On a pseudomonotone evolution equation with multiplicative noise

by

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Abstract

Let \((\Omega, \mathcal{F}, P)\) be a complete, countably generated probability space, \(T > 0, D \subset \mathbb{R}^d\) be a bounded Lipschitz domain, \(Q_T := (0,T) \times D\) and \(p > 2\). Our aim is the study of the problem

\[
\begin{align*}
\frac{du}{dt} - \text{div}(|\nabla u|^{p-2}\nabla u + F(u)) &= H(u) \, dW & \text{in } \Omega \times Q_T \\
u &= 0 & \text{on } \Omega \times (0,T) \times \partial D \\
u(0,\cdot) &= u_0 \in W_0^{1,p}(D) & \text{in } \Omega \times D
\end{align*}
\]

for a cylindrical Wiener process \(W\) in \(L^2(D)\) with respect to a filtration \((\mathcal{F}_t)\) satisfying the usual assumptions and \(F : \mathbb{R} \to \mathbb{R}^d\) Lipschitz continuous. We consider the case of multiplicative noise with \(H : L^2(D) \to HS(L^2(D))\), \(HS(L^2(D))\) being the space of Hilbert-Schmidt operators, satisfying appropriate regularity conditions. By an implicit time discretization of \((P)\), we obtain approximate solutions. Using the theorems of Skorokhod and Prokhorov, we are able to pass to the limit and show existence of martingale solutions. Using an argument of pathwise uniqueness, we show existence and uniqueness of strong solutions.

Keywords: pseudomonotone problem, multiplicative noise, cylindrical Wiener process, martingale solution, pathwise uniqueness, strong solution

AMS Classification: 35K92, 35K55, 60H15

1 Introduction

Let \((\Omega, \mathcal{F}, P)\) be a complete, countably generated probability space (for example the classical Wiener space), \(D \subset \mathbb{R}^d\) be a bounded Lipschitz domain, \(T > 0, Q_T := D \times (0,T)\) and \(p > 2\). For a separable Hilbert spaces \(\mathcal{U}, \mathcal{H}\), we denote the space of Hilbert-Schmidt operators from \(\mathcal{U}\) to \(\mathcal{H}\) by \(HS(\mathcal{U}; \mathcal{H})\). We are interested in existence and uniqueness of a solution to

\[
\frac{du}{dt} - \text{div}(|\nabla u|^{p-2}\nabla u + F(u)) = H(u) \, dW & \text{in } \Omega \times Q_T \\
u &= 0 & \text{on } \Omega \times (0,T) \times \partial D \\
u(0,\cdot) = u_0 \in W_0^{1,p}(D)
\]

for \(F : \mathbb{R} \to \mathbb{R}^d\) Lipschitz continuous. We will give the precise assumptions on \(H : L^2(D) \to HS(L^2(D))\) in the next section. \(W(t)\) is a cylindrical Wiener process with values in \(L^2(D)\) with respect to a filtration \((\mathcal{F}_t)\) satisfying the usual assumptions. More precisely: Let \((e_n)_{n \in \mathbb{N}}\) be an orthonormal basis of \(L^2(D)\) and \((\beta_n(t))_{n \in \mathbb{N}}\) a sequence of independent, real-valued brownian motions adapted to \((\mathcal{F}_t)\). We (formally) define

\[
W(t) := \sum_{n=1}^{\infty} e_n \beta_n(t).
\]

