z)\| + \|h(x,z)\| \leq \sigma(x)(\|z\|^{\gamma_{j,h}} + 1)$ for all $(x,z) \in \mathbb{R}^n \times \mathbb{R}^s$.

Remark 3. *Since (U1) implies $f_{opt} = f_{pes}$, it is not necessary to distinguish between the optimistic and the pessimistic setting.*

Remark 4. *Note that the uniqueness of solutions in (U1) is especially given, if D is positive definite. In that case, (U1) can weakened to*

(U1*) $\{y \in \mathbb{R}^m \mid Ay \leq h(x,z)\} \neq \emptyset$ *for any* $(x,z) \in \mathbb{R}^n \times \mathbb{R}^s$.

For details, refer to [5].

For any $\gamma > 0$, a subset $\mathcal{M}$ of $\mathcal{M}_s^{\gamma p} := \{\nu \in \mathcal{P}(\mathbb{R}^s) \mid \int_{\mathbb{R}^s} \|t\|^{\gamma p} \nu(dt) < \infty\}$ is called locally uniformly $\|\cdot\|^{\gamma p}$−integrating, if for any $\nu \in \mathcal{M}$ there exists some open neighborhood $\mathcal{N}$ of ν with respect to the topology of weak convergence such that

$$\lim_{a \to \infty} \sup_{\mu \in \mathcal{N} \cap \mathcal{M}} \int_{\mathbb{R}^s} \|z\|^{\gamma p} \cdot 1_{]a,\infty[}(\|z\|^{\gamma p}) \; \mu(dz) = 0.$$

The recent work [16] establishes several equivalent characterizations and gives numerous examples of locally uniformly $\|\cdot\|^{\gamma p}$−integrating subsets. Note that such sets are known to be useful when examining stability of stochastic programming problems (see e.g. [24]).

Theorem 1. *Set $\gamma := \max\{\gamma_{j,h} * \gamma_q, \gamma_c\}$ and let $\mathcal{M} \subseteq \mathcal{M}_s^{\gamma p}$ be locally uniformly $\|\cdot\|^{\gamma p}$−integrating. Assume that (B1), (B2), (U1) and (U2) are fulfilled and $x_0 \in X$ and $\mu \in \mathcal{M}$ are such that $\delta_{x_0} \otimes \mu(D_f) = 0$, where $D_f \subseteq \mathbb{R}^n \times \mathbb{R}^s$ denotes the set of discontinuities of f. Then the mapping $Q|_{X \times \mathcal{M}}$ is continuous at (x_0, μ) with respect to the product topology of the standard topology on $\mathbb{R}^n$ and the relative topology of weak convergence on $\mathcal{M}$.*

Proof 1. *For any $(x,z) \in \operatorname{dom} C$, set $C(x,z) = \{y(x,z)\}$. By assumption $(U1)$ and Corollary 5.1 in [14] there exists a constant L_C such that*

$$\|y(x,z) - y(x',z')\| \leq L_C(\|j(x,z) - j(x',z')\| + \|h(x,z) - h(x',z')\|)$$

for all $(x,z),(x',z') \in X \times \mathbb{R}^s \subseteq \operatorname{dom} C$. *Consequently,*

$$\begin{aligned}|f(x,z)-c(x,z)| = |q(y(x,z))| &\le \underbrace{|q(y(0,0))|}_{=:C_0} + |q(y(x,z))-q(y(0,0))| \\ &\le C_0 + C_q + L_q\|y(x,z)-y(0,0)\|^{\gamma_q} \\ &\le C_0 + C_q + L_q L_C^{\gamma_q}(\|j(x,z)-j(0,0)\| + \|h(x,z)-h(0,0)\|)^{\gamma_q} \\ &\le \underbrace{C_0 + C_q + 2^{\gamma_q} L_q L_C^{\gamma_q}(\|j(0,0)\| + \|h(0,0)\|)^{\gamma_q}}_{=:C'} \\ &\quad + 2^{\gamma_q} L_q L_C^{\gamma_q}(\|j(x,z)\| + \|h(x,z)\|)^{\gamma_q} \\ &\le C' + 2^{\gamma_q} L_q L_C^{\gamma_q}\sigma(x)^{\gamma_q}(\|z\|^{\gamma}+1)^{\gamma_q} \le \bar{\sigma}(x)(\|z\|^{\gamma_{j,h}*\gamma_q}+1),\end{aligned}$$

where $\bar{\sigma}(x) := C' + 4^{\gamma_q} L_q L_C^{\gamma_q}\max\{1,\sigma(x)^{\gamma_q}\}$ *is locally bounded. (B1) therefore yields*

$$|f(x,z)| \le (\bar{\sigma}(x)+\kappa(x))(\|z\|^{\gamma_{j,h}*\gamma_q} + \|z\|^{\gamma_c}+1) \le \eta(x)(\|z\|^{\max\{\gamma_{j,h}*\gamma_q,\gamma_c\}}+1),$$

where $\eta(x) := 2(\bar{\sigma}(x)+\kappa(x))$ *is locally bounded. Hence, the growth condition (G) holds and the application of Theorem 2.2 in [9] completes the proof.*

