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Martingale solutions for a pseudomonotone evolution equation with multiplicative noise

by

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Abstract

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical Wiener space endowed with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, $T > 0$ with the usual assumptions, $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $Q := (0,T) \times D$ and $p > 2$. Our aim is the study of the problem

$$(P)\begin{cases} du - \operatorname{div}(|\nabla u|^{p-2}\nabla u + F(u))\ dt = H(u)\ dW & \text{in } \Omega \times (0,T) \times D \\ u = 0 & \text{on } \Omega \times (0,T) \times \partial D \\ u(0,\cdot) = u_0 & \text{in } \Omega \times D \end{cases}$$

for a cylindrical Wiener process in $L^2(D)$ and $F : \mathbb{R} \to \mathbb{R}^d$ Lipschitz continuous. We consider the case of multiplicative noise with $H : L^2(D) \to HS(L^2(D))$, $HS(L^2(D))$ being the space of Hilbert-Schmidt operators, satisfying appriopriate regularity conditions. By an implicit time discretization of (P), we obtain approximate solutions. Using the theorems of Skorokhod and Prokhorov, we are able to pass to the limit and show existence of martingale solutions.

Keywords: pseudomonotone problem, multiplicative noise, cylindrical Wiener process, martingale solution

AMS Classification: 35K92, 35K55, 60H15

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete, countably generated probability space (for example the classical Wiener space) endowed with a filtration $(\mathcal{F}_t^W)_{t\in[0,T]}$, $T > 0$ with the usual assumptions, $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $Q := (0,T) \times D$ and $p > 2$. For a separable Hilbert spaces $\mathcal{U}, \mathcal{H}$, we denote the space of Hilbert-Schmidt operators from $\mathcal{U}$ to $\mathcal{H}$ by $HS(\mathcal{U}; \mathcal{H})$. We are interested in existence of a solution to

$$(P)\begin{cases} du - \operatorname{div}(|\nabla u|^{p-2}\nabla u + F(u))\ dt = H(u)\ dW & \text{in } \Omega \times (0,T) \times D \\ u = 0 & \text{on } \Omega \times (0,T) \times \partial D \\ u(0,\cdot) = u_0 & \text{in } \Omega \times D \end{cases}$$

for $u_0 \in W_0^{1,p}(D)$, $F : \mathbb{R} \to \mathbb{R}^d$ Lipschitz continuous. We will give the precise assumptions on $H : L^2(D) \to HS(L^2(D))$ in the next section. $W(t)$ is a cylindrical Wiener process with values in $L^2(D)$. More precisely: Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(D)$ and $(\beta_k(t))_{k\in\mathbb{N}}$ be a family of independant, real-valued brownian motions adapted to $(\mathcal{F}_t^W)$. We (formally) define

$$W(t) := \sum_{k=1}^{\infty} e_k \beta_k(t). \tag{1}$$

It is well-known that the sum on the right-hand side of (1) does not converge in $L^2(D)$, therefore we have to give a meaning to (1) following the ideas of [5] and [11]: For $u = \sum_{k=1}^{\infty} u_k e_k$ and $v = \sum_{k=1}^{\infty} v_k e_k$

$$(u,v)_U := \sum_{k=1}^{\infty} \frac{u_k v_k}{k^2}$$

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is a scalar product on $L^2(D)$. Now we define the (bigger) Hilbert space U as the completion of $L^2(D)$ with respect to the norm $\|\cdot\|_U$ induced by $(\cdot,\cdot)_U$. It is then easy to see that (ke_k) is an orthonormal basis of U. Note that

$$W(t) = \sum_{k=1}^{\infty} e_k\beta_k(t) = \sum_{k=1}^{\infty} \frac{1}{k} ke_k\beta_k(t) \tag{2}$$

and therefore $W(t)$ can be interpreted as Q-Wiener process with covariance Matrix $Q = \mathrm{diag}(\frac{1}{k^2})$ with values in U. Since $Q^{\frac{1}{2}}(U) = L^2(D)$, for all square integrable and predictable $\Phi : \Omega \times (0,T) \to HS(L^2(D))$ the stochastic integral with respect to the cylindrical Wiener process $W(t)$ can be defined by

$$\begin{aligned}\int_0^t \Phi\, dW &= \sum_{k=1}^{\infty} \int_0^t \Phi(e_k)\, d\beta_k \\ &= \sum_{k=1}^{\infty} \int_0^t \Phi(\frac{1}{k}\cdot ke_k)\, d\beta_k \\ &= \sum_{k=1}^{\infty} \int_0^t \Phi\circ Q^{1/2}(ke_k)\, d\beta_k. \end{aligned} \tag{3}$$

Since $\Phi\circ Q^{\frac{1}{2}} \in HS(U;L^2(D))$,

$$\sum_{k=1}^{\infty} \int_0^t \Phi\circ Q^{1/2}(ke_k)\, d\beta_k \in L^2(\Omega; C([0,T];L^2(D))).$$

In particular, for all $k\in\mathbb{N}$, $\Phi(e_k)\in L^2(\Omega\times(0,T);L^2(D))$ is predictable process, i.e. $\Phi(e_k)$ is $\mathcal{P}_T/\mathcal{B}(L^2(D))$-measurable where $\mathcal{P}_T$ is the (predictable) σ-field on $\Omega\times(0,T)$ generated by

$$(s,t]\times A, \quad 0\le s<t\le T, \quad A\in\mathcal{F}_s^W.$$

1.1 Strong and martingale solutions

In the theory of stochastic evolution equations two notions of solutions are typically considered for equations with multiplicative noise namely strong solutions and martingale solutions. A strong solution is defined as follows:

Definition 1.1. *A solution to (P) is a predictable process $u : \Omega\times[0,T]\to L^2(D)$ with a.e. paths*

$$u(\omega,\cdot)\in\mathcal{C}([0,T];W^{-1,p'}(D))\cap L^\infty(0,T;L^2(D)),$$

such that $u\in L^p(\Omega;L^p(0,T;W_0^{1,p}(D)))$, $u(0,\cdot)=u_0$ in $L^2(D)$ and

$$u(t)-u_0-\int_0^t \mathrm{div}(|\nabla u|^{p-2}\nabla u + F(u))\, ds = \int_0^t H(u)\, dW,$$

in $L^2(D)$ for all $t\in[0,T]$, a.s. in Ω.

In the former definition, the probabilistic quantities $(\Omega,\mathcal{F},P)$, $(\mathcal{F}_t^W)$ and W are fixed. In many cases, it is necessary that $(\Omega,\mathcal{F},P)$, $(\mathcal{F}_t^W)$ enter as unknowns into the problem, for example, if one uses the theorems of Prokhorov and Skorokhod to obtain a.s convergence of approximative solutions. More precisely,

Definition 1.2 (see, e.g. [5], [6], [8])**.** *We say that (P) has a martingale solution, iff there exist a probability space $(\overline{\Omega},\overline{\mathcal{F}},\overline{P})$, a filtration $(\overline{\mathcal{F}_t})$, a cylindrical Wiener process $\overline{W}$ and a predictable process $u:\overline{\Omega}\times[0,T]\to L^2(D)$ with a.e. paths*

$$u(\omega,\cdot)\in\mathcal{C}([0,T];W^{-1,p'}(D))\cap L^\infty(0,T;L^2(D)),$$

such that $u \in L^p(\overline{\Omega}; L^p(0,T; W_0^{1,p}(D)))$, $u(0,\cdot) = u_0$ *in* $L^2(D)$ *and*

$$u(t) - u_0 - \int_0^t \operatorname{div}(|\nabla u|^{p-2}\nabla u + F(u))\, ds = \int_0^t H(u)\, d\overline{W} \tag{4}$$

holds in $L^2(D)$ *for all* $t \in [0,T]$, *a.s. in* $\overline{\Omega}$.

1.2 Main results and outline

Our aim is to prove the following result:

Theorem 1.1. *For any* $u_0 \in W_0^{1,p}(D)$ *and any* $H : L^2(D) \to HS(L^2(D))$ *as defined in Section 2 there exists a martingale solution to* (P).

The proof of Theorem 1.1 is based on a approximation procedure by an implicit time discretization corresponding to (P), which will be introduced in Section 3.1. Since there is a lack of compactness with respect to $\omega \in \Omega$, will have to use the theorems of Prokhorov and Skorokhod that allow us to find a.s. convergence of approximate solutions u_N to a measurable function u_∞ in a new probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ (see Subsection 3.4). Passing to the limit we have to face two different difficulties: Firstly, we have to show that the limit of the stochastic integrals is a stochastic integral with respect to a cylindrical Wiener process defined on a possibly enlarged probability space. This can be done using the Martingale Representation Theorem from [5]. Secondly, since weak convergence is not compatible with nonlinear operators, we have to identify the weak limit of $|\nabla u_N|^{p-2}\nabla u_N$ with $|\nabla u_\infty|^{p-2}\nabla u_\infty$. Once we have identified the stochastic perturbation at the limit, we may use the Itô formula for the identification of the nonlinearity. Subsection 3.5 is devoted to the solution of these two problems.

2 Technical assumptions

For an orthonormal basis (e_n) of $L^2(D)$, $u \in L^2(D)$ let us define

$$H(u)(e_n) := \{x \mapsto h_n(u(x))\},$$

where, for any $n \in \mathbb{N}$, $h_n : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function such that $h_n(0) = 0$ satisfying

$(H1)$ There exists $C_1 > 0$ not depending on μ, λ such that

$$\sum_{n=1}^{\infty} |h_n(\lambda) - h_n(\mu)|^2 \leq C_1 |\lambda - \mu|^2$$

for all $\mu, \lambda \in \mathbb{R}$.

$(H2)$ There exists $C_2 > 0$ such that

$$\sum_{n=1}^{\infty} \|h_n'\|_\infty^2 \leq C_2.$$

For example, $h_n(\lambda) = a_n\lambda$ or $h_n(\lambda) = a_n \sin(\lambda)$ with $n \in \mathbb{N}$ and $(a_n) \in l^2(\mathbb{N})$ are satisfying $(H1)$ and $(H2)$. In particular for any $u \in L^2(D)$ thanks to $(H1)$ we have

$$\begin{aligned}\|H(u)\|_{HS(L^2(D))}^2 &= \sum_{n=1}^{\infty} \|H(u)(e_n)\|_{L^2(D)}^2 = \int_D \sum_{n=1}^{\infty} |h_n(u(x))|^2\, dx \\ &\leq C_1 \|u\|_{L^2(D)}^2 \end{aligned} \tag{5}$$

and herefore $H(u)$ is a Hilbert-Schmidt operator in $L^2(D)$ and $H : L^2(D) \to HS(L^2(D))$ is continuous. Thanks to $(H2)$, we also have the following result:

Proposition 2.1. *$H : W_0^{1,p}(D) \to HS(L^2(D); H_0^1(D))$ is continuous.*

Proof: Let us fix $(u_j) \subset W_0^{1,p}(D)$ such that there exists $u \in W_0^{1,p}(D)$ with $u_j \to u$ in $W_0^{1,p}(D)$ for $j \to \infty$. Then,

$$\begin{aligned} \|H(u_j) - H(u)\|^2_{HS(L^2(D);H_0^1(D))} &= \sum_{n=1}^{\infty} \|h_n(u_j) - h_n(u)\|^2_{H_0^1(D)} \\ &= \sum_{n=1}^{\infty} \int_D |h_n'(u_j)\nabla(u_j - u) + \nabla u(h_n'(u_j) - h_n'(u))|^2 \, dx. \end{aligned} \tag{6}$$

We can extract a not relabeled subsequence (u_j) such that $|\nabla u_j| \le g$ a.e. in D for all $j \in \mathbb{N}$ and some $g \in L^p(D)$ and

$$u_j \to u,$$
$$\nabla u_j \to \nabla u$$

for $j \to \infty$ a.e. in D. For any fixed $n \in \mathbb{N}$, since h_n' is continuous we have,

$$|h_n'(u_j)\nabla(u_j - u) + \nabla u(h_n'(u_j) - h_n'(u))|^2 \to 0 \tag{7}$$

for $j \to \infty$ a.e. in D. Let $C \ge 0$ be a constant not depending on j and n that may change from line to line. By $(H2)$ we have

$$\begin{aligned} & |h_n'(u_j)\nabla(u_j - u) + \nabla u(h_n'(u_j) - h_n'(u))|^2 \\ \le\ & C\|h_n'\|^2_\infty \left(|\nabla u_j - \nabla u|^2 + |\nabla u|^2\right) \\ \le\ & CC_2(|g|^2 + |\nabla u|^2) \end{aligned} \tag{8}$$

and the right-hand side of (8) is in $L^1(D)$. Therefore, by Lebesgue dominated convergence theorem,

$$\lim_{j\to\infty} \int_D |h_n'(u_j)\nabla(u_j - u) + \nabla u(h_n'(u_j) - h_n'(u))|^2 \, dx = 0 \tag{9}$$

for every $n \in \mathbb{N}$. Since such a subsequence with can be extracted from every subsequence of (u_j), (7) holds for the whole sequence (u_j). In particular, for any $N \in \mathbb{N}$, we have

$$\lim_{j\to\infty} \sum_{n=1}^{N} \int_D |h_n'(u_j)\nabla(u_j - u) + \nabla u(h_n'(u_j) - h_n'(u))|^2 \, dx = 0. \tag{10}$$

Let us fix $\varepsilon > 0$. For any $N \in \mathbb{N}$, we have

$$\begin{aligned} & \sum_{n=N}^{\infty} \int_D |h_n'(u_j)\nabla(u_j - u) + \nabla u(h_n'(u_j) - h_n'(u))|^2 \, dx \\ \le\ & \sum_{n=N}^{\infty} 4 \int_D \|h_n'\|^2_\infty |\nabla(u_j - u)|^2 + 4|\nabla u|^2 \|h_n'\|^2_\infty \, dx \\ \le\ & \sum_{n=N}^{\infty} 16\|h_n'\|^2_\infty \left(\int_D |\nabla(u_j - u)|^2 + |\nabla u|^2 \, dx \right). \end{aligned} \tag{11}$$

By $(H2)$,

$$\sum_{n=1}^{\infty} \|h_n'\|^2_\infty < \infty,$$

thus there exists $N_0 \in \mathbb{N}$ such that

$$\sum_{n=N}^{\infty} \|h_n'\|^2_\infty < \varepsilon$$

for all $N \geq N_0$. Therefore, now we get

$$\begin{aligned}
& \|H(u_j) - H(u)\|^2_{HS(L^2(D);H^1_0(D))} \\
= & \sum_{n=1}^{N_0} \int_D |h'_n(u_j)\nabla(u_j - u) + \nabla u(h'_n(u_j) - h'_n(u))|^2 \, dx \\
+ & \sum_{n=N_0+1}^{\infty} \int_D |h'_n(u_j)\nabla(u_j - u) + \nabla u(h'_n(u_j) - h'_n(u))|^2 \, dx \\
\leq & \sum_{n=1}^{N_0} \int_D |h'_n(u_j)\nabla(u_j - u) + \nabla u(h'_n(u_j) - h'_n(u))|^2 \, dx \\
+ & \varepsilon \left(\int_D |\nabla(u_j - u)|^2 + |\nabla u|^2 \, dx \right) \qquad (12)
\end{aligned}$$

using (10) in (12) now it follows that

$$\lim_{j\to\infty} \|H(u_j) - H(u)\|^2_{HS(L^2(D);H^1_0(D))} = 0. \qquad (13)$$

In particular, for any $u \in L^p(\Omega \times (0,T); W^{1,p}_0(D))$, using $(H2)$ we get

$$\begin{aligned}
& E\int_0^T \|H(u)\|^p_{HS(L^2(D);H^1_0(D))} \, dt \\
= & E\int_0^T \left(\sum_{n=1}^{\infty} \|h_n(u)\|^2_{H^1_0(D)} \right)^{p/2} dt \\
\leq & E\int_0^T \left(\sum_{n=1}^{\infty} \|h'_n\|^2 \int_D |\nabla u|^2 \, dx \right)^{p/2} \\
\leq & C_2^{p/2} C_p E \int_0^T \|\nabla u\|_p^p \, dt \qquad (14)
\end{aligned}$$

where $C_p \geq 0$ is a constant which is independent of u.

3 Proof of Theorem 1.1

3.1 Time discretization

For $N \in \mathbb{N}$ let $0 = t_0 < t_1 < \ldots < t_N = T$ be an equidistant subdivision of the interval $[0,T]$ with $\tau := T/N = t_{k+1} - t_k$ for all $k = 0, \ldots, N-1$. Let us introduce the implicit Euler scheme

$$(D) \begin{cases} u_0 = u(t_0) \in W^{1,p}_0(D) \\ u^{k+1} - u^k - \tau \operatorname{div}(|\nabla u^{k+1}|^{p-2}\nabla u^{k+1} + F(u^{k+1})) = H(u^k)\Delta_{k+1}W \end{cases}$$

where $\Delta_{k+1}W := W(t_{k+1}) - W(t_k)$ for $k = 0, \ldots, N-1$.