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It is well-known that the sum on the right-hand side of (2) does not converge in $L^2(D)$, therefore we have to give a meaning to (2) following the ideas of [8] and [18]: For $u = \sum_{n=1}^{\infty} u_n e_n$ and $v = \sum_{n=1}^{\infty} v_n e_n$

$$(u, v)_U := \sum_{n=1}^{\infty} \frac{u_n v_n}{n^2}$$

is a scalar product on $L^2(D)$. Now we define the (bigger) Hilbert space $U$ as the completion of $L^2(D)$ with respect to the norm $\| \cdot \|_U$ induced by $(\cdot, \cdot)_U$. It is then easy to see that $(n e_n)$ is an orthonormal basis of $U$. Note that

$$W(t) = \sum_{n=1}^{\infty} e_n \beta_n(t) = \sum_{n=1}^{\infty} \frac{1}{n} (n e_n) \beta_n(t)$$

and therefore $W(t)$ can be interpreted as $Q$-Wiener process with covariance Matrix $Q = \text{diag}(\frac{1}{n^2})$ with values in $U$. Since $Q^{1/2}(U) = L^2(D)$, for all square integrable and predictable $\Phi : \Omega \times (0,T) \to HS(L^2(D))$ the stochastic integral with respect to the cylindrical Wiener process $W(t)$ can be defined by

$$\int_0^t \Phi \ dW = \sum_{n=1}^{\infty} \int_0^t \Phi(e_n) \ d\beta_n$$

$$= \sum_{n=1}^{\infty} \int_0^t \Phi(\frac{1}{n} \cdot n e_n) \ d\beta_n$$

$$= \sum_{n=1}^{\infty} \int_0^t \Phi \circ Q^{1/2}(n e_n) \ d\beta_n.$$  

(4)

Since $\Phi \circ Q^{1/2} \in HS(U; L^2(D))$,

$$\sum_{n=1}^{\infty} \int_0^t \Phi \circ Q^{1/2}(n e_n) \ d\beta_n \in L^2(\Omega; C([0,T]; L^2(D))).$$

In particular, for all $n \in \mathbb{N}$, $\Phi(e_n) \in L^2(\Omega \times (0,T); L^2(D))$ is predictable process, i.e. $\Phi(e_n)$ is $\mathcal{P}_T / \mathcal{B}(L^2(D))$-measurable where $\mathcal{P}_T$ is the predictable $\sigma$-field on $\Omega \times (0,T)$ generated by

$$\{(s,t] \times A, \ 0 \leq s < t \leq T, \ A \in \mathcal{F}_s\}.$$  

1.1 Strong and martingale solutions

In the theory of stochastic evolution equations two notions of solutions are typically considered for equations with multiplicative noise namely strong solutions and martingale solutions. A strong solution to (1) is defined as follows:

**Definition 1.1.** A solution to (1) is a predictable process $u : \Omega \times [0,T] \to L^2(D)$ with a.e. paths

$$u(\omega, \cdot) \in C([0,T]; W^{1-p'}(D)) \cap L^\infty(0,T; L^2(D)),$$

such that $u \in L^p(\Omega; L^p(0,T; W^{1,p'}_0(D)))$, $u(0, \cdot) = u_0$ in $L^2(D)$ and

$$u(t) - u_0 = \int_0^t \text{div}(|\nabla u|^{p-2} \nabla u + F(u)) \ ds = \int_0^t H(u) \ dW,$$

in $L^2(D)$ for all $t \in [0,T]$, a.s. in $\Omega$. 

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Remark 1.1. According to [24], Lemma 1.4, p.263,
\[ u(\omega, \cdot) \in C([0, T]; W^{-1,p'}(D)) \cap L^\infty(0, T; L^2(D)) \text{ a.s in } \Omega \]
implies
\[ u \in C_w([0, T]; L^2(D)) \text{ a.s in } \Omega \]
where \( C_w([0, T]; L^2(D)) \) denotes the Bochner space of weakly continuous functions with values in \( L^2(D) \). Therefore, \( u(t) \) is in \( L^2(D) \) for all \( t \in [0, T] \) and \( u \) is a stochastic process with values in \( L^2(D) \).