Corollary 1. *Let* $\mathcal{M} \subseteq \mathcal{M}_s^{\gamma_p}$ *be locally uniformly* $\|\cdot\|^{\gamma_p}$*−integrating, assume that (B1), (B2), (U1) and (U2) are fulfilled and that the mappings* c, q, j *and* h *are continuous. Then the mapping* $Q|_{X\times\mathcal{M}}$ *is continuous on* $X \times \mathcal{M}$ *with respect to the product topology of the standard topology on* $\mathbb{R}^n$ *and the relative topology of weak convergence on* $\mathcal{M}$.

Proof 2. *The continuity of* c, q, j *and* h *yields that* $D_f = \emptyset$.

5. Quadratic lower level problem with right-hand side uncertainty. Consider the special case of (2), where $j(x,z) = j \in \mathbb{R}^m$ for all $(x,z) \in \mathbb{R}^n \times \mathbb{R}^s$ and D is positive semidefinite. Note that this setting covers the linear case $D = 0 \in \mathbb{R}^{m\times m}$. Assume that

- **(R1)** the feasible set of the lower level problem is always nonempty, i.e. $\{y \in \mathbb{R}^m \mid Ay \le h(x,z)\} \neq \emptyset$ for all $(x,z) \in X \times \mathbb{R}^s$,
- **(R2)** there exists $(x,z) \in X \times \mathbb{R}^s$ such that the lower level problem is solvable and
- **(R3)** there is a constant $\gamma_h > 0$ and a locally bounded mapping $\sigma : \mathbb{R}^n \to \mathbb{R}$ such that $\|h(x,z)\| \le \sigma(x)(\|z\|^{\gamma_h}+1)$ for all $(x,z) \in \mathbb{R}^n \times \mathbb{R}^s$.

Furthermore, assume

- **(O1)** $\min\{q(y) \mid y \in C(x,z)\}$ is solvable for all $(x,z) \in X \times \mathbb{R}^s$ and
- **(O2)** there exists $(x_0,z_0) \in X \times \mathbb{R}^s$ such that $\sup\{q(y) \mid y \in C(x_0,z_0)\} < \infty$ and

in the optimistic and

- **(P1)** $\max\{q(y) \mid y \in C(x,z)\}$ is solvable for all $(x,z) \in X \times \mathbb{R}^s$ and
- **(P2)** there exists $(x_0,z_0) \in X \times \mathbb{R}^s$ such that $\inf\{q(y) \mid y \in C(x_0,z_0)\} > -\infty$

in the pessimistic setting.

Remark 5. *(R1) and (R2) imply* $\operatorname{dom} C = X \times \mathbb{R}^s$*, i.e. the solvability of the lower level problem on the whole parameter space. Furthermore, under these conditions there exists a finite set* $E \subsetneq \mathbb{R}^m$ *such that* $C(x,z)$ *is a polyhedron having exactly the elements of* E *as its extreme directions for all* $(x,z) \in X \times \mathbb{R}^s$ *(see § 3 in [10]).*

Remark 6. *If* q *is linear, (O1) and (P1) can be weakened to*

(O1*) *there exists* $(x,z) \in X \times \mathbb{R}^s$ *such that* $\min\{q(y) \mid y \in C(x,z)\}$ *is solvable and*
(P1*) *there exists* $(x,z) \in X \times \mathbb{R}^s$ *such that* $\max\{q(y) \mid y \in C(x,z)\}$ *is solvable,*

respectively.

REMARK 7. *If there exists a vector* $(x,z) \in \mathbb{R}^n \times \mathbb{R}^s$ *such that* $C(x,z)$ *is bounded, (O1), (O2), (P1) and (P2) can be dropped completely (see the proof of Theorem 2).*

THEOREM 2. *Set* $\gamma := \max\{\gamma_h * \gamma_q, \gamma_c\}$ *and let* $\mathcal{M} \subseteq \mathcal{M}_s^{\gamma p}$ *be locally uniformly* $\|\cdot\|^{\gamma p}-$*integrating. If (B1), (B2), (R1)-(R3) as well as (O1) and (O2) in the optimistic or (P1) and (P2) in the pessimistic setting are fulfilled and* $x_0 \in X$ *and* $\mu \in \mathcal{M}$ *are such that* $\delta_{x_0} \otimes \mu(D_f) = 0$*, the mapping* $Q|_{X \times \mathcal{M}}$ *is continuous at* (x_0, μ) *with respect to the product topology of the standard topology on* $\mathbb{R}^n$ *and the relative topology of weak convergence on* $\mathcal{M}$*.*