Remark 3.1. *Since $\Delta_{k+1}W$ takes values in the Hilbert space U, we have*

$$\begin{aligned}
\int_{t_k}^{t_{k+1}} H(u^k) \, dW &= \sum_{n=1}^{\infty} H(u^k)(\frac{1}{n} \cdot ne_n)(\beta_n(t_{k+1}) - \beta_n(t_k)) \\
&= \sum_{n=1}^{\infty} H(u^k) \circ Q^{\frac{1}{2}}(ne_n)(\beta_n(t_{k+1}) - \beta_n(t_k)) \qquad (15)
\end{aligned}$$

for all $k = 0, \dots, N-1$. Since $H(u^k) \circ Q^{\frac{1}{2}} \in HS(U; L^2(D))$, the last expression converges in $L^2(\Omega; \mathcal{C}([0,T]; L^2(D)))$. Therefore we will use the formal notation

$$H(u^k)\Delta_{k+1}W := \int_{t_k}^{t_{k+1}} H(u^k)\, dW = H(u^k) \circ Q^{\frac{1}{2}}(W(t_{k+1}) - W(t_k)).$$

Lemma 3.1. *For any $k = 0, \dots, N-1$, there exists a unique, $\mathcal{F}^W_{t_{k+1}}$-measurable function $u^{k+1} : \Omega \to W^{1,p}_0(D)$ such that for a.e. $\omega \in \Omega$*

$$u^{k+1} - u^k - \tau \mathrm{div}(|\nabla u^{k+1}|^{p-2}\nabla u^{k+1} + F(u^{k+1})) = H(u^k)\Delta_{k+1}W \tag{16}$$

in $L^2(D)$.

Proof: We fix $\tau > 0$. Since $p > 2$, the operator $A_\tau : W^{1,p}_0(D) \to W^{-1,p'}(D)$ defined by

$$\langle A_\tau(u), v\rangle_{W^{-1,p'}(D), W^{1,p}_0(D)} := (u,v)_2 + \tau \int_D (|\nabla u|^{p-2}\nabla u + F(u)) \cdot \nabla v \, dx$$

for u, $v \in W^{1,p}_0(D)$ is a pseudomonotone operator and therefore, by Brezis' theorem, A_τ is onto $W^{-1,p'}(D)$.
In order to show that A_τ is injective, we fix $f \in W^{-1,p'}(D)$ and assume that u_1, u_2 are two solutions to $A_\tau u = f$ in $W^{-1,p'}(D)$. Then we take $v = \mathrm{sign}_\delta$ as a test function, where sign_δ is a Lipschitz continuous approximation of the sign function and obtain $u_1 = u_2$ by passing to the limit when δ goes to 0.
It is left to show that $A_\tau^{-1} : W^{-1,p'}(D) \to W^{1,p}_0(D)$ is continuous. For $f \in W^{-1,p'}(D)$ and u such that $A_\tau(u) = f$, using the Gauss-Green theorem on the convection term we get

$$\|u\|_2^2 + \tau\|\nabla u\|_p^p = \langle f, u\rangle_{W^{-1,p'}(D), W^{1,p}_0(D)} \le \frac{\tau}{2}\|\nabla u\|_p^p + C_\tau \|f\|^{p'}_{W^{-1,p'}(D)}. \tag{17}$$

Let $(f_n) \subset W^{-1,p'}(D)$ be a sequence converging to f in $W^{-1,p'}(D)$. For for all $n \in \mathbb{N}$, we define

$$u_n := A_\tau^{-1}(f_n). \tag{18}$$

From (17) it follows that there exists a not relabeled subsequence of (u_n), $u \in W^{1,p}_0(D)$ and B in $L^{p'}(D)^d$ such that $u_n \rightharpoonup u$ in $W^{1,p}_0(D)$, $u_n \to u$ in $L^p(D)$ and $|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup B$ in $L^{p'}(\Omega)^d$ for $n \to \infty$. Using these convergence results and (18), we get

$$\begin{aligned} & \|u\|_2^2 + \tau \limsup_{n\to\infty} \int_D |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla u_n \, dx \\ = \; & \langle f, u\rangle_{W^{-1,p'}(D), W^{1,p}_0(D)} \\ = \; & \|u\|_2^2 + \tau \int_D B \cdot \nabla u \, dx, \end{aligned} \tag{19}$$

thus from (19) it follows that

$$\begin{aligned} & \limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle_{W^{-1,p'}(D), W^{1,p}_0(D)} \\ = \; & \limsup_{n\to\infty} \int_D |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla u_n \, dx - \int_D B \cdot \nabla u \, dx \\ = \; & 0 \end{aligned} \tag{20}$$

and since A_τ is pseudomonotone, (20) implies $A_\tau u = f$. In particular, $B = |\nabla u|^{p-2}\nabla u$ and

$$\begin{aligned} & \lim_{n\to\infty} \int_D |\nabla u - \nabla u_n|^p \, dx \\ \le \; & 2^{p-2} \limsup_{n\to\infty} \int_D (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \cdot \nabla(u_n - u) \, dx \\ = \; & 0. \end{aligned} \tag{21}$$

From (21) it follows that our not relabeled subsequence (u_n) converges strongly to u in $W_0^{1,p}(D)$ for $n \to \infty$. Since u is unique it follows that the whole sequence (u_n) converges to u in $W_0^{1,p}(D)$ for $n \to \infty$ and A_τ^{-1} is continuous.
Since, for all $k = 0, \dots, N-1$,

$$\begin{aligned} & u^{k+1} - u^k + \tau - \operatorname{div}(|\nabla u^{k+1}|^{p-2}\nabla u^{k+1} + F(u^{k+1})) = H(u^k)\Delta_{k+1}W \\ \Leftrightarrow \quad & u_{k+1} = A_\tau^{-1}(H(u^k)\Delta_{k+1}W + u^k), \end{aligned} \tag{22}$$

and the argument on the right-hand side of (22) is $\mathcal{F}^W_{t_{k+1}}$-measurable assuming that u^k is $\mathcal{F}^W_{t_k}$-measurable, the assertion follows by induction.

3.2 Estimates

Lemma 3.2. *Let (u^{k+1}) be a solution to (D). Then,*

$$\begin{aligned} & \frac{1}{2}E\left(\|u^{k+1}\|_2^2 - \|u^k\|_2^2\right) + \frac{1}{4}E\|u^{k+1}-u^k\|_2^2 + \tau E\int_D |\nabla u^{k+1}|^p\,dx \\ \leq \quad & \tau E\|H(u^k)\|^2_{HS(L^2(D))} \end{aligned} \tag{23}$$

for all $k = 0, 1, \dots, N-1$.

Proof: Taking the L^2-scalar product with u^{k+1} in (16), we get

$$\begin{aligned} & \|u^{k+1}\|_2^2 - (u^k, u^{k+1})_2 - \tau(\operatorname{div}(|\nabla u^{k+1}|^{p-2}\nabla u^{k+1} + F(u^{k+1})), u^{k+1})_2 \\ = \quad & (H(u^k)\Delta_{k+1}W, u^{k+1})_2 \\ \Leftrightarrow \quad & I_1 + I_2 + I_3 = I_4 \end{aligned} \tag{24}$$

where

$$\begin{aligned} I_1 &:= \|u^{k+1}\|_2^2 - (u^k, u^{k+1})_2 = \frac{1}{2}\left(\|u^{k+1}\|_2^2 - \|u^k\|_2^2 + \|u^{k+1}-u^k\|_2^2\right), \\ I_2 &:= \tau\int_D |\nabla u^{k+1}|^p\,dx, \\ I_3 &= \tau\int_D F(u^{k+1})\nabla u^{k+1}\,dx = 0, \\ I_4 &:= (H(u^k)\Delta_{k+1}W, u^{k+1}-u^k)_2 + (H(u^k)\Delta_{k+1}W, u^k)_2. \end{aligned}$$

Taking expectation on both sides of (24) we arrive at

$$\begin{aligned} & \frac{1}{2}E\left(\|u^{k+1}\|_2^2 - \|u^k\|_2^2 + \|u^{k+1}-u^k\|_2^2\right) + \tau E\int_D |\nabla u^{k+1}|^p\,dx \\ = \quad & E(H(u^k)\Delta_{k+1}W, u^{k+1}-u^k)_2 + E(H(u^k)\Delta_{k+1}W, u^k)_2. \end{aligned} \tag{25}$$

Since $u_k, H(u^k)$ are $\mathcal{F}^W_{t_k}$-measurable and $W(t_{k+1}) - W(t_k)$ is $\mathcal{F}^W_{t_k}$-independent, we have

$$\begin{aligned} & E(H(u^k)\Delta_{k+1}W, u^k)_2 = EE\left[\left(H(u^k)\circ Q^{1/2}(W(t_{k+1})-W(t_k)), u^k\right)_2 |\mathcal{F}_{t_k}\right] \\ = \quad & E\left(u^k, E\left[\int_{t_k}^{t_{k+1}} H(u^k)\,dW|\mathcal{F}_{t_k}\right]\right)_2 = 0. \end{aligned} \tag{26}$$

Using Hölder and Young inequality it follows that for any $\alpha > 0$

$$\begin{aligned} & E(H(u^k)\Delta_{k+1}W, u^{k+1}-u^k)_2 \leq E(\|\Phi_k\Delta_{k+1}W\|_2 \cdot \|u^{k+1}-u^k\|_2) \\ \leq \quad & \frac{1}{2}\left(\frac{1}{\alpha}E\left\|\int_{t_k}^{t_{k+1}} H(u^k)\,dW\right\|_2^2 + \alpha E\|u^{k+1}-u^k\|_2^2\right) \end{aligned} \tag{27}$$

By Itô isometry and for $\alpha = \frac{1}{2}$ from (27) it follows that

$$\begin{aligned} & E(H(u^k)\Delta_{k+1}W, u^{k+1} - u^k) \\ \leq\ & E\int_{t_k}^{t_{k+1}} \|H(u^k)\|^2_{HS(L^2(D))}\ dt + \frac{1}{4}E\|u^{k+1} - u^k\|_2^2 \\ =\ & \tau E\|H(u^k)\|^2_{HS(L^2(D))} + \frac{1}{4}E\|u^{k+1} - u^k\|_2^2 \end{aligned} \tag{28}$$

and therefore we arrive at

$$\begin{aligned} & \frac{1}{2}E\left(\|u^{k+1}\|_2^2 - \|u^k\|_2^2 + \|u^{k+1} - u^k\|_2^2\right) + \tau E\int_D |\nabla u^{k+1}|^p\ dx \\ \leq\ & \tau E\|H(u^k)\|^2_{HS(L^2(D))} + \frac{1}{4}E\|u^{k+1} - u^k\|_2^2, \end{aligned} \tag{29}$$

hence (23) holds.

Definition 3.1. *For $N \in \mathbb{N}$, $\tau > 0$ we introduce the right-continuous step function*

$$u_N(t) = \sum_{k=0}^{N-1} u^{k+1}\chi_{[t_k,t_{k+1})}(t),\ t \in [0,T],$$

the left-continuous, $\mathcal{F}_t^W$-adapted step function

$$u_\tau(t) = \sum_{k=0}^{N-1} u^k\chi_{(t_k,t_{k+1}]}(t),\ t \in (0,T],\ u_\tau(0) = u_0,$$

the continuous, square-integrable $\mathcal{F}_t^W$-martingale

$$B_N(t) = \int_0^t H(u_\tau)\ dW,\ t \in [0,T]$$

and the piecewise affine functions

$$\widetilde{u}_N(t) := \sum_{k=0}^{N-1}\left(\frac{u^{k+1} - u^k}{\tau}(t - t_k) + u^k\right)\chi_{[t_k,t_{k+1})}(t),\ t \in [0,T),\ \widetilde{u}_N(T) = u^N,$$

$$\widetilde{B}_N(t) = \sum_{k=0}^{N-1}\left(\frac{B_N(t_{k+1}) - B_N(t_k)}{\tau}(t - t_k) + B_N(t_k)\right)\chi_{[t_k,t_{k+1})}(t),\ t \in [0,T].$$

Lemma 3.3. *There exists a constant $K \geq 0$ not depending on the discretization parameters such that*

$$\max_{n=1,\dots,N} \|u^n\|_2^2 \leq K, \tag{30}$$

$$\sum_{k=0}^{N-1} E\|u^{k+1} - u^k\|_2^2 \leq 4K + 2\|u_0\|_2^2. \tag{31}$$

In particular, by $(H1)$ there exists $K(C_1, C_2, \|u_0\|, T) > 0$ such that

$$E\int_0^T \|H(u_\tau)\|^2_{HS(L^2(D))}\ dt \leq K(C_1, C_2, \|u_0\|, T). \tag{32}$$

Moreover we have

$$E\sup_{t\in[0,T]} \|\widetilde{u}_N\|_2^2 = E\sup_{t\in[0,T]} \|u_N\|_2^2 \leq K, \tag{33}$$

$$E\int_0^T\int_D |\nabla u_N|^p\ dx\ dt \leq K + \frac{1}{2}\|u_0\|_2^2. \tag{34}$$

Proof: We fix $n \in \{1, \dots, N\}$, take the sum over $0, \dots, n-1$ in (23) to get

$$\begin{aligned}
&\frac{1}{2}E\|u^n\|_2^2 - \frac{1}{2}E\|u_0\|_2^2 + \frac{1}{4}\sum_{k=0}^{n-1} E\|u^{k+1} - u^k\|_2^2 + \sum_{k=0}^{n-1} \tau E \int_D |\nabla u^{k+1}|^p \, dx \\
\leq \; & \sum_{k=0}^{n-1} \tau E\|H(u^k)\|_{HS(L^2(D))}^2 \, ds \qquad (35)
\end{aligned}$$

Discarding nonnegative terms by $(H1)$ it follows that

$$\frac{1}{2}E\|u^n\|_2^2 \leq \frac{1}{2}E\|u_0\|_2^2 + \sum_{k=0}^{n-1} C_1 \tau E\|u^k\|_2^2 \qquad (36)$$

Applying the discrete Gronwall inequality in (36) yields

$$E\|u^n\|_2^2 \leq \|u_0\|_2^2 e^{2C_1T} \qquad (37)$$

and (30) follows from (37) with $K := \|u_0\|_2^2 e^{2C_1T}$. Now (31) follows from (30) and (35) by taking $n = N$ and keeping the nonnegative term

$$\frac{1}{4}\sum_{k=0}^{n-1} E\|u^{k+1} - u^k\|_2^2.$$

(32) is a direct consequence of $(H1)$ and (30). Moreover,

$$E \sup_{t\in[0,T]} \|\widetilde{u}_N\|_2^2 = E \sup_{t\in[0,T]} \|u_N\|_2^2 \leq E \max_{k=1,\dots,N} \|u^k\|_2^2 \leq K. \qquad (38)$$

Finally, (34) follows now from (35) and (30) by keeping the nonnegative term

$$\sum_{k=0}^{n-1} \tau E \int_D |\nabla u^{k+1}|^p \, dx$$

and taking $n = N$.