In the former definition, the probabilistic quantities \( (\Omega, \mathcal{F}, P), (\mathcal{F}_t) \) and \( W \) are fixed. In many cases, it is necessary that \( (\Omega, \mathcal{F}, P), (\mathcal{F}_t) \) and \( W \) enter as unknowns into the problem, for example, if one uses the theorems of Prokhorov and Skorokhod to obtain a.s convergence of approximative solutions. More precisely,

**Definition 1.2** (see, e.g. [8], [9], [11]). We say that (1) has a martingale solution, iff there exist a probability space \( (\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}) \), a filtration \( (\mathcal{F}_t) \), a cylindrical Wiener process \( \overline{W} \) and a predictable process \( u : \overline{\Omega} \times [0, T] \rightarrow L^2(D) \) with a.e. paths
\[ u(\omega, \cdot) \in C([0, T]; W^{-1,p'}(D)) \cap L^\infty(0, T; L^2(D)), \]
such that \( u \in L^p(\overline{\Omega}; L^p(0, T; W_0^{1,p}(D))) \), \( u(0, \cdot) = u_0 \) in \( L^2(D) \) and
\[ u(t) - u_0 = \int_0^t \text{div}(\nabla u|^{p-2}\nabla u + F(u)) \, ds = \int_0^t H(u) \, d\overline{W} \tag{5} \]
holds in \( L^2(D) \) for all \( t \in [0, T] \), a.s. in \( \overline{\Omega} \).

### 1.2 Main results and outline

Our aim is to prove the following results:

**Theorem 1.1.** For any \( u_0 \in W_0^{1,p}(D) \) and any \( H : L^2(D) \rightarrow HS(L^2(D)) \) as defined in Section 2 there exists a martingale solution to (1).

**Theorem 1.2.** For any \( u_0 \in W_0^{1,p}(D) \) and any \( H : L^2(D) \rightarrow HS(L^2(D)) \) as defined in Section 2 there exists a unique strong solution to (1).

The proof of Theorem 1.1 is based on a approximation procedure by an implicit time discretization corresponding to (1), which will be introduced in Section 3.1. Since there is a lack of compactness with respect to \( \omega \in \Omega \), we use the theorems of Prokhorov and Skorokhod to get a.s. convergence of a sequence of approximate solutions \( \{\tilde{u}_N\} \) to a measurable function \( u_\infty \) on a new probability space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}) \) (see Subsection 3.4). Passing to the limit we have to face two different difficulties: Firstly, we have to show that the limit of the stochastic integrals is a stochastic integral with respect to a cylindrical Wiener process defined on a possibly enlarged probability space. This can be done using the martingale representation theorem from [8]. Secondly, since weak convergence is not compatible with nonlinear operators, we have to identify the weak limit of \( |\nabla u_N|^{p-2}\nabla u_N \) with \( |\nabla u_\infty|^{p-2}\nabla u_\infty \). Once we have identified the stochastic perturbation at the limit, we may use the Itô formula for the identification of the nonlinearity. Subsection 3.5 is devoted to the solution of these two problems. The proof of Theorem 1.2 is contained in Section 4. We adapt the argument of [14]: Firstly, we prove uniqueness of solutions (see Proposition 4.1) and secondly we construct two sequences of approximate solutions which converge on the same probability space. We adapt the technique of [17], [6], [5] (see also [15], [16], [3]) and avoid the application of the martingale representation theorem.
2 Technical assumptions

For an orthonormal basis \((e_n)\) of \(L^2(D)\), \(u \in L^2(D)\) let us define
\[
H(u)(e_n) := \{x \mapsto h_n(u(x))\},
\]
where, for any \(n \in \mathbb{N}\), \(h_n : \mathbb{R} \to \mathbb{R}\) is a Lipschitz continuous function such that \(h_n(0) = 0\) satisfying

\(H1\) There exists \(C_1 > 0\) not depending on \(\mu, \lambda\) such that
\[
\sum_{n=1}^{\infty} |h_n(\lambda) - h_n(\mu)|^2 \leq C_1 |\lambda - \mu|^2
\]
for all \(\mu, \lambda \in \mathbb{R}\).