PROOF 3. *By Theorem 4.2 in [15] there exists a constant* $L > 0$ *such that*

$$\mathrm{d}_\infty(C(x,z), C(x',z')) \leq L\|h(x,z) - h(x',z')\| \ \textit{ for all } (x,z),(x',z') \in X \times \mathbb{R}^s,$$

where d_∞ *denotes the Hausdorff distance. Consider* $(x,z) \in X \times \mathbb{R}^s$ *and* $y \in C(x,z)$ *satisfying* $f(x,z) = c(x,z) + q(y)$*. Then there exists a vector* $y_0 \in C(x_0,z_0)$ *such that* $\|y - y_0\| \leq L\|h(x,z) - h(x_0,z_0)\|$ *and the following holds*

$$\begin{aligned}
|f(x,z) - c(x,z)| &= |q(y)| \leq |q(y_0) - q(y)| + |q(y_0)| \\
&\leq L_q\|y_0 - y\|^{\gamma_q} + \underbrace{C_q + \sup\{|q(y')| \mid y' \in C(x_0,z_0)\}}_{=:\bar{q}} \\
&\leq \bar{q} + L_q L^{\gamma_q}\|h(x,z) - h(x_0,z_0)\|^{\gamma_q} \\
&\leq \underbrace{\bar{q} + 2^{\gamma_q} L_q L^{\gamma_q}\|h(x_0,z_0)\|^{\gamma_q}}_{=:q'} + 2^{\gamma_q} L_q L^{\gamma_q}\|h(x,z)\|^{\gamma_q} \\
&\leq q' + 2^{\gamma_q} L_q L^{\gamma_q} \sigma(x)^{\gamma_q}(\|z\|^{\gamma_h} + 1)^{\gamma_q} \leq \bar{\sigma}(x)(\|z\|^{\gamma_h * \gamma_q} + 1),
\end{aligned}$$

where $\bar{\sigma}(x) := q' + 4^{\gamma_q} L_q L^{\gamma_q} \max\{1, \sigma(x)^{\gamma_q}\}$ *is locally bounded. By (B1),*

$$|f(x,z)| \leq \eta(x)(\|z\|^{\max\{\gamma_h * \gamma_q, \gamma_c\}} + 1),$$

where $\eta(x) := 2(\bar{\sigma}(x) + \kappa(x))$ *is locally bounded. Hence, (G) holds and Theorem 2.2 from [9] completes the proof.*

COROLLARY 2. *Let* $\mathcal{M} \subseteq \mathcal{M}_s^{\gamma p}$ *be locally uniformly* $\|\cdot\|^{\gamma p}-$*integrating, assume that (B1), (B2), (R1)-(R3) as well as (O1) and (O2) in the optimistic or (P1) and (P2) in the pessimistic setting are fulfilled and that the mappings* c, q *and* h *are continuous. Then the mapping* $Q|_{X \times \mathcal{M}}$ *is continuous on* $X \times \mathcal{M}$ *with respect to the product topology of the standard topology on* $\mathbb{R}^n$ *and the relative topology of weak convergence on* $\mathcal{M}$*.*

6. Implications. Let $\varphi : \mathcal{M}_s^{\gamma p} \to \bar{\mathbb{R}}$, $\varphi(\mu) := \inf\{Q(x,\mu) \mid x \in X\}$ denote the optimal value function of the mean risk stochastic bilevel program.

COROLLARY 3. *Let the assumptions of Corollary 1 or Corollary 2 be fulfilled. Then* $\varphi|_{\mathcal{M}}$ *is upper semicontinuous on* $\mathcal{M}$ *with respect to the relative topology of weak convergence.*

PROOF 4. *Under the given assumptions,* $Q|_{X \times \mathcal{M}}$ *is continuous on* $X \times \mathcal{M}$*. Since the feasible set* X *is fixed, that yields the upper semicontinuity of* $\varphi|_{\mathcal{M}}$ *(see section 4.1 in [6]).*

Corollary 4. *Let the assumptions of Corollary 1 or Corollary 2 be fulfilled and assume that X is compact. Then $\varphi|_{\mathcal{M}}$ is continuous on $\mathcal{M}$ and the restricted optimal solution set mapping $\psi|_{\mathcal{M}} : \mathcal{M} \to 2^{\mathbb{R}^n}$, $\psi|_{\mathcal{M}}(\mu) = \operatorname{argmin}\{Q(x,\mu) \mid x \in X\}$ is upper semicontinuous on $\mathcal{M}$ with respect to the relative topology of weak convergence, i.e. for any $\mu_0 \in \mathcal{M}$ and any open set $\mathcal{O} \subseteq \mathbb{R}^n$ such that $\psi|_{\mathcal{M}}(\mu_0) \subseteq \mathcal{O}$ there exists a neighborhood $\mathcal{N}$ of μ_0 with respect to the topology of weak convergence such that $\psi|_{\mathcal{M}}(\mu) \subseteq \mathcal{O}$ for all $\mu \in \mathcal{N}$.*

Proof 5. *See Proposition 1.1 in [26].*

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