Lemma 3.4. *There exists $C \geq 0$ not depending on $N \in \mathbb{N}$ such that*

$$\begin{aligned}
&E \int_0^T \|\frac{d}{dt}(\widetilde{u}_N - \widetilde{B}_N)\|_{W^{-1,p'}(D)}^{p'} \, dt \\
\leq \; & C\left(E\int_0^T \|u_N\|_2^2 + \|\nabla u_N\|_p^p \, dt + 1\right). \qquad (39)
\end{aligned}$$

Proof: For all $t \in (t_k, t_{k+1})$, and all $k = 0, \dots, N-1$

$$\begin{aligned}
\frac{d}{dt}(\widetilde{u}_N - \widetilde{B}_N) &= \frac{u^{k+1} - u^k - H(u^k)\Delta_{k+1}W}{\tau} \\
&= \operatorname{div}(|\nabla u^{k+1}|^{p-2}\nabla u^{k+1} + F(u^{k+1})). \qquad (40)
\end{aligned}$$

Since $p \geq 2$, there exists a constant $C \geq 0$ not depending on $N \in \mathbb{N}$ that may change from line to line such that

$$\begin{aligned}
&\|\frac{d}{dt}(\widetilde{u}_N - \widetilde{B}_N)\|_{W^{-1,p'}(D)} \\
= &\sup_{\|\varphi\|_{W_0^{1,p}(D)}\le 1} \int_D \Big[|\nabla u^{k+1}|^{p-2}\nabla u^{k+1} + F(u^{k+1})\Big]\nabla\varphi\, dx \\
\le &\sup_{\|\varphi\|_{W_0^{1,p}(D)}\le 1} \Big(\|\nabla u^{k+1}\|_p^{p-1}\|\nabla\varphi\|_p + \|F(u^{k+1})\|_2\|\nabla\varphi\|_2\Big) \\
\le &\sup_{\|\varphi\|_{W_0^{1,p}(D)}\le 1} \Big(\|\nabla u^{k+1}\|_p^{p-1} + C\|F(u^{k+1})\|_2\|\nabla\varphi\|_p\Big) \\
\le &\|\nabla u^{k+1}\|_p^{p-1} + \|F(u^{k+1})\|_2. \qquad (41)
\end{aligned}$$

Therefore, for $p' \le 2$, $L > 0$ the Lipschitz constant of F

$$\begin{aligned}
\|\frac{d}{dt}(\widetilde{u}_N - \widetilde{B}_N)\|_{W^{-1,p'}(D)}^{p'} &\le \Big(\|\nabla u^{k+1}\|_p^{p-1} + C\|F(u^{k+1})\|_2\Big)^{p'} \\
&\le 2^{p'}(\|\nabla u^{k+1}\|_p^p + CL\|u^{k+1}\|_2^{p'}) \\
&\le 2^{p'}\|\nabla u^{k+1}\|_p^p + C(1 + \|u^{k+1}\|_2^2).
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_0^T \|\frac{d}{dt}(\widetilde{u}_N - \widetilde{B}_N)\|_{W^{-1,p'}(D)}^{p'}\, dt \\
\le &\ 2^{p'}\tau\sum_{k=0}^{N-1}\|\nabla u^{k+1}\|_p^p + C\left(1 + \tau\sum_{k=0}^{N-1}\|u^{k+1}\|_2^2\right),
\end{aligned}$$

and

$$\begin{aligned}
&E\int_0^T \|\frac{d}{dt}(\widetilde{u}_N - \widetilde{B}_N)\|_{W^{-1,p'}(D)}^{p'}\, dt \\
\le &\ 2^{p'}\tau\sum_{k=0}^{N-1}E\|\nabla u^{k+1}\|_p^p + C(1 + \tau\sum_{k=0}^{N-1}E\|u^{k+1}\|_2^2) \\
\le &\ C\left(E\int_0^T \|u_N\|_2^2 + \|\nabla u_N\|_p^p\, dt + 1\right).
\end{aligned}$$

From Lemma 3.3 and Lemma 3.4 we get

Lemma 3.5. *There exists a constant $C \ge 0$ not depending on $N \in \mathbb{N}$ such that*

$$E\int_0^T \|\frac{d}{dt}(\widetilde{u}_N - \widetilde{B}_N)\|_{W^{-1,p'}(D)}^{p'} \le C. \qquad (42)$$

Lemma 3.6. *For $T > 0$, $N \in \mathbb{N}$ we define an equidistant subdivision of $[0,T]$ by*

$$0 = t_0 < t_1 < \ldots < t_N = T$$

with $\tau = \frac{T}{N} = t_{k+1} - t_k$ for $k = 0, \ldots, N-1$. Let $\mathcal{K}$, $\mathcal{H}$ be separable Hilbert spaces and W be a Wiener process in $\mathcal{K}$ with covariance operator Q. For a $\mathcal{F}_{t_k}^W$-measurable random variable Φ_k with values in $HS(Q^{1/2}(\mathcal{K}), \mathcal{H})$ we define the left-continuous, $\mathcal{F}_t^W$-adapted process

$$\Phi_\tau := \sum_{k=0}^{N-1}\Phi_k\chi_{(t_k,t_{k+1}]}.$$

For any $p > 2$, there exists constants $\gamma > 0$ and $C_\gamma \geq 0$ not depending on $N \in \mathbb{N}$ and an integrable, real-valued random variable X such that

$$\begin{aligned} & \sup_{k\in\{0,\dots,N-1\}} \sup_{s\in[t_k,t_{k+1}]} \| \int_{t_k}^{s} \Phi_\tau \, dW \|_{\mathcal{H}} \\ \leq \; & C_\gamma \tau^\gamma \left(\sup_{k\in\{0,\dots,N-1\}} \tau \|\Phi_k\|^p_{HS(\mathcal{K},\mathcal{H})} + 1 + X \right). \end{aligned}$$

Moreover, there exists a constant $C \geq 0$ such that

$$E(X) \leq C \mathrm{tr}(Q). \tag{43}$$

Proof: Let us fix $s \in [t_k, t_{k+1}]$ and $k \in \{0, \dots, N-1\}$. Then we have

$$\| \int_{t_k}^{s} \Phi_\tau \, dW \|_{\mathcal{H}} \leq \|\Phi_k\|_{HS(\mathcal{K},\mathcal{H})} \|W(s) - W(t_k)\|_{\mathcal{K}}.$$

Now, from [10], [16, Ex. 2.4.1] (see Lemma 4.6 in the Appendix) for any $q \geq 1$ and $\alpha > \frac{1}{q}$ it follows that

$$\begin{aligned} & \|W(s) - W(t_k)\|_{\mathcal{K}} \\ \leq \; & C_{\alpha,q}^{1/q} \tau^{\alpha - 1/q} \left(\int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \frac{\|W(t) - W(r)\|^q_{\mathcal{K}}}{|t-r|^{\alpha q + 1}} \, dt \, dr \right)^{1/q} \\ \leq \; & C_{\alpha,q}^{1/q} \tau^{\alpha - 1/q} \left(\int_0^T \int_0^T \frac{\|W(t) - W(r)\|^q_{\mathcal{K}}}{|t-r|^{\alpha q + 1}} \, dt \, dr \right)^{1/q} \\ = \; & C_{\alpha,q}^{1/q} \tau^{\alpha - 1/q} X^{1/q} \end{aligned} \tag{44}$$

where

$$X := \int_0^T \int_0^T \frac{\|W(t) - W(r)\|^q_{\mathcal{K}}}{|t-r|^{\alpha q + 1}} \, dt \, dr$$

is a real-valued random variable. Thus,

$$\begin{aligned} & \sup_{k\in\{0,\dots,N-1\}} \sup_{s\in[t_k,t_{k+1}]} \| \int_{t_k}^{s} \Phi_\tau \, dW \|_{\mathcal{H}} \\ \leq \; & \sup_{k\in\{0,\dots,N-1\}} \sup_{s\in[t_k,t_{k+1}]} \|\Phi_k\|_{HS(\mathcal{K},\mathcal{H})} \|W(s) - W(t_k)\|_{\mathcal{K}} \\ \leq \; & \left(\sup_{k\in\{0,\dots,N-1\}} \|\Phi_k\|_{HS(\mathcal{K},\mathcal{H})} \right) \sup_{k\in\{0,\dots,N-1\}} \sup_{s\in[t_k,t_{k+1}]} \|W(s) - W(t_k)\|_{\mathcal{K}} \end{aligned}$$

and from (44) it follows that

$$\begin{aligned} & \sup_{k\in\{0,\dots,N-1\}} \sup_{s\in[t_k,t_{k+1}]} \| \int_{t_k}^{s} \Phi_\tau \, dW \|_{\mathcal{H}} \\ \leq \; & C_{\alpha,q}^{1/q} \tau^{\alpha - 1/q} X^{1/q} \sup_{k\in\{0,\dots,N-1\}} \|\Phi_k\|_{HS(\mathcal{K},\mathcal{H})} \\ = \; & C_{\alpha,q}^{1/q} \tau^{\alpha - 1/q - 1/p} \left(\sup_{k\in\{0,\dots,N-1\}} \tau^{1/p} \|\Phi_k\|_{HS(\mathcal{K},\mathcal{H})} X^{1/q} \right) \\ \leq \; & C_{\alpha,q}^{1/q} \tau^{\alpha - 1/q - 1/p} \left(\sup_{k\in\{0,\dots,N-1\}} \tau \|\Phi_k\|^p_{HS(\mathcal{K},\mathcal{H})} + X^{p'/q} \right) \\ \leq \; & C_{\alpha,q}^{1/q} \tau^{\alpha - 1/q - 1/p} \left(\sup_{k\in\{0,\dots,N-1\}} \tau \|\Phi_k\|^p_{HS(\mathcal{K},\mathcal{H})} + 1 + X \right) \end{aligned}$$

where $q \geq 1$ is such that

$$\gamma := \alpha - 1/q - 1/p > 0, \quad p'/q \leq 1.$$

Moreover,

$$E(X) = \int_0^T \int_0^T \frac{E\|W(t) - W(r)\|_{\mathcal{K}}^q}{|t-r|^{\alpha q+1}} \, dt \, dr.$$

Since

$$W(t) - W(s) \sim \mathcal{N}(0, Q(t-s)),$$

it follows that there exists $C_q \geq 0$ such that

$$E\|W(t) - W(r)\|_{\mathcal{K}}^q \leq C_q \mathrm{tr}(Q)|t-r|^{q/2},$$

and one gets, choosing q such that $q > p > 2$ and $\alpha \in (\frac{1}{p} + \frac{1}{q}, \frac{1}{2})$

$$E(X) \leq C_q \mathrm{tr}(Q) \int_0^T \int_0^T |t-r|^{q/2-\alpha q-1} dt \, dr =: C\mathrm{tr}(Q).$$

3.3 Regularity of approximate solutions

Lemma 3.7. *There exists a constant $K_1 > 0$ not depending on the discretization parameters such that*

$$E\int_0^T \|H(u_\tau)\|_{HS(L^2(D);H_0^1(D))}^p \, dt \leq K_1. \tag{45}$$

Proof: We fix an orthonormal basis $(e_n) \subset L^2(D)$. Then,

$$E\int_0^T \|H(u_\tau)\|_{HS(L^2(D);H_0^1(D))}^p \, dt = E\sum_{k=0}^{N-1} \tau\|H(u^k)\|_{HS(L^2(D);H_0^1(D))}^p$$

$$= E\tau \sum_{k=0}^{N-1} \left(\sum_{n=1}^{\infty} \|H(u^k)(e_n)\|_{H_0^1(D)}^2 \right)^{p/2} \tag{46}$$

Now we use $(H2)$ to estimate

$$\begin{aligned} E\tau \sum_{k=0}^{N-1} \left(\sum_{n=1}^{\infty} \|H(u^k)(e_n)\|_{H_0^1(D)}^2 \right)^{p/2} &\leq EC_2^{p/2}\tau \sum_{k=0}^{N-1} \|\nabla u^k\|_2^p \\ \leq\ & C_2^{p/2}\tau E\left(\sum_{k=0}^{N-1} \|\nabla u^{k+1}\|_2^p + \|\nabla u_0\|_2^p \right) \\ \leq\ & C_2^{p/2} C_p E\int_0^T \|\nabla u_N\|_p^p + \|\nabla u_0\|_p^p \, dt \\ =\ & C_2^{p/2} C_p E\int_0^T \|\nabla u_N\|_p^p \, dt + T\|\nabla u_0\|_p^p \, dt \end{aligned} \tag{47}$$

where $C_p \geq 0$ is a constant not depending on the discretization parameters. According to Lemma 3.3, (34), from (47) it follows that

$$E\int_0^T \|H(u_\tau)\|_{HS(L^2(D);H_0^1(D))}^p \, dt \leq K_1 \tag{48}$$

with $K_1 := \max(C_2^{p/2}C_p(K + \|u_0\|_2^2), C_2^{p/2}C_p\|\nabla u_0\|_p^p)$.

Definition 3.2. *For a Banach space V, $T>0$, $0<\alpha<1$ and $1\le p<\infty$ we recall the definition of the fractional Sobolev space (see also [1], p.111, [15] for more information):*

$$W^{\alpha,p}(0,T;V) := \{f \in L^p(0,T;V) \mid \|f\|_{W^{\alpha,p}(0,T;V)} < +\infty\},$$

where

$$\|f\|_{W^{\alpha,p}(0,T;V)} = \left(\int_0^T\int_0^T \frac{\|f(r)-f(t)\|_V^p}{|t-r|^{\alpha p+1}}\,dr\,dt\right)^{1/p}.$$

Lemma 3.8. *For any $\alpha\in(0,\frac{1}{2})$ there exists a constant $C(\alpha,p)\ge 0$ such that*

$$E\|\int_0^{\cdot} H(u_\tau)\,dW\|^p_{W^{\alpha,p}(0,T;H_0^1(D))} \le C(\alpha,p)K_1, \tag{49}$$

where $K_1\ge 0$ is defined in Lemma 3.7. In particular,

$$\int_0^{\cdot} H(u_\tau)\,dW$$

is bounded in $L^p(\Omega;W^{\alpha,p}(0,T;H_0^1(D)))$.

Proof: We recall that u_τ is a left-continuous, $\mathcal{F}_t^W$-adapted process with values in $W_0^{1,p}(D)$ and $H: W_0^{1,p}(D)\to HS(L^2(D);H_0^1(D))$ is continuous. Thus, $H(u_\tau)$ is a left-continuous, $\mathcal{F}_t^W$-adapted process and therefore it is progressively measurable. From [8], Lemma 2.1., p.369 (Lemma 4.7 in the Appendix) it follows that there exists $C(\alpha,p)\ge 0$ such that

$$\begin{aligned} & E\|\int_0^{\cdot} H(u_\tau)\,dW\|^p_{W^{\alpha,p}(0,T;H_0^1(D))} \\ \le\ & C(\alpha,p)E\int_0^T \|H(u_\tau)\|^p_{HS(L^2(D);H_0^1(D))}\,dt. \end{aligned} \tag{50}$$

Now, the assertion is a direct consequence of Lemma 3.7.

Lemma 3.9. *$(\widetilde{B}_N)$ is uniformly bounded in $L^p(\Omega;W^{\alpha,p}(0,T;H_0^1(D)))$ for any $\alpha\in(0,\gamma)$ and $\gamma=\frac{1}{2}-\frac{1}{p}$.*

Proof: We have to verify the assumptions of Lemma [2], Lemma 3.2 (Lemma 4.8 in the Appendix) for $\mathcal{G}=\widetilde{B}_N$: For any $l\in\{0,\dots,N\}$ we have

$$\begin{aligned} & \tau\sum_{k=0}^{N-l}\|\widetilde{B}_N(t_{k+l})-\widetilde{B}_N(t_k)\|^p_{L^p(\Omega;H_0^1(D))} \\ =\ & \tau\sum_{k=0}^{N-l}E\|\int_{t_k}^{t_{k+l}} H(u_\tau)\,dW\|^p_{H_0^1(D)} \\ =\ & \tau\sum_{k=0}^{N-l}E\|\int_0^{t_{k+l}} H(u_\tau)\chi_{(t_k,t_{k+l}]}\,dW\|^p_{H_0^1(D)}. \end{aligned} \tag{51}$$

We use the Burkholder-Davies-Gundy and the Hölder inequality to get

$$\begin{aligned} & \tau\sum_{k=0}^{N-l}\|\widetilde{B}_N(t_{k+l})-\widetilde{B}_N(t_k)\|^p_{L^p(\Omega;H_0^1(D))} \\ \le\ & \tau\sum_{k=0}^{N-l}E\left(\int_{t_k}^{t_{k+l}}\|H(u_\tau)\|^2_{HS(L^2(D);H_0^1(D))}\,dt\right)^{p/2} \\ \le\ & E\tau\sum_{k=0}^{N-l}(t_{k+l}-t_k)^{\frac{p}{2}-1}\left(\int_0^T\|H(u_\tau)\|^p_{HS(L^2(D);H_0^1(D))}\,dt\right). \end{aligned} \tag{52}$$

From (51) and Lemma 3.7 it follows that there exists a constant $K_1 > 0$ not depending on the discretization parameters such that

$$\begin{aligned}
&\tau \sum_{k=0}^{N-l} \|\widetilde{B}_N(t_{k+l}) - \widetilde{B}_N(t_k)\|^p_{L^2(\Omega;H_0^1(D))} \\
\leq \;& \tau(N-l) t_l^{\frac{p}{2}-1} E \int_0^T \|H(u_\tau)\|^p_{HS(L^2(D);H_0^1(D))}\, dt \\
\leq \;& (T+t_l) t_l^{\frac{p}{2}-1} K_1 \leq 2TK_1 t_l^{\frac{p-2}{2}}
\end{aligned} \tag{53}$$

For $\gamma := \frac{1}{2} - \frac{1}{p} > 0$, and $C := (2TK_1)^{1/p}$ from (53) it follows that

$$\tau \sum_{k=0}^{N-l} \|\widetilde{B}_N(t_{k+l}) - \widetilde{B}_N(t_k)\|^p_{L^p(\Omega;H_0^1(D))} \leq C^p t_l^{\gamma p}. \tag{54}$$

According to Lemma [2], Lemma 3.2 (Lemma 4.8 in the Appendix), from (54) it follows that $(\widetilde{B}_N)$ is uniformly bounded in the Nikolskii space

$$N^{\gamma,p}(0,T;L^p(\Omega;H_0^1(D))) \hookrightarrow W^{\alpha,p}(0,T;L^p(\Omega \times H_0^1(D)))$$

with continuous imbedding for any $\alpha \in (0,\gamma)$ (see [1], p.111, [15]). Thanks to the Fubini theorem this implies

$$\|\widetilde{B}_N\|_{L^p(\Omega;W^{\alpha,p}(0,T;H_0^1(D)))} \leq C$$

for all $N \in \mathbb{N}$ and a constant $C \geq 0$ not depending on $N \in \mathbb{N}$.