\(H2\) There exists \(C_2 > 0\) such that
\[
\sum_{n=1}^{\infty} \|h'_n\|_{\infty}^2 \leq C_2.
\]

For example, \(h_n(\lambda) = a_n \lambda\) or \(h_n(\lambda) = a_n \sin(\lambda)\) with \(n \in \mathbb{N}\) and \((a_n) \in l^2(\mathbb{N})\) are satisfying \((H1)\) and \((H2)\). In particular for any \(u \in L^2(D)\) thanks to \((H1)\) we have
\[
\|H(u)\|_{HS(L^2(D))}^2 = \sum_{n=1}^{\infty} \|H(u)(e_n)\|_{L^2(D)}^2 = \int_D \sum_{n=1}^{\infty} |h_n(u(x))|^2\, dx
\leq C_1 \|u\|_{L^2(D)}^2
\]
and therefore \(H(u)\) is a Hilbert-Schmidt operator in \(L^2(D)\) and \(H : L^2(D) \to HS(L^2(D))\) is continuous. Thanks to \((H2)\), we also have the following result:

**Proposition 2.1.** \(H : W^{1,p}_0(D) \to HS(L^2(D); H^1_0(D))\) is continuous.

Proof: Let us fix \((u_j) \subset W^{1,p}_0(D)\) such that there exists \(u \in W^{1,p}_0(D)\) with \(u_j \to u\) in \(W^{1,p}_0(D)\) for \(j \to \infty\). Then,
\[
\|H(u_j) - H(u)\|_{HS(L^2(D); H^1_0(D))}^2 = \sum_{n=1}^{\infty} \|h_n(u_j) - h_n(u)\|_{H^1_0(D)}^2
\]
\[
= \sum_{n=1}^{\infty} \int_D |h'_n(u_j) \nabla (u_j - u) + \nabla u (h'_n(u_j) - h'_n(u))|^2\, dx.
\]
We can extract a not relabeled subsequence \((u_j)\) such that \(|\nabla u_j| \leq g\) a.e. in \(D\) for all \(j \in \mathbb{N}\) and some \(g \in L^p(D)\) and
\[
u_j \rightarrow u,
\]
\[
\nabla u_j \rightarrow \nabla u
\]
for \(j \rightarrow \infty\) a.e. in \(D\). For any fixed \(n \in \mathbb{N}\), since \(h'_n\) is continuous we have,
\[
|h'_n(u_j) \nabla (u_j - u) + \nabla u (h'_n(u_j) - h'_n(u))|^2 \rightarrow 0
\]
for \(j \rightarrow \infty\) a.e. in \(D\). Let \(C \geq 0\) be a constant not depending on \(j\) and \(n\) that may change from line to line. By \((H2)\) we have
\[
|h'_n(u_j) \nabla (u_j - u) + \nabla u (h'_n(u_j) - h'_n(u))|^2
\leq C \|h'_n\|_{L^2(D)}^2 \left( |\nabla u_j - \nabla u|^2 + |\nabla u|^2 \right)
\leq CC_2(|g|^2 + |\nabla u|^2)
\]
and the right-hand side of (9) is in \( L^1(D) \). Therefore, by Lebesgue dominated convergence theorem,
\[
\lim_{j \to \infty} \int_D |h_n'(u_j) \nabla (u_j - u) + \nabla u (h_n'(u_j) - h_n'(u))|^2 \, dx = 0
\] (10)
for every \( n \in \mathbb{N} \). Since such a subsequence with can be extracted from every subsequence of \((u_j)\), (8) holds for the whole sequence \((u_j)\). In particular, for any \( N \in \mathbb{N} \), we have
\[
\lim_{j \to \infty} \sum_{n=1}^N \int_D |h_n'(u_j) \nabla (u_j - u) + \nabla u (h_n'(u_j) - h_n'(u))|^2 \, dx = 0.
\] (11)

Let us fix \( \varepsilon > 0 \). For any \( N \in \mathbb{N} \), we have
\[
\sum_{n=N}^\infty \int_D |h_n'(u_j) \nabla (u_j - u) + \nabla u (h_n'(u_j) - h_n'(u))|^2 \, dx \\
\leq \sum_{n=N}^\infty 4 \int_D |h_n'\|_\infty^2| \nabla (u_j - u)|^2 + 4|\nabla u|^2 \|h_n'\|_\infty^2 \, dx \\
\leq \sum_{n=N}^\infty 16\|h_n'\|_\infty^2 \left( \int_D |\nabla (u_j - u)|^2 + |\nabla u|'^2 \, dx \right).
\] (12)

By \((H2)\),
\[
\sum_{n=1}^\infty \|h_n'\|_\infty^2 < \infty,
\]
thus there exists \( N_0 \in \mathbb{N} \) such that
\[
\sum_{n=N}^\infty \|h_n'\|_\infty^2 < \varepsilon
\]
for all \( N \geq N_0 \). Therefore, now we get
\[
\|H(u_j) - H(u)\|^2_{HS(L^2(D);H^1_0(D))} \\
= \sum_{n=1}^{N_0} \int_D |h_n'(u_j) \nabla (u_j - u) + \nabla u (h_n'(u_j) - h_n'(u))|^2 \, dx \\
+ \sum_{n=N_0+1}^\infty \int_D |h_n'(u_j) \nabla (u_j - u) + \nabla u (h_n'(u_j) - h_n'(u))|^2 \, dx \\
\leq \sum_{n=1}^{N_0} \int_D |h_n'(u_j) \nabla (u_j - u) + \nabla u (h_n'(u_j) - h_n'(u))|^2 \, dx \\
+ \varepsilon \int_D |\nabla (u_j - u)|^2 + |\nabla u|'^2 \, dx
\] (13)

using (11) in (13) now it follows that
\[
\lim_{j \to \infty} \|H(u_j) - H(u)\|^2_{HS(L^2(D);H^1_0(D))} = 0.
\] (14)
In particular, for any \( u \in L^p(\Omega \times (0,T);W_{0}^{1,p}(D)) \), using (H2) we get
\[
E \int_{0}^{T} \|H(u)\|^p_{HS(L^2(D);H^1_0(D))} dt = E \int_{0}^{T} \left( \sum_{n=1}^{\infty} \|h_n(u)\|^2_{H^1_0(D)} \right)^{p/2} dt \leq E \int_{0}^{T} \left( \sum_{n=1}^{\infty} \|h_n'(u)\|^2 \int_{D} |\nabla u|^2 dx \right)^{p/2} \leq C_2^{p/2}C_p E \int_{0}^{T} \|\nabla u\|^p dt
\]
where \( C_p \geq 0 \) is a constant which is independent of \( u \).

3 Proof of Theorem 1.1

3.1 Time discretization

For \( N \in \mathbb{N} \) let \( 0 = t_0 < t_1 < \ldots < t_N = T \) be an equidistant subdivision of the interval \([0,T]\) with \( \tau := T/N = t_{k+1} - t_k \) for all \( k = 0, \ldots, N - 1 \). For \( u_0 \in L^2(D) \), we introduce the implicit Euler scheme
\[
u^{k+1} - \nu^k - \tau \text{div}(|\nabla \nu^{k+1}|^{p-2} \nabla \nu^{k+1} + F(u^{k+1})) = H(u^k)\Delta_{k+1}W
\]
with \( \Delta_{k+1}W := W(t_{k+1}) - W(t_k) \) for \( k = 0, \ldots, N - 1 \).