Remark 3.2. *It is well-known (see, e.g. [13], Lemma 7.1, p.202 and Lemma 7.7, p.208) that the space*

$$\mathcal{W} := \{v \in L^p(0,T;H_0^1(D)) \mid \frac{d}{dt} v \in L^{p'}(0,T;W^{-1,p'}(D))\}$$

is continuously embedded into $\mathcal{C}([0,T];W^{-1,p'}(D))$ *and compactly embedded into* $L^2(0,T;L^2(D))$.

Lemma 3.10. *There exists a constant* $C \geq 0$ *such that*

$$\|\widetilde{u}_N\|_{L^p(\Omega;L^p(0,T;W_0^{1,p}(D)))} + \|\widetilde{u}_N - \widetilde{B}_N\|_{L^{p'}(\Omega;\mathcal{W})} \leq C \tag{55}$$

for all $N \in \mathbb{N}$.

Proof: Elementary calculations yield that there exists a constant $\tilde{C} > 0$ not depending on the discretization parameters such that

$$\begin{aligned}
E\|\widetilde{u}_N\|^p_{L^p(0,T;W_0^{1,p}(D))} \;&\leq\; \tilde{C} E \tau \sum_{k=0}^{N} \|u^k\|^p_{W_0^{1,p}(D)} \\
&\leq\; \tilde{C} E \left(\int_0^T \|\nabla u_N\|_p^p \, dt + \|\nabla u_0\|_p^p \right)
\end{aligned} \tag{56}$$

and by Lemma 3.3 the right-hand side of (56) is bounded. From Lemma 3.9 it follows that $(\widetilde{B}_N)$ is bounded in $L^p(\Omega;W^{\alpha,p}(0,T;H_0^1(D)))$ for $\alpha \in (0,\frac{1}{2}-\frac{1}{p})$. Thus, $(\widetilde{u}_N - \widetilde{B}_N)$ is bounded in $L^p(\Omega;L^p(0,T;H_0^1(D)))$. Now, the assertion is a direct consequence of Lemma 3.5.

3.4 Tightness

Next, we set

$$\mathcal{X} := \mathcal{C}([0,T];L^2(D)) \times L^2(0,T;L^2(D)).$$

For $N \in \mathbb{N}$, we denote the law of $\widetilde{u}_N$ by $\mu_{\widetilde{u}_N} = P \circ (\widetilde{u}_N)^{-1}$ on $L^2(0,T;L^2(D))$ and the law of B_N by $\mu_{B_N} = P \circ (B_N)^{-1}$ on $\mathcal{C}([0,T];L^2(D))$. Their joint law on $\mathcal{X}$ is denoted by $\mu_N = (\mu_{B_N}, \mu_{\widetilde{u}_N})$.

Proposition 3.11. *The sequence $(\mu_{\widetilde{u}_N})$ is tight on $L^2(0,T;L^2(D))$ and the sequence (μ_{B_N}) is tight on $\mathcal{C}([0,T];L^2(D))$. In particular, the sequence of their joint laws (μ_N) is tight on $\mathcal{X}$.*

Proof: For $\alpha \in (0, \frac{1}{2})$, the linear space

$$\mathcal{V} := \{u = v + w,\ v \in \mathcal{W},\ w \in W^{\alpha,p}(0,T;H_0^1(D))\}$$

endowed with the norm

$$\|u\|_{\mathcal{V}} := \inf_{\substack{v\in\mathcal{W},\\ w\in W^{\alpha,p}(0,T;H_0^1(D)),\\ u=v+w}} \max(\|v\|_{\mathcal{W}}, \|w\|_{W^{\alpha,2}})$$

is a Banach space which is compactly embedded into $L^2(0,T;L^2(D))$ (see Lemma 4.9 in the Appendix). Since

$$\widetilde{u}_N = (\widetilde{u}_N - \widetilde{B}_N) + \widetilde{B}_N$$

for all $N \in \mathbb{N}$, it follows from Lemma 3.9 and Lemma 3.10 that $(\widetilde{u}_N)$ is bounded in $L^{p'}(\Omega;\mathcal{V})$. Now, let us fix $\varepsilon > 0$. For any $R > 0$ the set

$$B_{\mathcal{V}}(R,0) := \{u \in \mathcal{V} \mid \|u\|_{\mathcal{V}} \leq R\}$$

is compact in $L^2(0,T;L^2(D))$. There exists a constant $C > 0$ not depending $R > 0$, such that for any $R > 0$, and any $N \in \mathbb{N}$

$$\begin{aligned}
\mu_{\widetilde{u}_N}(B_{\mathcal{V}}(R,0)) &= 1 - \mu_{\widetilde{u}_N}(B^c_{\mathcal{V}}(R,0)) \\
&= 1 - \int_{\{\omega\in\Omega \ \mid \ \|\widetilde{u}_N\|_{\mathcal{V}}>R\}} 1 \, dP \\
&\geq 1 - \frac{1}{R^{p'}} \int_{\{\omega\in\Omega \ \mid \ \|\widetilde{u}_N\|_{\mathcal{V}}>R\}} \|\widetilde{u}_N\|^{p'}_{\mathcal{V}} \, dP \\
&\geq 1 - \frac{1}{R^{p'}} E(\|\widetilde{u}_N\|^{p'}_{\mathcal{V}}) = 1 - \frac{C}{R^{p'}} \qquad (57)
\end{aligned}$$

and from (57) it follows that we can find $R_\varepsilon > 0$ such that

$$\mu_{\widetilde{u}_N}(B_{\mathcal{V}}(R_\varepsilon, 0)) \geq 1 - \varepsilon$$

for all $N \in \mathbb{N}$.

According to [14], p.82, Corollary 2,

$$W^{\alpha,p}(0,T;H_0^1(D)) \hookrightarrow \mathcal{C}([0,T];L^2(D))$$

with compact imbedding for all $\alpha \in (\frac{1}{p}, \frac{1}{2})$. Thus, for any $R > 0$ and any $\alpha \in (\frac{1}{p}, \frac{1}{2})$,

$$B_{W^{\alpha,p}}(R,0) := \{u \in W^{\alpha,p}(0,T;H_0^1(D)) \mid \|u\|_{W^{\alpha,p}(0,T;H_0^1(D))} \leq R\}$$

is compact in $\mathcal{C}([0,T];L^2(D))$.
By Lemma 3.8 (B_N) is uniformly bounded in $L^p(\Omega;W^{\alpha,p}(0,T;H_0^1(D)))$ for $\alpha\in(0,\frac{1}{2})$, hence there exists a constant $C>0$ not depending $R>0$ such that

$$\begin{aligned}\mu_{B_N}(B_{W^{\alpha,p}}(R,0)) &= 1-\mu_{B_N}(B^c_{W^{\alpha,p}}(R,0))\\ \geq\ & 1-\frac{1}{R^p}E(\|B_N\|^p_{W^{\alpha,p}(0,T;H_0^1(D))}) = 1-\frac{C}{R^p}\end{aligned} \tag{58}$$

Thanks to (58), for any $\varepsilon>0$ we can find $R_\varepsilon>0$ such that

$$\mu_{B_N}(B_{W^{\alpha,p}}(R_\varepsilon,0))\geq 1-\varepsilon.$$

Remark 3.3. *From Prokhorov theorem (see Theorem 4.1 and 4.3 in the Appendix for references) and Proposition 3.11 it follows that the sequence $(\mu_N)=(\mu_{B_N},\mu_{\widetilde{u}_N})$ is relatively compact, i.e. there exists a (not relabeled) subsequence of (μ_N) and a probability measure $\mu_\infty=(\mu^1_\infty,\mu^2_\infty)$ on $\mathcal{X}$, such that*

$$\lim_{N\to\infty}\int_{\mathcal{C}([0,T];L^2(D))}\psi\ d\mu_{B_N}=\int_{\mathcal{C}([0,T];L^2(D))}\psi\ d\mu^1_\infty \tag{59}$$

for all bounded, continuous functions $\psi:\mathcal{C}([0,T];L^2(D))\to\mathbb{R}$,

$$\lim_{N\to\infty}\int_{L^2(0,T;L^2(D))}\varphi\ d\mu_{\widetilde{u}_N}=\int_{L^2(0,T;L^2(D))}\varphi\ d\mu^2_\infty \tag{60}$$

for all bounded, continuous functions $\varphi:L^2(0,T;L^2(D))\to\mathbb{R}$. In particular,

$$\int_{L^2(0,T;L^2(D))}\varphi\ d\mu_{\widetilde{u}_N}=\int_\Omega\varphi(\widetilde{u}_N)\ dP=E[\varphi(\widetilde{u}_N)],$$

$$\int_{\mathcal{C}([0,T];L^2(D))}\psi\ d\mu_{B_N}=\int_\Omega\psi(B_N)\ dP=E[\psi(B_N)],$$

hence (59) implies $B_N\mathscr{L}\!\!\rightsquigarrow\mu^1_\infty$ and (60) implies $\widetilde{u}_N\mathscr{L}\!\!\rightsquigarrow\mu^2_\infty$.

3.5 Existence of martingale solutions

Now, we use the following version of the theorem of Skorokhod (see [18], Theorem 1.10.4 and Addendum 1.10.5, p.59 and [1], Theorem 2.3, p.119-120), which can be found in the Appendix, to conclude:

There exists a sequence of measurable mappings

$$(\hat{B}_N,\hat{u}_N):(\hat{\Omega},\hat{\mathcal{F}},\hat{P})\to\mathcal{X},\ \ N\in\mathbb{N}\cup\{\infty\},$$

on some probalility space $(\hat{\Omega},\hat{\mathcal{F}},\hat{P})$ such that

i.) $\hat{u}_N\to u_\infty$ in $L^2(0,T;L^2(D))$ for $N\to\infty$ a.s. in $\hat{\Omega}$,

ii.) $\hat{B}_N\to B_\infty$ in $\mathcal{C}([0,T];L^2(D))$ for $N\to\infty$ a.s. in $\hat{\Omega}$

iii.) The joint laws $\mu_N=(\mu_{\mu_{B_N}},\mu_{\widetilde{u}_N})$ and $\hat{\mu}_N=(\mu_{\hat{B}_N},\mu_{\hat{u}_N})$ are the same for all $N\in\mathbb{N}\cup\{\infty\}$. In particular, this implies

$$E[\Psi(B_N,\widetilde{u}_N)]=\int_{\mathcal{X}}\Psi\ d\mu_N=\int_{\mathcal{X}}\Psi\ d\hat{\mu}_N=E[\Psi(\hat{B}_N,\hat{u}_N)]$$

for all $N\in\mathbb{N}$ and all $\Psi\in\mathcal{C}_b(\mathcal{X})$.

iv.) There exist measurable mappings $\phi_N : (\hat{\Omega}, \hat{\mathcal{F}}) \to (\Omega, \mathcal{F})$ such that $\hat{u}_N = \widetilde{u}_N \circ \phi_N$, $\hat{B}_N = B_N \circ \phi_N$ and $P = \hat{P} \circ \phi_N^{-1}$ for all $N \in \mathbb{N}$.

We can assume without loss of generality (see Remark 4.2 in the Appendix for a reference) that $\hat{\mathcal{F}}$ is countably generated.

Definition 3.3. *For $N \in \mathbb{N}$ we define $W_N := W \circ \phi_N$ and*

$$v^k := u^k \circ \phi_N, \ k = 0, \dots, N.$$

For all $t \in [0,T]$, we introduce the left-continuous function

$$v_\tau(t) := \sum_{k=0}^{N-1} v^k \chi_{(t_k, t_{k+1}]}(t), \ t \in (0,T], \ v_\tau(0) = u_0,$$

the right-continuous function

$$v_N(t) := \sum_{k=0}^{N-1} v^{k+1} \chi_{[t_k, t_{k+1})}(t), \ t \in [0,T]$$

and the piecewise affine function

$$b_N(t) := \sum_{k=0}^{N-1} \left(\frac{\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k)}{\tau} (t - t_k) + \hat{B}_N(t_k) \right) \chi_{[t_k, t_{k+1})}(t), \ t \in [0,T].$$

Lemma 3.12. *For any $N \in \mathbb{N}$ and any $k = 0, \dots, N-1$ we have*

$$v^{k+1} - v^k - \tau \mathrm{div}(|\nabla v^{k+1}|^{p-2} \nabla v^{k+1} + F(v^{k+1})) - H(v^k) \Delta_{k+1} W_N = 0 \tag{61}$$

a.s. in $\hat{\Omega}$.

Proof: Since $P = \hat{P} \circ \phi_N^{-1}$, by definition of the image measure for any $\hat{A} \in \hat{\mathcal{F}}$ we have

$$\begin{aligned} & \int_{\hat{A}} v^{k+1} - v^k - \tau \mathrm{div}(|\nabla v^{k+1}|^{p-2} \nabla v^{k+1} + F(v^{k+1})) - H(v^k) \Delta_{k+1} W_N \, d\hat{P} \\ = & \int_{\phi_N(\hat{A})} u^{k+1} - u^k - \tau \mathrm{div}(|\nabla u^{k+1}|^{p-2} \nabla u^{k+1} + F(u^{k+1})) - H(u^k) \Delta_{k+1} W \, dP \\ = & \ 0 \end{aligned} \tag{62}$$

Lemma 3.13. *For any $N \in \mathbb{N}$, W_N is a Q-Wiener process in U with $Q = \mathrm{diag}(\frac{1}{k^2})$, thus a cylindrical Wiener process in $L^2(D) = Q^{1/2}(U)$ adapted to the natural filtration $(\mathcal{F}_t^{W_N})$.*

Proof: For all $t \in [0,T]$ and all $N \in \mathbb{N}$, $W_N(t)$ is $\hat{\mathcal{F}}/\mathcal{B}(L^2(D))$-measurable as the composition of the $\hat{\mathcal{F}}/\mathcal{F}$-measurable function ϕ_N with the $\hat{\mathcal{F}}/\mathcal{B}(L^2(D))$-measurable function $Q^{1/2} \circ W(t)$. Thus, $W_N : \hat{\Omega} \times [0,T] \to L^2(D)$ is a stochastic process. All further properties follow from the representation

$$W_N(\hat{\omega}, t) = \sum_{k=0}^{\infty} \frac{1}{k} \beta_k(\phi_N(\hat{\omega}), t)$$

in $L^2(\hat{\Omega}; \mathcal{C}([0,T]; L^2(D)))$, for an orthonormal basis (e_k) of $L^2(D)$ and (β_k) independant, real-valued Brownian motions on $(\Omega, \mathcal{F}, P)$.

Lemma 3.14. *We have*

$$\hat{B}_N(t) = \int_0^t H(v_\tau) \, dW_N, \tag{63}$$

for all $t \in [0,T]$, *a.s. in* $\hat{\Omega}$,

$$\hat{u}_N(t) = \frac{v^{k+1} - v^k}{\tau}(t - t_k) + v^k, \tag{64}$$

for all $t \in [t_k, t_{k+1})$, $k = 0, \dots, N-1$ *and* $\hat{u}_N(T) = v^N$ *a.s. in* $\hat{\Omega}$. *Moreover, there exist constants* $\hat{K}$, $\hat{K}_1 \geq 0$ *such that*

$$E \sup_{t\in[0,T]} \|\hat{u}_N(t)\|_2^2 = E \sup_{t\in[0,T]} \|v_N(t)\|_2^2 \leq \hat{K}, \tag{65}$$

$$E \int_0^T \int_D |\nabla v_N|^p \; dx \; dt \leq \hat{K} + \frac{1}{2}\|u_0\|_2^2. \tag{66}$$

$$E \int_0^T \|H(v_\tau)\|^p_{H_0^1(D)} \; dt \leq \hat{K}_1 \tag{67}$$

for all $N \in \mathbb{N}$.

Proof: For any $t \in [t_k, t_{k+1})$ and $k = 0, \dots, N-1$ we have

$$\begin{aligned}
\hat{B}_N(t) &= (B_N \circ \phi_N)(t) \\
&= \sum_{l=0}^{k-1} H(u^l \circ \phi_N) \circ Q^{1/2}(W(t_{l+1}) \circ \phi_N - W(t_l) \circ \phi_N) \\
&+ H(u^k \circ \phi_N) \circ Q^{1/2}(W(t) \circ \phi_N - W(t_k) \circ \phi_N) \\
&= \sum_{l=0}^{k-1} H(v^l) \circ Q^{1/2}(W_N(t_{l+1}) - W_N(t_l)) \\
&+ H(v^k) \circ Q^{1/2}(W_N(t) - W_N(t_k)) \\
&= \int_0^t H(v_\tau) \; dW_N.
\end{aligned} \tag{68}$$

(64) follows since

$$\begin{aligned}
\hat{u}_N(\hat{\omega}, t) &= \widetilde{u}_N(\phi_N(\hat{\omega}), t) \\
&= \frac{u^{k+1}(\phi_N(\hat{\omega})) - u^k(\phi_N(\hat{\omega}))}{\tau}(t - t_k) + u^k(\phi_N(\hat{\omega}))
\end{aligned} \tag{69}$$

for a.e. $\hat{\omega} \in \hat{\Omega}$, all $t \in [t_k, t_{k+1})$, $k = 0, \dots N-1$. Moreover,

$$\hat{u}_N(\hat{\omega}, T) = \widetilde{u}_N(\phi_N(\hat{\omega}), T) = u^N(\phi_N(\hat{\omega})) = v^N.$$

Thanks to (61), (65) and (66) follow repeating the arguments in the proof of Lemma 3.3 with recpect to v^{k+1}. Then, (67) follows repeating the arguments in the proof of Lemma 3.7 with respect to v_τ.