Remark 3.1. Since \( \Delta_{k+1}W \) takes values in the Hilbert space \( U \), we have
\[
\int_{t_k}^{t_{k+1}} H(u^k) \ dW = \sum_{n=1}^{\infty} H(u^k)(\frac{1}{n} \cdot ne_n)(\beta_n(t_{k+1}) - \beta_n(t_k)) = \sum_{n=1}^{\infty} H(u^k) \circ Q^{1/2}(ne_n)(\beta_n(t_{k+1}) - \beta_n(t_k))
\]
for all \( k = 0, \ldots, N - 1 \). Since \( H(u^k) \circ Q^{1/2} \in HS(U; L^2(D)) \), the last expression converges in \( L^2(\Omega; C([0,T]; L^2(D))) \). Therefore we will use the formal notation
\[
H(u^k)\Delta_{k+1}W := \int_{t_k}^{t_{k+1}} H(u^k) \ dW = H(u^k) \circ Q^{1/2}(W(t_{k+1}) - W(t_k)).
\]

Lemma 3.1. For any \( k = 0, \ldots, N - 1 \) and any \( u_0 \in L^2(D) \) there exists a unique, \( F_{k+1} \)-measurable function \( u^{k+1} : \Omega \rightarrow W_{0}^{1,p}(D) \) such that for a.e. \( \omega \in \Omega \)
\[
u^{k+1} - \nu^k - \tau \text{div}(|\nabla \nu^{k+1}|^{p-2} \nabla \nu^{k+1} + F(u^{k+1})) = H(u^k)\Delta_{k+1}W
\]
in \( L^2(D) \).

Proof: We fix \( \tau > 0 \). Since \( p > 2 \), the operator \( A_{\tau} : W_{0}^{1,p}(D) \rightarrow W^{-1,p'}(D) \) defined by
\[
\langle A_{\tau}(u), v \rangle_{W^{-1,p'}(D), W_{0}^{1,p}(D)} := (u, v)_2 + \tau \int_{D} (|\nabla u|^{p-2} \nabla u + F(u)) \cdot \nabla v \ dx
\]
for \( u, v \in W_{0}^{1,p}(D) \) is a pseudomonotone operator and therefore, by Brezis’ theorem, \( A_{\tau} \) is onto \( W^{-1,p}(D) \).

In order to show that \( A_{\tau} \) is injective, we fix \( f \in W^{-1,p'}(D) \) and assume that \( u_1, u_2 \) are two
solutions to $A_\tau u = f$ in $W^{-1,p'}(D)$. Then we take $v = \text{sign}_\delta$ as a test function, where $\text{sign}_\delta$ is a Lipschitz continuous approximation of the sign function and obtain $u_1 = u_2$ by passing to the limit when $\delta$ goes to 0.

It is left to show that $A_{\tau}^{-1} : W^{-1,p'}(D) \to W^{1,p'}_0(D)$ is continuous. For $f \in W^{-1,p'}(D)$ and $u$ such that $A_\tau(u) = f$, using the Gauss-Green theorem on the convection term we get

$$
\|u\|_2^2 + \tau \|\nabla u\|_p^p = \langle f, u \rangle_{W^{-1,p'}(D), W^{1,p'}_0(D)} \leq \frac{\tau}{2} \|\nabla u\|_p^p + C_\tau \|f\|_{W^{-1,p'}(D)}^p.
$$

(19)

Let $(f_n) \subset W^{-1,p'}(D)$ be a sequence converging to $f$ in $W^{-1,p'}(D)$. For all $n \in \mathbb{N}$, we define

$$
u_n := A_{\tau}^{-1}(f_n).
$$

(20)

From (19) it follows that there exists a not relabeled subsequence of $(u_n)$, $u \in W^{1,p}_0(D)$ and $B$ in $L^{p'}(D)^d$ such that $u_n \rightharpoonup u$ in $W^{1,p}_0(D)$, $u_n \to u$ in $L^p(D)$ and $|\nabla u_n|^{p-2}\nabla u_n \to B$ in $L^{p'}(\Omega)^d$ for $n \to \infty$. Using these convergence results and (20), we get

$$
\limsup_{n \to \infty} \langle A(u_n), u_n - u \rangle_{W^{-1,p'}(D), W^{1,p'}_0(D)} = \limsup_{n \to \infty} \int_D |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla u_n \, dx - \int_D B \cdot \nabla u \, dx
$$