Lemma 3.15. *For* $N \to \infty$, *we have the following convergences:*

1.) $\hat{B}_N \to B_\infty$ *in* $L^q(\hat{\Omega}; \mathcal{C}([0,T]; L^2(D)))$ *for all* $1 \leq q < p$,

2.) $\hat{B}_N \rightharpoonup B_\infty$ *in* $L^p(\hat{\Omega}; W^{\alpha,p}(0,T; H_0^1(D)))$,

3.) $\hat{u}_N \to u_\infty$ *in* $L^q(\hat{\Omega}; L^2(0,T; L^2(D)))$ *for all* $1 \leq q < p$,

4.) $v_N \to u_\infty$ *in* $L^2(\hat{\Omega} \times Q)$,

5.) $v_\tau \to u_\infty$ *in* $L^2(\hat{\Omega} \times Q)$.

6.) $\hat{u}_N \overset{*}{\rightharpoonup} u_\infty$ *in* $L^2_w(\hat{\Omega}; L^\infty(0,T;L^2(D)))$*, where*

$$L^2_w(\hat{\Omega}; L^\infty(0,T;L^2(D))) \simeq \left(L^2(\hat{\Omega}; L^1(0,T;L^2(D)))\right)^*$$

and the space on the left-hand side contains all weak-$*$ measurable mappings

$$u : \hat{\Omega} \to L^\infty(0,T;L^2(D)),\ E\|u\|_{L^\infty(0,T;L^2(D))} < \infty$$

(see [7], Th. 8.20.3, p.606).

Proof: For $\alpha \in (\frac{1}{p}, \frac{1}{2})$,

$$L^p(\hat{\Omega}; W^{\alpha,p}(0,T;H^1_0(D))) \hookrightarrow L^p(\hat{\Omega}; \mathcal{C}([0,T];L^2(D)))$$

with continuous imbedding. Thus, using Lemma 3.14, (67) and [8], Lemma 2.1., p.369 (Lemma 4.7 in the Appendix) it follows that there exists $C \geq 0$ such that

$$\|\hat{B}_N\|_{L^p(\hat{\Omega};\mathcal{C}([0,T];L^2(D)))} + \|\hat{B}_N\|_{L^p(\hat{\Omega};W^{\alpha,p}(0,T;H^1_0(D)))} \leq C \tag{70}$$

for all $N \in \mathbb{N}$ and therefore $(\hat{B}_N)$ is equi-integrable in $L^q(\hat{\Omega}; \mathcal{C}([0,T];L^2(D)))$ for all $1 \leq q < p$. Since $\hat{B}_N \to B_\infty$ in $\mathcal{C}([0,T];L^2(D))$ for $N \to \infty$ a.s. in $\hat{\Omega}$, 1.) follows from the Vitali theorem. Passing to a not relabeled subsequence, from (70) also follows that there exists $g \in L^p(\hat{\Omega}; W^{\alpha,p}(0,T;H^1_0(D)))$, such that

$$\hat{B}_N \rightharpoonup g \text{ in } L^p(\hat{\Omega}; W^{\alpha,p}(0,T;H^1_0(D)))$$

for $N \to \infty$. Since

$$L^p(\hat{\Omega}; W^{\alpha,p}(0,T;H^1_0(D))) \hookrightarrow L^2(\hat{\Omega}; L^2(Q))$$

with continuous imbedding and $\hat{B}_N \to B_\infty$ in $L^2(\hat{\Omega} \times Q)$ for $N \to \infty$, it follows that $g = B_\infty$ a.s. in $\hat{\Omega} \times Q$. Thus, the whole sequence $(\hat{B}_N)$ converges weakly to B_∞ in $L^p(\hat{\Omega}; W^{\alpha,p}(0,T;H^1_0(D)))$ and we have shown 2.).
There exists a constant $\hat{C} \geq 0$ not depending on $N \in \mathbb{N}$ such that

$$E\|\hat{u}_N\|^p_{L^p(0,T;W^{1,p}_0(D))} \leq \hat{C} E\left(\int_0^T \|\nabla v_N\|_p^p\, dt + \|\nabla u_0\|_p^p\right). \tag{71}$$

By Lemma (66) and the Poincaré inequality it follows that $(\hat{u}_N)$ is bounded in $L^p(\hat{\Omega}; L^2(Q))$ and therefore equi-integrable in $L^q(\hat{\Omega}; L^2(Q))$ for all $1 \leq q < p$. Together with the a.s. convergence of $(\hat{u}_N)$ to u_∞ in $L^2(Q)$ for $N \to \infty$, 2.) follows from the Vitali theorem.
From Lemma 3.12 and Lemma 3.14 it follows with similar arguments as in Lemma 3.3 that there exists a constant $C \geq 0$ such that

$$\sum_{k=0}^{N-1} E\|v^{k+1} - v^k\|_2^2 \leq C. \tag{72}$$

For any $N \in \mathbb{N}$ we have

$$\begin{aligned}
& E\int_0^T \|\hat{u}_N(t) - v_N(t)\|_2^2\, dt \\
= \;& E\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \left\|\frac{v^{k+1} - v^k}{\tau}(t - t_k) + v^k - v^{k+1}\right\|_2^2\, dt \\
= \;& E\sum_{k=0}^{N-1} \|v^{k+1} - v^k\|_2^2 \int_{t_k}^{t_{k+1}} \left(\frac{t - t_k}{\tau} - 1\right)^2\, dt \\
= \;& \frac{\tau}{3}\sum_{k=0}^{N-1} E\|v^{k+1} - v^k\|_2^2 \leq \tau\frac{C}{3}
\end{aligned} \tag{73}$$

therefore 3.) follows. Finally, from (72) we also have

$$\begin{aligned} E\int_0^T \|v_\tau - v_N\|_2^2\, dt &= E\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} \|v^{k+1}-v^k\|_2^2\, dt \\ &= E\tau\sum_{k=0}^{N-1}\|v^{k+1}-v^k\|_2^2 \le C\tau \end{aligned} \tag{74}$$

and 4.) follows from (74).
Using Lemma 3.14, (65), from the Banach-Alaouglu theorem it follows that there exists $f \in L^2_w(\hat\Omega; L^\infty(0,T;L^2(D)))$ such that, passing to a not relabeled subsequence, $\hat u_N \overset{*}{\rightharpoonup} f$ in $L^2_w(\hat\Omega; L^\infty(0,T;L^2(D)))$ for $N\to\infty$. Now, taking test functions $\chi_A\psi$ with $\psi\in\mathcal{D}(Q)$ and $A\in\hat{\mathcal{F}}$, it follows that $f = u_\infty$ a.s. in $\hat\Omega\times Q$.

Lemma 3.16. *$u_\infty \in L^\infty(0,T;L^2(D))$ a.s. in $\hat\Omega$.*

Proof: Since $u_\infty \in L^2_w(\hat\Omega; L^\infty(0,T;L^2(D)))$, the mapping

$$\hat\Omega \ni \hat\omega \mapsto \|u(\hat\omega)\|_{L^\infty(0,T;L^2(D))} \in \mathbb{R}$$

is $\hat{\mathcal{F}}$-measurable and therefore the assertion follows.

The next lemma is a direct consequence of Lemma 3.14, (66):

Lemma 3.17. *There exists a not relabeled subsequence of (v_N) such that*

$$\nabla v_N \rightharpoonup \nabla u_\infty \text{ in } L^p(\hat\Omega\times Q)^d \tag{75}$$

for $N\to\infty$. Moreover, there exists $G\in L^{p'}(\hat\Omega\times Q)^d$ such that

$$|\nabla v_N|^{p-2}\nabla v_N \rightharpoonup G \text{ in } L^{p'}(\hat\Omega\times Q)^d \tag{76}$$

for the same subsequence and $N\to\infty$.

Lemma 3.18. *There exist constants $\gamma>0$, $C_\gamma\ge 0$, $C\ge 0$ such that*

$$\begin{aligned} &E\sup_{t\in[0,T]}\|\hat B_N(t) - b_N(t)\|_{H^1_0(D)} \\ \le\ & C_\gamma\tau^\gamma\left(E\int_0^T \|H(v_\tau)\|^p_{HS(L^2(D);H^1_0(D))}\, dt + 1 + C\mathrm{tr}(Q)\right), \end{aligned} \tag{77}$$

for all $N\in\mathbb{N}$, where $Q = \mathrm{diag}(\frac{1}{k^2})$.

Proof: We fix $N\in\mathbb{N}$. For $k\in\{0,\dots,N-1\}$ and $t\in[t_k,t_{k+1})$ we have a.s. in $\hat\Omega$

$$\begin{aligned} &\|\hat B_N(t) - b_N(t)\|_{H^1_0(D)} \\ =\ & \|\int_0^t H(v_\tau)\, dW_N - \frac{\hat B_N(t_{k+1}) - \hat B_N(t_k)}{\tau}(t-t_k) - \hat B_N(t_k)\|_{H^1_0(D)} \\ =\ & \|\int_{t_k}^t H(v_\tau)\, dW_N - \frac{t-t_k}{\tau}\int_{t_k}^{t_{k+1}} H(v_\tau)\, dW_N\|_{H^1_0(D)} \\ \le\ & \|\int_{t_k}^t H(v_\tau)\, dW_N\|_{H^1_0(D)} + \|\int_{t_k}^{t_{k+1}} H(v_\tau)\, dW_N\|_{H^1_0(D)} \end{aligned} \tag{78}$$

and therefore

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\hat{B}_N(t)-b_N(t)\|_{H_0^1(D)} \\
=& \sup_{k=0,\dots,N-1}\ \sup_{t\in[t_k,t_{k+1})} \|\hat{B}_N(t)-b_N(t)\|_{H_0^1(D)} \\
\leq& \ 2 \sup_{k=0,\dots,N-1}\ \sup_{t\in[t_k,t_{k+1})} \|\int_{t_k}^t H(v_\tau)\ dW_N\|_{H_0^1(D)} \qquad (79)
\end{aligned}$$

By Lemma 3.13, W_N is a Q-Wiener process on U, thus according to Lemma 3.6, there exists $\gamma>0$, $C_\gamma\geq 0$ not depending on $N\in\mathbb{N}$ such that

$$\begin{aligned}
&E\sup_{t\in[0,T]} \|\hat{B}_N(t)-b_N(t)\|_{H_0^1(D)} \\
\leq& \ 2C_\gamma\tau^\gamma\left(E\sup_{k=0,\dots,N-1}\tau\|H(v^k)\|^p_{HS(L^2(D);H_0^1(D))}+1+C\mathrm{tr}(Q)\right) \\
\leq& \ 2C_\gamma\tau^\gamma\left(E\int_0^T\|H(v_\tau)\|^p_{HS(L^2(D);H_0^1(D))}\ dt+1+C\mathrm{tr}(Q)\right). \qquad (80)
\end{aligned}$$

Corollary 3.19. *From Lemma 3.14, (67) and Lemma 3.18 it follows that*

$$E\sup_{t\in[0,T]}\|\hat{B}_N(t)-b_N(t)\|_{H_0^1(D)}\leq C_\gamma\tau^\gamma(\hat{K}_1+1+C\mathrm{tr}(Q))$$

for all $N\in\mathbb{N}$.

Proposition 3.20. *$u_\infty:\hat{\Omega}\times[0,T]\to L^2(D)$ is a stochastic process with $u_\infty(0)=u_0$ such that*

$$u_\infty(t)=B_\infty(t)+u_0+\int_0^t \mathrm{div}(G+F(u_\infty))\ ds \qquad (81)$$

holds in $L^2(D)$ a.s. in $\hat{\Omega}$ for all $t\in[0,T]$.

Proof: For all $k=0,\dots,N-1$ from (61) it follows that

$$\frac{v^{k+1}-v^k-(\hat{B}_N(t_{k+1})-\hat{B}_N(t_k))}{\tau}=\mathrm{div}(|\nabla v^{k+1}|^{p-2}\nabla v^{k+1}+F(v^{k+1})) \qquad (82)$$

Multiplying (82) with χ_A for $A\in\hat{\mathcal{F}}$, $\psi\in W_0^{1,p}(D)$, $\xi\in\mathcal{D}(0,T)$, integrating over $[t_k,t_{k+1}]\times D\times\hat{\Omega}$ and summing over $k=1,\dots,N-1$ it follows that

$$\begin{aligned}
&\int_A\int_0^T\int_D(\hat{u}_N-b_N)\xi_t\psi\ dx\ dt\ d\hat{P} \\
=& \int_A\int_0^T\int_D(|\nabla v_N|^{p-2}\nabla v_N+F(v^N))\cdot\nabla\psi\xi\ dx\ dt\ d\hat{P} \qquad (83)
\end{aligned}$$

Let us write (83) as

$$I_1+I_2=I_3+I_4, \qquad (84)$$

where

$$\begin{aligned}
I_1 &= \int_A\int_0^T\int_D(\hat{u}_N-\hat{B}_N)\xi_t\psi(x)\ dx\ dt\ d\hat{P}, \\
I_2 &= \int_A\int_0^T\int_D(\hat{B}_N-b_N)\xi_t\psi(x)\ dx\ dt\ d\hat{P} \\
I_3 &= \int_A\int_0^T\int_D|\nabla v_N|^{p-2}\nabla v_N\cdot\nabla\psi\xi\ dx\ dt\ d\hat{P} \\
I_4 &= \int_A\int_0^T\int_D F(v_N)\cdot\nabla\psi\xi\ dx\ dt\ d\hat{P} \qquad (85)
\end{aligned}$$

Since $\hat{B}_N - \hat{u}_N \to B_\infty - u_\infty$ in $L^2(\hat{\Omega} \times Q)$ for $N \to \infty$, it follows that

$$\lim_{N\to\infty} I_1 = \int_A \int_0^T \int_D (u_\infty - B_\infty)\xi_t \psi(x) \, dx \, dt \, d\hat{P}. \tag{86}$$

Moreover, by Hölder inequality,

$$\begin{aligned} |I_2| &\leq \int_A \int_0^T \|\xi_t \psi\|_2 \|\hat{B}_N - b_N\|_2 \, dt \, d\hat{P} \\ &\leq \int_{\hat{\Omega}} \sup_{t\in[0,T]} \|\hat{B}(t)_N - b_N(t)\|_2 \, d\hat{P} \int_0^T \|\xi_t \psi\|_2 \, dt \\ &\leq C_D E \sup_{t\in[0,T]} \|\hat{B}_N(t) - b_N(t)\|_{H_0^1(D)} \int_0^T \|\xi_t \psi\|_2 \, dt \end{aligned} \tag{87}$$

where $C_D \geq 0$ is a constant not depending on $N \in \mathbb{N}$. From Corollary 3.19 it now follows that

$$|I_2| \leq C_D C_\gamma \tau^\gamma (\hat{K}_1 + 1 + C\text{tr}(Q)) \int_0^T \|\xi_t \psi\|_2 \, dt, \tag{88}$$

therefore $\lim_{N\to\infty} I_2 = 0$. Since

$$|\nabla v_N|^{p-2} \nabla v_N \rightharpoonup G \text{ in } L^{p'}(\hat{\Omega} \times Q)^d$$

for $N \to \infty$ (see Lemma 3.18), we get

$$\lim_{N\to\infty} I_3 = \int_A \int_0^T \int_D G \cdot \nabla \psi \xi \, dx \, dt \, d\hat{P}. \tag{89}$$

From Lemma 3.15 it follows that $v_N \to u_\infty$ in $L^2(\hat{\Omega} \times Q)$ for $N \to \infty$, thus we can extract a not relabeled subsequence such that

$$v_N \to u_\infty \text{ a.e. in } \hat{\Omega} \times Q$$

and there exists $g \in L^2(\hat{\Omega} \times Q)$ such that $|v_N| \leq g$ for all $N \in \mathbb{N}$ a.e. in $\hat{\Omega} \times Q$. Since F is Lipschitz continuous, it follows by Lebesgue dominated convergence theorem that

$$\lim_{N\to\infty} F(v_N) = F(u_\infty). \tag{90}$$

in $L^2(\hat{\Omega} \times Q)^d$. Since this argument can be repeated with any arbitrary subsequence of (v_N), (90) holds for the whole sequence and therefore

$$\lim_{N\to\infty} I_4 = \int_A \int_0^T \int_D F(u_\infty) \cdot \nabla \psi \xi \, dx \, dt \, d\hat{P}. \tag{91}$$

Now from (86)-(91) it follows that

$$-\int_A \int_0^T \int_D (u_\infty - B_\infty)\xi_t \psi + (G + F(u_\infty)) \cdot \nabla \psi \xi \, dx \, dt \, d\hat{P} = 0 \tag{92}$$

for all $A \in \hat{\mathcal{F}}$, $\xi \in \mathcal{D}(0,T)$ and all $\psi \in W_0^{1,p}(D)$. (92) implies that

$$\frac{d}{dt}(u_\infty - B_\infty) = \text{div}(G + F(u_\infty)) \tag{93}$$

in $L^{p'}(\hat{\Omega}; L^{p'}(0,T; W^{-1,p'}(D)))$. Moreover, from Lemma 3.18 (75) and Lemma 3.15, 2.) it follows that

$$u_\infty - B_\infty \in L^p(\hat{\Omega}; L^p(0,T; H_0^1(D))),$$

thus $u_\infty - B_\infty \in L^{p'}(\hat\Omega; \mathcal{C}([0,T]; W^{-1,p'}(D)))$. Thanks to Lemma 3.16 and [17], Lemma 1.4, p.263, it follows that $(u_\infty - B_\infty)$ is weakly continuous with values in $L^2(D)$ a.s. in $\hat\Omega$. Consequently,

$$(u_\infty - B_\infty)(t) \in L^2(D)$$

for all $t \in [0,T]$, a.s. in $\hat\Omega$, hence

$$\langle (B_\infty - u_\infty)(t), \psi\rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} = \int_D (u_\infty - B_\infty)(t)\psi \, dx \tag{94}$$

for all $\psi \in W_0^{1,p}(D)$, a.s. in $\hat\Omega$ for all $t \in [0,T]$. With this information we may fix $t \in [0,T)$ and choose a test function $\xi \in \mathcal{D}([t,T))$ with $\xi(t) = 1$. Then, for any $\psi \in W_0^{1,p}(D)$, a.s. in $\hat\Omega$ using (93) and (94) we get

$$\begin{aligned}
& \int_t^T \int_D (u_\infty - B_\infty)\xi_t \psi \, dx \, dr \\
= & \int_t^T \xi_t \langle (B_\infty - u_\infty)(r), \psi\rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} \, dr \\
= & \int_t^T \xi_t \left\langle (u_\infty - B_\infty)(t) + \int_t^r \operatorname{div}(G + F(u_\infty))(s) \, ds, \psi \right\rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} dr,
\end{aligned}$$

and using Fubini theorem we get

$$\begin{aligned}
& \int_t^T \int_D (u_\infty - B_\infty)\xi_t \psi \, dx \, dr + \int_D (u_\infty - B_\infty)(t)\psi \, dx \\
= & \int_t^T \int_t^r \xi_t(r) \langle \operatorname{div}(G + F(u_\infty))(s), \psi\rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} \, ds \, dr \\
= & \int_t^T \langle \operatorname{div}(G + F(u_\infty))(s), \psi\rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} \int_s^T \xi_t(r) \, dr \, ds \\
= & \int_t^T \int_D (G + F(u_\infty)) \cdot \nabla\psi \xi \, dr.
\end{aligned} \tag{95}$$

From (95) it follows that

$$\begin{aligned}
& -\int_A \int_t^T \int_D (\hat u_N - b_N)\xi_t \psi \, dx \, dt \, d\hat P - \int_A \int_D (\hat u_N - b_N)(t)\psi \, dx \, d\hat P \\
& + \int_A \int_t^T \int_D (|\nabla v_N|^{p-2}\nabla v_N + F(v_N)) \cdot \nabla\psi\xi \, dx \, dt \, d\hat P = 0 \\
& = -\int_A \int_t^T \int_D (u_\infty - B_\infty)\xi_t \psi \, dx \, dr \, d\hat P - \int_A \int_D (u_\infty - B_\infty)(t)\psi \, dx \, d\hat P \\
& + \int_A \int_t^T \int_D (G + F(u_\infty)) \cdot \nabla\psi\xi \, dx \, dr \, d\hat P.
\end{aligned} \tag{96}$$

From Lemma 3.14, (65) it follows that there exists a subsequence $(u_{N_t}(t))$ of $(u_N(t))$ converging weakly some $\chi(t)$ in $L^2(\hat\Omega \times D)$. With respect to this subsequence we have

$$\int_A \int_D (\hat u_{N_t} - b_N)(t)\psi \, dx \, d\hat P = I_1 + I_2 \tag{97}$$

where, for $N \to \infty$,

$$I_1 = \int_A \int_D (\hat u_{N_t} - \hat B_N)(t)\psi \, dx \, d\hat P \to \int_A \int_D (\chi(t) - B_\infty(t))\psi \, dx \, d\hat P$$

and, using Corollary 3.19,

$$\begin{aligned}|I_2| &\leq \int_A \|\hat{B}_N(t) - b_N(t)\|_2 \|\psi\|_2 \, d\hat{P} \\ &\leq \|\psi\|_2 C_D E \sup_{t\in[0,T]} \|\hat{B}_N(t) - b_N(t)\|_{H_0^1(D)} \\ &\leq C_D C_\gamma \tau^\gamma (\hat{K}_1 + 1 + C\mathrm{tr}(Q)) \to 0.\end{aligned}$$

Passing to the subsequence N_t, we can pass to the limit with $N_t \to \infty$ in (96) and it follows that

$$\int_D \chi(t)\psi \, dx = \int_D u_\infty(t)\psi \, dx \tag{98}$$

a.s. in $\hat{\Omega}$. Thus, $\chi(t) = u_\infty(t)$ a.s. in $\hat{\Omega} \times D$ for all $t \in [0, T)$. In particular, for $t = 0$, we get $u_\infty(0) = u_0$ in $L^2(D)$ and equation (81) holds true. Moreover, for any $t \in [0, T)$ the weak convergence to $\chi(t)$ holds for the whole sequence $(u_N(t))$. With this information, using the weak continuity of u_∞ and $\hat{u}_N$ we can prove that $\chi(T) = u_\infty(T)$ a.s. in $\hat{\Omega} \times D$ and we have

Corollary 3.21. *For all $t \in [0, T]$, $u_N(t) \rightharpoonup u_\infty(t)$ in $L^2(\hat{\Omega} \times D)$.*

With the proof of the following lemma the proof of Proposition 3.20 is completed:

Lemma 3.22. *u_∞ is a stochastic process with values in $L^2(D)$.*

Proof: Since u_∞ is weakly continuous with values in $L^2(D)$ for a.s. in $\hat{\Omega}$, it follows that

$$\hat{\Omega} \ni \hat{\omega} \mapsto u_\infty(\hat{\omega})(t) \in L^2(D)$$

for all $t \in [0, T]$. We fix $t \in [0, T]$ and prove that $u_\infty(t)$ is a random variable: By Pettis theorem, $u_\infty(t)$ is measurable, if it is weakly measurable, i.e. the mapping

$$\hat{\Omega} \ni \hat{\omega} \mapsto (u(t)(\hat{\omega}), h)_2$$

is measurable for all $h \in L^2(D)$. Recall that

$$B_\infty \in L^2(\hat{\Omega}; \mathcal{C}([0, T]; L^2(D)))$$

and

$$B_\infty - u_\infty \in L^{p'}(\hat{\Omega}; \mathcal{C}([0, T]; W^{-1,p'}(D))),$$

hence it follows that $u_\infty \in L^{p'}(\hat{\Omega}; \mathcal{C}([0, T]; W^{-1,p'}(D)))$, thus for all $h \in W_0^{1,p}(D)$

$$\hat{\Omega} \ni \hat{\omega} \mapsto (u(t)(\hat{\omega}), h)_2 = \langle u(t)(\hat{\omega}), h\rangle_{W^{-1,p'}(D), W_0^{1,p}(D)}$$

is measurable. Now, the assertion follows since any $h \in L^2(D)$ can be approximated by a sequence $(h_n) \subset W_0^{1,p}(D)$ in $L^2(D)$.

Proposition 3.23. *B_∞ is an $\mathcal{F}_t^\infty$-martingale with respect to the augmentation $(\mathcal{F}_t^\infty)$ of the natural filtration $\hat{\mathcal{F}}_t^\infty := \sigma(B_\infty(s), u_\infty(s))_{0\leq s\leq t}$, $t \in [0, T]$ (i.e. the smallest complete, right-continuous filtration containing $(\hat{\mathcal{F}}_t^\infty)$) such that*

$$\ll B_\infty \gg_t = \int_0^t H(u_\infty) \circ H^*(u_\infty) \, ds \tag{99}$$

for all $t \in [0, T]$.

Proof: To show that B_∞ is a $\mathcal{F}_t^\infty$-martingale, it is enough to show that it is a $\hat{\mathcal{F}}_t^\infty$-martingale (see [4], p.75). By definition, B_∞ is adapted to $(\hat{\mathcal{F}}_t^\infty)$. Thus we have to prove that

$$E((B_\infty(t) - B_\infty(s))\chi_A) = 0 \tag{100}$$

for all $A \in \hat{\mathcal{F}}_s^\infty$ and all $0 \le s \le t$. (100) is equivalent to

$$E[(B_\infty(t) - B_\infty(s), h)_2 \psi(B_\infty, u_\infty)] = 0 \tag{101}$$

for all $t \in [0,T]$, $0 \le s \le t$, $\psi \in \mathcal{C}_b(\mathcal{C}([0,s]; L^2(D)) \times L^2(0,s; L^2(D)))$ and all $h \in L^2(D)$. By Lemma 3.15, 4.), we may pass to a not relabeled subsequence of (v_τ), such that $v_\tau \to u_\infty$ for $N \to \infty$ in $L^2(\hat{\Omega}; L^2(Q))$ and a.s. in $L^2(Q)$. We will show that

$$\begin{aligned} & E[(B_\infty(t) - B_\infty(s), h)_2 \psi(B_\infty, u_\infty)] \\ = & \lim_{N\to\infty} E[(\hat{B}_N(t) - \hat{B}_N(s), h)_2 \psi(\hat{B}_N, v_\tau)] = 0. \end{aligned} \tag{102}$$

for all $t \in [0,T]$, $0 \le s \le t$, $\psi \in \mathcal{C}_b(\mathcal{C}([0,s]; L^2(D)) \times L^2(0,s; L^2(D)))$ and all $h \in L^2(D)$.

For any $N \in \mathbb{N}$, for $t \in [0,T]$ the process

$$\hat{B}_N(t) = \int_0^t H(v_\tau)\, dW_N \tag{103}$$

is a continuous, square-integrable martingale with respect to $(\mathcal{F}_t^{W_N})$. Moreover, $\hat{B}_N$ is $\mathcal{F}_t^\tau := \sigma(\hat{B}_N(s), v_\tau(s))_{0\le s\le t} \subset \mathcal{F}_t^{W_N}$-adapted and for all $t \in [0,T]$, for all $A \in \mathcal{F}_s^\tau$ we have

$$\begin{aligned} E[(\hat{B}_N(t) - \hat{B}_N(s))\chi_A] &= E[E((\hat{B}_N(t) - \hat{B}_N(s))\chi_A | \mathcal{F}_s^\tau)] \\ &= E[\chi_A E((\hat{B}_N(t) - \hat{B}_N(s)) | \mathcal{F}_s^\tau)] \\ &= E[\chi_A E(E((\hat{B}_N(t) - \hat{B}_N(s)) | \mathcal{F}_s^{W_N}) | \mathcal{F}_s^\tau)] \\ &= 0. \end{aligned} \tag{104}$$

Thus $\hat{B}_N$ is also a $\mathcal{F}_t^\tau$-martingale with

$$\ll \hat{B}_N \gg_t = \int_0^t H(v_\tau) \circ H^*(v_\tau)\, ds. \tag{105}$$

For any $N \in \mathbb{N}$, (104) is equivalent to

$$E[(\hat{B}_N(t) - \hat{B}_N(s), h)_2 \psi(\hat{B}_N, v_\tau)] = 0 \tag{106}$$

for any $t \in [0,T]$, $0 \le s \le t$, $\psi \in \mathcal{C}_b(\mathcal{C}([0,s]; L^2(D)) \times L^2(0,s; L^2(D)))$, $h \in L^2(D)$.
We fix $t \in [0,T]$, $0 \le s \le t$, $\psi \in \mathcal{C}_b(\mathcal{C}([0,s]; L^2(D)) \times L^2(0,s; L^2(D)))$ and $h \in L^2(D)$. Our aim is to pass to the limit with $N \to \infty$ on the left-hand side of

$$E[(\hat{B}_N(t) - \hat{B}_N(s), h)_2 \psi(\hat{B}_N, v_\tau)] = 0. \tag{107}$$

To this end, we will show that

i.) $(\hat{B}_N(t) - \hat{B}_N(s), h)_2 \to (B_\infty(t) - B_\infty(s), h)_2$ in $L^2(\hat{\Omega})$,

ii.) $\psi(\hat{B}_N, v_\tau) \to \psi(B_\infty, u_\infty)$ in $L^2(\hat{\Omega})$.

For all $t \in [0,T]$, $0 \le s \le t$, $\delta_{t-s} : L^2(\hat{\Omega}; \mathcal{C}([0,T]; L^2(D))) \to L^2(\hat{\Omega} \times D)$ defined by $\delta_{t-s}(f) = f(t) - f(s)$ is a continuous, linear mapping. We recall that by Lemma 3.15, 1.), $\hat{B}_N \to B_\infty$ in $L^2(\hat{\Omega}; \mathcal{C}([0,T]; L^2(D)))$ for $N \to \infty$, thus

$$\hat{B}_N(t) - \hat{B}_N(s) = \delta_{t-s}(\hat{B}_N) \to \delta_{t-s}(B_\infty) = B_\infty(t) - B_\infty(s)$$

for $N \to \infty$ in $L^2(\hat{\Omega} \times D)$ and we have shown $i.)$.
To show $ii.)$, we recall that $\psi(\hat{B}_N, v_\tau) \to \psi(B_\infty, u_\infty)$ a.s. in $\hat{\Omega}$ for $N \to \infty$. With Lebesgue's dominated convergence theorem it follows that

$$\lim_{N\to\infty} \psi(\hat{B}_N, v_\tau) \to \psi(B_\infty, u_\infty)$$

in $L^2(\hat{\Omega})$. The convergences in $i.)$ and $ii.)$ are sufficient to pass to the limit with $N \to \infty$ in (107) and we obtain (102). In particular, (102) implies that B_∞ is a martingale with respect to $(\hat{\mathcal{F}}_t^\infty)$.

Now let us calculate the quadratic variation process of B_∞: Let (e_n) be an orthonormal basis of $L^2(D)$. To prove (99), we recall that for any $N \in \mathbb{N}$ (105) is equivalent to

$$0 = E[\Big((\hat{B}_N(t), e_k)_2 - (\hat{B}_N(s), e_j)_2 - \Lambda(s,t,v_\tau,e_k,e_j)\Big)\, \psi(\hat{B}_N, v_\tau)] \tag{108}$$

for all $k, j \in \mathbb{N}$, $t \in [0,T]$, $0 \le s \le t$, $\psi \in \mathcal{C}_b(\mathcal{C}([0,s]; L^2(D)) \times L^2(0,s; L^2(D)))$ and

$$\Lambda(s,t,u,e_k,e_j) := \left(\left[\int_s^t H(u) \circ H^*(u)\ dr\right](e_k), e_j\right)_2 \tag{109}$$

for $u \in L^2(D)$. We show that

$$\begin{aligned} &\lim_{N\to\infty} E[\Big((\hat{B}_N(t), e_k)_2 - (\hat{B}_N(s), e_j)_2 - \Lambda(s,t,v_\tau,e_k,e_j)\Big)\, \psi(\hat{B}_N, v_\tau)] \\ =\ & E[\big((B_\infty(t), e_k)_2 - (B_\infty(s), e_j)_2 - \Lambda(s,t,u_\infty,e_k,e_j)\big)\, \psi(\hat{B}_\infty, u_\infty)] \end{aligned} \tag{110}$$

for all $k, j \in \mathbb{N}$, $t \in [0,T]$, $0 \le s \le t$, $\psi \in \mathcal{C}_b(\mathcal{C}([0,s]; L^2(D)) \times L^2(0,s; L^2(D)))$ for a suitable, not relabeled subsequence. To this end, we fix $k, j \in \mathbb{N}$, $t \in [0,T]$, $0 \le s \le t$, $\psi \in \mathcal{C}_b(\mathcal{C}([0,s]; L^2(D)) \times L^2(0,s; L^2(D)))$ and pass to a not relabeled subsequence of (v_τ), such that $v_\tau \to u_\infty$ for $N \to \infty$ in $L^2(\hat{\Omega}; L^2(Q))$ and a.s. in $L^2(Q)$. Since $\psi(\hat{B}_N, v_\tau)$ is uniformly bounded in $L^\infty(\hat{\Omega})$ and $\psi(\hat{B}_N, v_\tau) \to \psi(B_\infty, u_\infty)$ a.s. in $\hat{\Omega}$,

$$\psi(\hat{B}_N, v_\tau) \overset{*}{\rightharpoonup} \psi(B_\infty, u_\infty) \tag{111}$$

for $N \to \infty$ in $L^\infty(\hat{\Omega})$. We will show

$$\begin{aligned} &(\hat{B}_N(t), e_k)_2 - (\hat{B}_N(s), e_j)_2 - \Lambda(s,t,v_\tau,e_k,e_j) \\ &\to (B_\infty(t), e_k)_2 - (B_\infty(s), e_j)_2 - \Lambda(s,t,u_\infty,e_k,e_j) \end{aligned} \tag{112}$$

for $N \to \infty$ in $L^1(\hat{\Omega})$. For any $n \in \mathbb{N}$, the mapping

$$L^2(\hat{\Omega} \times D) \ni u \mapsto (u, e_n)_2 \in L^2(\hat{\Omega})$$

is continuous. Since $\hat{B}_N(r) \to B_\infty(r)$ for $N \to \infty$ in $L^2(\hat{\Omega} \times D)$ for any $r \in [0,T]$, it follows that

$$(\hat{B}_N(t), e_k)_2 - (\hat{B}_N(s), e_j)_2 \to (B_\infty(t), e_k)_2 - (B_\infty(s), e_j)_2$$

for $N \to \infty$ in $L^2(\hat{\Omega})$, thus also in $L^1(\hat{\Omega})$. Now, the term $\Lambda(s,t,v_\tau,e_k,e_j)$ deserves our attention a.s. in $\hat{\Omega}$:

$$\begin{aligned} \Lambda(s,t,v_\tau,e_k,e_j) &= \left(\left[\int_s^t H(v_\tau) \circ H^*(v_\tau)\ dr\right](e_k), e_j\right)_2 \\ &= \left(\int_s^t H(v_\tau) \circ H^*(v_\tau)(e_k)\ dr, e_j\right)_2 \\ &= \int_s^t (H(v_\tau) \circ H^*(v_\tau)(e_k), e_j)_2\ dr. \end{aligned} \tag{113}$$

Now from Cauchy inequality it follows that

$$\begin{aligned}
& E|\Lambda(s,t,v_\tau,e_k,e_j) - \Lambda(s,t,u_\infty,e_k,e_j)| \\
\le\ & E\int_s^t |([H(v_\tau)\circ H^*(v_\tau) - H(u_\infty)\circ H^*(u_\infty)](e_k),e_j)_2|\ dr \\
\le\ & E\int_s^t \|[H(v_\tau)\circ H^*(v_\tau) - H(u_\infty)\circ H^*(u_\infty)](e_k)\|_2\ dr \\
\le\ & CE\int_s^t \|H(v_\tau)\circ H^*(v_\tau) - H(u_\infty)\circ H^*(u_\infty)\|_{HS(L^2(D))}\ dr
\end{aligned} \tag{114}$$

for a constant $C \ge 0$. Therefore,

$$\begin{aligned}
& E|\Lambda(s,t,v_\tau,e_k,e_j) - \Lambda(s,t,u_\infty,e_k,e_j)| \\
\le\ & CE\int_s^t \|H(v_\tau)\circ [H^*(v_\tau) - H^*(u_\infty)]\|_{HS(L^2(D))}\ dr \\
+\ & CE\int_s^t \|[H(v_\tau) - H(u_\infty)]\circ H^*(u_\infty)\|_{HS(L^2(D))}\ dr \\
\le\ & CE\int_s^t \|H(v_\tau)\|_{HS(L^2(D))}\|H^*(v_\tau) - H^*(u_\infty)\|_{HS(L^2(D))}\ dr \\
+\ & CE\int_s^t \|H(v_\tau) - H(u_\infty)\|_{HS(L^2(D))}\|H^*(u_\infty)\|_{HS(L^2(D))}\ dr
\end{aligned} \tag{115}$$

Using Hölder inequality, from (114) we get

$$\begin{aligned}
& E|\Lambda(s,t,v_\tau,e_k,e_j) - \Lambda(s,t,u_\infty,e_k,e_j)| \\
\le\ & C\left(E\int_s^t \|H(v_\tau)\|^2_{HS(L^2(D))}\ dr\right)^{1/2} \\
\cdot\ & \left(E\int_s^t \|H^*(v_\tau) - H^*(u_\infty)\|_{HS(L^2(D))}\ dr\right)^{1/2} \\
+\ & C\left(E\int_s^t \|H^*(u_\infty)\|^2_{HS(L^2(D))}\ dr\right)^{1/2} \\
\cdot\ & \left(E\int_s^t \|H(v_\tau) - H(u_\infty)\|_{HS(L^2(D))}\ dr\right)^{1/2}
\end{aligned} \tag{116}$$

By Parseval identity it follows that a.s. in $\hat{\Omega}\times(0,T)$ we have

$$\|H^*(v_\tau) - H^*(u_\infty)\|_{HS(L^2(D))} = \|H(v_\tau) - H(u_\infty)\|_{HS(L^2(D))} \tag{117}$$

and from (117) using $(H1)$ we get

$$\begin{aligned}
& E\int_s^t \|H^*(v_\tau) - H^*(u_\infty)\|^2_{HS(L^2(D))}\ dr \\
=\ & E\int_s^t \|H(v_\tau) - H(u_\infty)\|^2_{HS(L^2(D))}\ dr \\
=\ & E\int_s^t \sum_{n=1}^{\infty} \|h_n(v_\tau) - h_n(u_\infty))\|^2_{L^2(D)}\ dr \\
\le\ & C_1 E\int_0^T \|v_\tau - u_\infty\|_2^2\ dr.
\end{aligned} \tag{118}$$

Since $v_\tau \to u_\infty$ in $L^2(\Omega \times Q)$ for $N \to \infty$, from (118) it follows that

$$H^*(v_\tau) \to H^*(u_\infty) \text{ and } H(v_\tau) \to H(u_\infty) \tag{119}$$

in $L^2(\hat{\Omega} \times (s,t); HS(L^2(D)))$ for $N \to \infty$. In particular, there exists $C \geq 0$ such that

$$\|H(v_\tau)\|^2_{L^2(\hat{\Omega}\times(s,t);HS(L^2(D)))} = E\int_s^t \|H(v_\tau)\|^2_{HS(L^2(D))}\, dr \leq C \tag{120}$$

for all $N \in \mathbb{N}$. Using (116), (119) and (120), it follows that

$$E|\Lambda(s,t,v_\tau,e_k,e_j) - \Lambda(s,t,u_\infty,e_k,e_j)| \to 0 \tag{121}$$

for $N \to \infty$ and therefore (112) holds true. The convergences in (111) and (112) are enough to conclude (110).

Proposition 3.24. *There exists an enlarged probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$, an enlarged filtration $(\overline{\mathcal{F}}_t)$ and a cylindrical Wiener process $\overline{W}$ with values in $L^2(D)$ defined on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$ and adapted to $(\overline{\mathcal{F}}_t)$, such that*

$$B_\infty(t) = \int_0^t H(u_\infty)\, d\overline{W} \tag{122}$$

for all $t \in [0,T]$ and a.s. in $\overline{\Omega}$.

Proof: According to Proposition 3.23, B_∞ is a $\mathcal{F}_t^\infty$-martingale with quadratic variation process

$$\ll B_\infty \gg_t = \int_0^t H(u_\infty) \circ H^*(u_\infty)\, ds.$$

Since u_∞ is a $\mathcal{F}_t^\infty$-adapted process with values in $L^2(D)$ and it is a.s. weakly continuous, for any $h \in L^2(D)$ the process $(u_\infty, h)_2$ is $\mathcal{F}_t^\infty$-adapted with values in $\mathbb{R}$ and a.s. continuous trajectories. Therefore, $(u_\infty, h)_2$ is a predictable process for all $h \in L^2(D)$ and by Pettis theorem one gets that u_∞ is a predictable process with values in $L^2(D)$. Now the assertion follows from [5], Theorem 8.2, p.220 (see Theorem 4.5 in the Appendix): There exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, a filtration $(\tilde{\mathcal{F}}_t)$ and a cylindrical Wiener process $\overline{W}$ with values in $L^2(D)$, defined on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$ for $\overline{\Omega} := \hat{\Omega} \times \tilde{\Omega}$, $\overline{\mathcal{F}} := \mathcal{F} \times \tilde{\mathcal{F}}$, $\overline{P} := P \times \tilde{P}$ adapted to $(\overline{\mathcal{F}}_t) := (\mathcal{F}_t^\infty \times \tilde{\mathcal{F}}_t)$, such that

$$B_\infty(t,\hat{\omega}) = B_\infty(t,\overline{\omega}) = \int_0^t H(u_\infty(\hat{\omega}))\, d\overline{W}(s,\overline{\omega})$$

for all $t \in [0,T]$, a.s. in $\overline{\Omega}$.

Remark 3.4. *Without changing notation, we can identify any random variable X in $\hat{\Omega}$ to a random variable in $\overline{\Omega}$ by setting $X(\hat{\omega}, \tilde{\omega}) := X(\hat{\omega})$ a.s. in $\overline{\Omega}$. In particular, all previous estimates and convergences remain true with respect to the probability space $\overline{\Omega}$. In particular, $u_\infty : \overline{\Omega} \times [0,T] \to L^2(D)$ is a predictable process with a.e. paths*

$$u_\infty(\omega,\cdot) \in \mathcal{C}([0,T]; W^{-1,p'}(D)) \cap L^\infty(0,T; L^2(D)),$$

such that $u_\infty \in L^p(\overline{\Omega}; L^p(0,T; W_0^{1,p}(D)))$, $u_\infty(0,\cdot) = u_0$ in $L^2(D)$ and

$$u_\infty(t) = u_0 + \int_0^t \operatorname{div}(G(s) + F(u_\infty(s)))\, ds + \int_0^t H(u_\infty)\, d\overline{W} \tag{123}$$

in $L^2(D)$ for all $t \in [0,T]$ a.s. in $\overline{\Omega}$.

Lemma 3.25. $G = |\nabla u_\infty|^{p-2}\nabla u_\infty$ *in* $L^{p'}(\overline{\Omega}\times Q)^d$

Proof: Taking the scalar product with v^{k+1} in (61), we get

$$\begin{aligned} & (v^{k+1}-v^k, v^{k+1})_2 - (\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k), v^{k+1})_2 \\ + \ & \tau\int_D (|\nabla v^{k+1}|^{p-2}\nabla v^{k+1} + F(v^{k+1}))\cdot\nabla v^{k+1}\ dx \\ = \ & 0. \end{aligned} \tag{124}$$

Using the identity

$$(a-b)a = \frac{1}{2}(|a|^2 - |b|^2) + \frac{1}{2}|a-b|^2,\ a, b\in\mathbb{R},$$

Gauss-Green theorem on the convection term and taking expectation we get

$$\begin{aligned} & E\left(\frac{1}{2}\|v^{k+1}\|_2^2 - \frac{1}{2}\|v^k\|_2^2\right) + E\frac{1}{2}\|v^{k+1}-v^k\|_2^2 \\ - \ & E(\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k), v^{k+1})_2 + \tau E\int_D |\nabla v^{k+1}|^{p-2}\nabla v^{k+1}\cdot\nabla v^{k+1}\ dx. \\ = \ & 0 \end{aligned} \tag{125}$$

Since v^k is $\mathcal{F}_{t_k}^{W_N}$-measurable, $E(\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k), v^k)_2 = 0$, thus using

$$-ab = -\frac{1}{2}|a|^2 - \frac{1}{2}|b|^2 + \frac{1}{2}|a-b|^2,\ a, b\in\mathbb{R}$$

we can write

$$\begin{aligned} & -E(\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k), v^{k+1})_2 = -E(\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k), v^{k+1}-v^k)_2 \\ = \ & -E\frac{1}{2}\|\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k)\|_2^2 - E\frac{1}{2}\|v^{k+1}-v^k\|_2^2 \\ + \ & E\frac{1}{2}\|\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k) - (v^{k+1}-v^k)\|_2^2 \end{aligned} \tag{126}$$

and therefore

$$\begin{aligned} & 0 = E\left(\frac{1}{2}\|v^{k+1}\|_2^2 - \frac{1}{2}\|v^k\|_2^2\right) + E\frac{1}{2}\|v^{k+1}-v^k\|_2^2 \\ - \ & E\frac{1}{2}\|\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k)\|_2^2 - E\frac{1}{2}\|v^{k+1}-v^k\|_2^2 \\ + \ & E\frac{1}{2}\|\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k) - (v^{k+1}-v^k)\|_2^2 \\ + \ & \tau E\int_D |\nabla v^{k+1}|^{p-2}\nabla v^{k+1}\cdot\nabla v^{k+1}\ dx \\ \geq \ & E\left(\frac{1}{2}\|v^{k+1}\|_2^2 - \frac{1}{2}\|v^k\|_2^2\right) - E\frac{1}{2}\|\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k)\|_2^2 \\ + \ & \tau E\int_D |\nabla v^{k+1}|^{p-2}\nabla v^{k+1}\cdot\nabla v^{k+1}\ dx. \end{aligned} \tag{127}$$

Summing over $k = 0, \ldots, N-1$ from (127) it follows that

$$\begin{aligned} & E\frac{1}{2}\|\hat{u}_N(T)\|_2^2 + E\int_0^T\int_D |\nabla v_N|^{p-2}\nabla v_N\cdot\nabla v_N\ dx\ dt \\ - \ & \frac{1}{2}\sum_{k=0}^{N-1} E\|\int_{t_k}^{t_{k+1}} H(v_\tau)\ dW_N\|_2^2 \\ \leq \ & \frac{1}{2}\|u_0\|_2^2 \end{aligned} \tag{128}$$

where, by Itô isometry,

$$\sum_{k=0}^{N-1} E\| \int_{t_k}^{t_{k+1}} H(v_\tau)\ dW_N\|_2^2 = \sum_{k=0}^{N-1} E\int_{t_k}^{t_{k+1}} \|H(v_\tau)\|^2_{HS(L^2(D))}\ dt$$
$$= \quad E\int_0^T \|H(v_\tau)\|^2_{HS(L^2(D))}\ dt. \tag{129}$$

On the other hand, by Itô formula from (123) it follows that

$$\frac{1}{2}\|u_\infty(T)\|_2^2 = \frac{1}{2}\|u_0\|_2^2 - \int_0^T \int_D G\cdot\nabla u_\infty\ dx\ dt$$
$$+ \quad \frac{1}{2}\int_0^T \|H(u_\infty)\|^2_{HS(L^2(D))}\ dt + \int_0^T (u_\infty, H(u_\infty)\ d\overline{W})_2\ dt, \tag{130}$$

a.s in $\overline{\Omega}$, therefore

$$\frac{1}{2}E\|u_\infty(T)\|_2^2 + E\int_0^T \int_D G\cdot\nabla u_\infty\ dx\ dt - \frac{1}{2}E\int_0^T \|H(u_\infty)\|^2_{HS(L^2(D))}\ dt$$
$$= \quad \frac{1}{2}\|u_0\|_2^2 \tag{131}$$

From (128), (129) and (131) it follows that

$$\frac{1}{2}E\|u_\infty(T)\|_2^2 + E\int_0^T \int_D G\cdot\nabla u_\infty\ dx\ dt - \frac{1}{2}E\int_0^T \|H(u_\infty)\|^2_{HS(L^2(D))}\ dt$$
$$\geq \quad E\frac{1}{2}\|\hat{u}_N(T)\|_2^2 + E\int_0^T \int_D |\nabla v_N|^{p-2}\nabla v_N\cdot\nabla v_N\ dx\ dt$$
$$- \quad \frac{1}{2}E\int_0^T \|H(v_\tau)\|^2_{HS(L^2(D))}\ dt, \tag{132}$$

hence

$$E\int_0^T \int_D G\cdot\nabla u_\infty\ dx\ dt \geq \frac{1}{2}E\left(\|\hat{u}_N(T)\|_2^2 - \|u_\infty(T)\|_2^2\right)$$
$$- \quad \frac{1}{2}E\int_0^T \|H(v_\tau)\|^2_{HS(L^2(D))} - \|H(u_\infty)\|^2_{HS(L^2(D))}\ dt$$
$$+ \quad E\int_0^T \int_D |\nabla v_N|^{p-2}\nabla v_N\cdot\nabla v_N\ dx\ dt. \tag{133}$$

Since the mapping $\|\cdot\|_2^2 : L^2(\overline{\Omega}\times D)\to[0,\infty)$ is continuous and convex, it is weakly l.s.c. and from Corollary 3.21 it follows that

$$0 \leq \liminf_{N\to\infty} E\|\hat{u}_N(T)\|_2^2 - E\|u_\infty(T)\|_2^2. \tag{134}$$

Moreover, from (118) in particular it follows that

$$\lim_{N\to\infty} H(v_\tau)\to H(u_\infty) \text{ in } L^2(\overline{\Omega}\times(0,T);HS(L^2(D))),$$

thus

$$E\int_0^T \|H(v_\tau)\|^2_{HS(L^2(D)}\ dt \to E\int_0^T \|H(u_\infty)\|^2_{HS(L^2(D)}\ dt \tag{135}$$

for $N \to \infty$. Therefore from (133) and (135) it follows that

$$
\begin{aligned}
& E\int_0^T \int_D G \cdot \nabla u_\infty \; dx \; dt \geq \frac{1}{2}\left(\liminf_{N\to\infty} E\|\hat{u}_N(T)\|_2^2 - E\|u_\infty(T)\|_2^2\right) \\
+ \quad & \limsup_{N\to\infty} E\int_0^T \int_D |\nabla v_N|^{p-2}\nabla v_N \cdot \nabla v_N \; dx \; dt \\
- \quad & \frac{1}{2}\lim_{N\to\infty}\left(E\int_0^T \|H(v_\tau)\|^2_{HS(L^2(D))} - \|H(u_\infty)\|^2_{HS(L^2(D))} \; dt\right) \\
\geq \quad & \limsup_{N\to\infty} E\int_0^T \int_D |\nabla v_N|^{p-2}\nabla v_N \cdot \nabla v_N \; dx \; dt. \qquad (136)
\end{aligned}
$$

Since $p > 2$, there exists a constant $C \geq 0$ not depending on $N \in \mathbb{N}$ such that

$$
\begin{aligned}
& C \limsup_{N\to\infty} E\int_0^T \int_D |\nabla v_N - \nabla u_\infty|^p \; dx \; dt \\
\leq \quad & \limsup_{N\to\infty} E\int_0^T \int_D (|\nabla v_N|^{p-2}\nabla v_N - |\nabla u_\infty|^{p-2}\nabla u_\infty) \cdot \nabla(v_N - u_\infty) \; dx \; dt \\
\leq \quad & \limsup_{N\to\infty} E\int_0^T \int_D |\nabla v_N|^{p-2}\nabla v_N \cdot \nabla v_N \; dx \; dt - E\int_0^T \int_D G \cdot \nabla u_\infty \; dx \; dt \\
\leq \quad & 0, \qquad (137)
\end{aligned}
$$

where the last inequality is a consequence of (136). From (137) it now follows that $\nabla v_N \to \nabla u_\infty$ in $L^p(\overline{\Omega} \times Q)^d$ for $N \to \infty$ and therefore

$$|\nabla v_N|^{p-2}\nabla v_N \to |\nabla u_\infty|^{p-2}\nabla u_\infty \text{ in } L^{p'}(\overline{\Omega} \times Q)^d.$$

4 Appendix

4.1 On Prokhorov compactness theorem

Definition 4.1 (see [3], p.59)**.** *Let Π be a family of probability measures on the metric space V with the Borel σ-algebra $\mathcal{B}(V)$. The family Π is tight iff, for every $\epsilon > 0$, there exists a compact set K_ϵ such that*

$$P(K_\epsilon) > 1 - \epsilon$$

for every $P \in \Pi$.

Tightness can be used as a compactness criterion in the narrow topology, this is the direct half of Prokhorov theorem:

Theorem 4.1. *[see [3], Theorem 5.1., p.59] If Π is tight, then it is relatively compact with respect to the narrow topology $\sigma(\mathcal{C}_b(V)', \mathcal{C}_b(V))$, i.e. for any subsequence $(P_n) \subset \Pi$ there exists a subsequence (P_{n_k}) and a probability measure μ such that*

$$\lim_{k\to\infty} \int_V f \; dP_{n_k} = \int_V f \; d\mu \qquad (138)$$

for all $f \in \mathcal{C}_b(V)$.

We have the following subsequence principle:

Corollary 4.2. *If the sequence of probability measures $(P_n)_{n\in\mathbb{N}}$ is tight, and if each subsequence that converges narrowly at all in fact converges narrowly to μ, then the entire sequence converges narrowly to μ.*

If, in addition, V is a Polish space, then the converse part of Prokhorov theorem also holds true:

Theorem 4.3. *[see [3], Theorem 5.2, p.60] Suppose that V is separable and complete. If Π is relatively compact with respect to the narrow topology $\sigma(\mathcal{C}_b(V)', \mathcal{C}_b(V))$, then it is tight.*

4.2 On Skorokhod representation theorem

Definition 4.2 (see [18] p.17)**.** *For $n \in \mathbb{N}$, let $X_n : (\Omega, \mathcal{F}, P) \to (V, \mathcal{B}(V))$ be a random variable with values in a metric space V. We say that (X_n) converges to a Borel measure μ[3] in law, (or distribution), and write $X_n \overset{\mathscr{L}}{\rightsquigarrow} \mu$, iff*

$$Ef(X_n) \to \int_V f \, d\mu$$

for any bounded, continuous function f on V.

Remark 4.1. *Note that $X_n \overset{\mathscr{L}}{\rightsquigarrow} \mu$ is equivalent to $P \circ X_n^{-1} \overset{*}{\rightharpoonup} \mu$ with respect to the narrow topology on the bounded Borel measures where $P \circ X_n^{-1}$ is the image measure of X_n for all $n \in \mathbb{N}$.*

Theorem 4.4 (see [18], Theorem 1.10.4, p.59)**.** *Let $(\Omega, \mathcal{F}, P)$ be a probability space, V a separable metric space and $X_n : \Omega \to V$ be a sequence of random variables such that $X_n \overset{\mathscr{L}}{\rightsquigarrow} X_\infty$. Then there exists a sequence of random variables $\hat{X}_n : \hat{\Omega} \to V$, $n \in \mathbb{N} \cup \{\infty\}$, on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ with the following properties:*

i.) $\hat{X}_n \to \hat{X}_\infty$ *in V for $N \to \infty$ a.s. in $\hat{\Omega}$*

ii.) *The laws of X_n and $\hat{X}_n$ are the same for all $n \in \mathbb{N} \cup \{\infty\}$. In particular, for any bounded measurable function $f : V \to \mathbb{R}$, $Ef(X_n) = Ef(\hat{X}_n)$ for all $n \in \mathbb{N}$.*

Remark 4.2. *According to [5], Theorem 2.4., p.33, we can assume that $\hat{\mathcal{F}}$ is countably generated.*

Remark 4.3. *According to [18], Addendum 1.10.5. p.59, there exist random variables $\phi_n : \hat{\Omega} \to \Omega$ such that $\hat{X}_n = X_n \circ \phi_n$ and $P = \hat{P} \circ \phi_n^{-1}$.*

4.3 Martingale representation theorem

Theorem 4.5 (see [5], Theorem 8.2, p.220)**.** *Assume $\mathcal{U}, \mathcal{H}$ are separable Hilbert spaces, M is a square-inegrable martingale with*

$$\ll M \gg_t = \int_0^t (\Phi \circ Q^{1/2}) \circ (\Phi \circ Q^{1/2})^* \, ds, \; t \in [0, T],$$

where $\mathcal{U}_0 = Q^{1/2}(\mathcal{U})$, Φ is a predictable, $HS(\mathcal{U}_0, \mathcal{H})$-valued process and Q a given, bounded, symmetric nonnegative operator in $\mathcal{U}$. Then there exists a probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$, a filtration $(\overline{\mathcal{F}}_t)$ and a Q-Wiener process $\overline{W}$ with values in $\mathcal{U}$, defined on $(\Omega \times \overline{\Omega}, \mathcal{F} \times \overline{\mathcal{F}}, P \times \overline{P})$ adapted to $(\mathcal{F}_t \times \overline{\mathcal{F}}_t)$, such that

$$M(t, \omega, \overline{\omega}) = \int_0^t \Phi(s, \omega, \overline{\omega}) \, d\overline{W}(s, \omega, \overline{\omega}), \; t \in [0, T],$$

for a.e. $(\omega, \overline{\omega}) \in (\Omega \times \overline{\Omega})$ where

$$M(t, \omega, \overline{\omega}) = M(t, \omega), \; \Phi(t, \omega, \overline{\omega}) = \Phi(t, \omega)$$

for all $t \in [0, T]$, a.s. in $\Omega \times \overline{\Omega}$.

[3]i.e. a measure on the Borel sets, finite on the compact ones.

4.4 Technical lemmas

4.4.1 On the Garsia-Rodemich-Rumsey inequality

Lemma 4.6. *(Garsia-Rodemich-Rumsey inequality, see [10],[16, Ex. 2.4.1]) Let* $q \geq 1$, $\alpha > 1/q$ *and* $f : [a, b] \to V$ *be continuous, then*

$$\|f(s) - f(s')\|_V^q \leq C_{\alpha,q}|s - s'|^{\alpha q-1} \int_a^b \int_a^b \frac{\|f(t) - f(r)\|_V^q}{|t - r|^{\alpha q+1}}\, dt\, dr. \tag{139}$$

4.4.2 $W^{\alpha,p}$-regularity

Lemma 4.7 ([8], Lemma 2.1., p.369)**.** *Let* $\mathcal{K}$, $\mathcal{H}$ *be separable Hilbert spaces and* W *be a cylindrical Wiener process in* $\mathcal{K}$. *Assume* $p \geq 2$, $\alpha \in (0, \frac{1}{2})$. *Then, for any progressively measurable process* $f \in L^p(\Omega \times (0, T); HS(\mathcal{K}; \mathcal{H}))$ *we have*

$$\int_0^\cdot f\, dW \in L^p(\Omega; W^{\alpha,p}(0, T; \mathcal{H}))$$

and there exists a constant $C(p, \alpha) > 0$ *such that*

$$E\| \int_0^\cdot f\, dW\|_{W^{\alpha,p}(0,T;\mathcal{H})}^p \leq C(\alpha, p)E \int_0^T \|f(t)\|_{HS(\mathcal{K};\mathcal{H})}^p\, dt.$$

Lemma 4.8 ([2], Lemma 3.2)**.** *Let* V *be a Banach space. Assume that* $\tau > 0$ *and that* $I_\tau = \{t_k\}_{k=0}^N$ *is an equidistant mesh of size* $\tau > 0$ *covering* $[0, T]$. *Assume that* $\mathcal{G} \in C([0, T]; V)$ *is such that, for every* $k \in \{0, \ldots, N - 1\}$ *the function*

$$[t_k, t_{k+1}) \ni t \mapsto \mathcal{G}(t)$$

is affine. Assume that, for some $p \geq 1$, $\alpha > 0$ *and* $C > 0$ *and every* $l \in \{1, \ldots N\}$,

$$\tau \sum_{k=0}^{N-l} \|\mathcal{G}(t_{k+l}) - \mathcal{G}(t_k)\|_V^p \leq C^p t_l^{\alpha p}.$$

Then, $\mathcal{G}$ *is uniformly bounded in the Nikolskii space* $N^{\alpha,p}(0, T; V)$ *and there exists a constant* $C = C(T) > 0$, *which does not depend on* $\tau > 0$ *such that*

$$\|\mathcal{G}\|_{N^{\alpha,p}(0,T;V)} = \sup_{s>0} s^{-\alpha}\|\mathcal{G}(\cdot + s) - \mathcal{G}(\cdot)\|_{L^p(-s,T-s;V)} \leq C.$$

4.4.3 Further results

Lemma 4.9. *Let* $\mathcal{W}$ *be a Banach space which is compactly embedded into* $L^2(0, T; L^2(D))$ *and* $p \geq 2$. *For* $\alpha \in (0, \frac{1}{2})$, *the linear space*

$$V := \{u = v + w,\ v \in \mathcal{W},\ w \in W^{\alpha,p}(0, T; H_0^1(D))\} \subset L^2(0, T; L^2(D))$$

endowed with the norm

$$\|u\|_V := \inf_{\substack{v\in\mathcal{W},\\ w\in W^{\alpha,p}(0,T;L^2(D)),\\ u=v+w}} \max(\|v\|_{\mathcal{W}}, \|w\|_{W^{\alpha,p}})$$

is a Banach space which is compactly embedded into $L^2(0, T; L^2(D))$.

Proof: It follows from [9], Remark 5.13, p.12-13 that $(V, \|\cdot\|_V)$ is a Banach space. There exists $C \geq 0$ such that for any $u \in V$ and any $v \in \mathcal{W}$, $w \in W^{\alpha,p}(0,T;H_0^1(D))$ with $u = v+w$

$$\|u\|_{L^2(0,T;L^2(D))} \leq C \max(\|v\|_{\mathcal{W}}, \|w\|_{W^{\alpha,p}}) \tag{140}$$

and therefore the imbedding $V \hookrightarrow L^2(0,T;L^2(D)))$ is continuous. Let (u_n) be a bounded sequence in V, i.e. there exists $R > 0$ such that $\|u_n\|_V \leq R$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be fixed. According to the definition of the norm in V, for any $k \in \mathbb{N}$, there exist $v_n^k \in \mathcal{W}$, $w_n^k \in W^{\alpha,p}(0,T;H_0^1(D))$ such that $u_n = v_n^k + w_n^k$ and

$$\|v_n^k\|_{\mathcal{W}} \leq R + \frac{1}{k}, \ \|w_n^k\|_{W^{\alpha,p}} \leq R + \frac{1}{k}.$$

Consequently, choosing $k = n$ we can construct $(v_n^n) \subset \mathcal{W}$, $(w_n^n) \subset W^{\alpha,p}(0,T;H_0^1(D))$ such that $u_n = v_n^n + w_n^n$ and

$$\|v_n^n\|_{\mathcal{W}} \leq R+1, \ \|w_n^n\|_{W^{\alpha,2}} \leq R+1$$

for all $n \in \mathbb{N}$. Passing to a not relabeled subsequence if necessary, there exists $v \in L^2(0,T;L^2(D))$ such that $v_n^n \to v$ in $L^2(0,T;L^2(D))$. Following [14], Corollary 2, p.82,

$$W^{\alpha,p}(0,T;H_0^1(D)) \hookrightarrow L^2(0,T;L^2(D))$$

with compact imbedding. Therefore passing to a not relabeled subsequence if necessary, there exists $w \in L^2(0,T;L^2(D))$ such that $w_n^n \to w$ in $L^2(0,T;L^2(D))$. Therefore, passing to a not relabeled subsequence if necessary,

$$u_n = v_n^n + w_n^n \to v + w$$

in $L^2(0,T;L^2(D))$ and therefore the imbedding $V \hookrightarrow L^2(0,T;L^2(D))$ is compact.

References

[1] L. Baňas, Z. Brzeźniak, M. Neklyudov and A. Prohl. Stochastic Ferromagnetism. De Gruyter, Berlin/Boston (2014).

[2] L. Baňas, Z. Brzeźniak, M. Neklyudov and A. Prohl. A convergent finite-element-based discretization of the stochastic Landau-Lifshitz-Gilbert equation. IMA J. Numer. Anal. 34 (2014), no. 2, 502-549.

[3] P. Billingsley. Convergence of probability measures. Wiley, New York, (1999).

[4] C. Dellacherie, P.A Meyer. Probabilités et potentiel. Chapitres V-VIII. Théorie des martingales. Revised edition. Actualités Scientifiques et Industrielles, 1385. Hermann, Paris (1980).

[5] G. Da Prato and J. Zabczyk. Stochastic equations in infinite dimensions. Volume 44 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge (1992).

[6] A. Debussche, N. Glatt-Holtz and R. Temam. Local martingale and pathwise solutions for an abstract fluids model. Phys. D 240 (2011), no. 14-15, 1123-1144.

[7] R. E. Edwards. *Functional analysis.* Dover Publications, Inc., New York, 1995.

[8] F. Flandoli and D. Gatarek. Martingale and stationary solutions for stochastic Navier-Stokes equations. Prob. Theory Relat. Fields, 102 (1995), 367-391.

[9] H. Gajewski, K. Gröger and K. Zacharias. Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen. Akademie Verlag, Berlin (1974).

[10] A. Garsia, E. Rodemich H. Rumsey. A real variable lemma and the continuity of paths of some Gaussian processes. Indiana Univ. Math. Journal 20 (1970/71), 565-578 (1970/71).

[11] C. Prévôt and M. Röckner. A concise course on stochastic partial differential equations. Volume 1905 of Lecture Notes in Mathematics. Springer, Berlin (2007).

[12] E. Pardoux. Équations aux dérivées partielles stochastiques non linéaires monotones. PhD thesis, Paris Sud-Orsay (1975).

[13] T. Roubíček. Nonlinear Partial Differential Equations with Applications. Springer, Basel (2013).

[14] J. Simon. Compact sets in the space $L^p(0, T; B)$. Ann. Mat. Pura Appl. (4) 146 (1987), 65–96.

[15] J. Simon. Sobolev, Besov and Nikolskii fractional spaces: imbeddings and comparisons for vector valued spaces on an interval. Ann. Mat. Pura Appl. 157 (4) (1990), 117-148.

[16] D.W. Stroock and S.R.S. Varadhan. Multidimensional diffusion processes. Springer-Verlag, Berlin-New York, (1979).

[17] R. Temam. Navier-Stokes equations. Theory and numerical analysis. North-Holland Publishing Co., Amsterdam-New York-Oxford, (1977).

[18] A.W van der Vaart and J.A. Wellner. Weak convergence and empirical processes. Springer, New York (1996).

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