(21)

and since $A_\tau$ is pseudomonotone, (22) implies $A_\tau u = f$. In particular, $B = |\nabla u|^{p-2}\nabla u$ and

$$
\lim_{n \to \infty} \int_D |\nabla u - \nabla u_n|^p \, dx
$$

$$
\leq 2^{p-2} \limsup_{n \to \infty} \int_D (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \cdot \nabla (u_n - u) \, dx
$$

$$
= 0.
$$

(23)

From (23) it follows that our not relabeled subsequence $(u_n)$ converges strongly to $u$ in $W^{1,p}_0(D)$ for $n \to \infty$. Since $u$ is unique it follows that the whole sequence $(u_n)$ converges to $u$ in $W^{1,p}_0(D)$ for $n \to \infty$ and $A_{\tau}^{-1}$ is continuous.

Since, for all $k = 0, \ldots, N-1$,

$$
\langle u^k+1 - u^k, \tau - \text{div}(|\nabla u^{k+1}|^{p-2}\nabla u^{k+1} + F(u^{k+1})) \rangle = H(u^k) \Delta_{k+1} W
$$

$$

\iff u_{k+1} = A_{\tau}^{-1}(H(u^k) \Delta_{k+1} W + u^k),
$$

(24)

and the argument on the right-hand side of (24) is $\mathcal{F}_{\text{H}^{k+1}}$-measurable assuming that $u^k$ is $\mathcal{F}_{\text{H}^k}$-measurable, the assertion follows by induction.

3.2 Estimates

Lemma 3.2. For $u_0 \in L^2(D)$, and $k = 0, 1, \ldots, N-1$ let $u^{k+1}$ be a solution to (16). Then,

$$
\frac{1}{2} E \left(\|u^{k+1}\|_2^2 - \|u^k\|_2^2\right) + \frac{1}{4} E \|u^{k+1} - u^k\|_2^2 + \tau E \int_D |\nabla u^{k+1}|^p \, dx
$$

$$
\leq \tau E \|H(u^k)\|_{HS(L^2(D))}^2
$$

(25)
Proof: Taking the $L^2$-scalar product with $u^{k+1}$ in (16), we get
\[
\|u^{k+1}\|_2^2 - (u^k, u^{k+1})_2 = \tau (\text{div}(|\nabla u^{k+1}|^{p-2}\nabla u^{k+1} + F(u^{k+1})), u^{k+1})_2
\]
\[
= (H(u^k)\Delta_{k+1}W, u^{k+1})_2
\]
\[
\Leftrightarrow I_1 + I_2 + I_3 = I_4
\]
where
\[
I_1 := \|u^{k+1}\|_2^2 - (u^k, u^{k+1})_2 = \frac{1}{2} \left( \|u^{k+1}\|_2^2 - \|u^k\|_2^2 + \|u^{k+1} - u^k\|_2^2 \right),
\]
\[
I_2 := \tau \int_D |\nabla u^{k+1}|^p \, dx,
\]
\[
I_3 = \tau \int_D F(u^{k+1})\nabla u^{k+1} \, dx = 0,
\]
\[
I_4 := (H(u^k)\Delta_{k+1}W, u^{k+1} - u^k)_2 + (H(u^k)\Delta_{k+1}W, u^k)_2.
\]
Taking expectation on both sides of (26) we arrive at
\[
\frac{1}{2} E \left( \|u^{k+1}\|_2^2 - \|u^k\|_2^2 + \|u^{k+1} - u^k\|_2^2 \right) + \tau E \int_D |\nabla u^{k+1}|^p \, dx = E(H(u^k)\Delta_{k+1}W, u^{k+1} - u^k)_2 + E(H(u^k)\Delta_{k+1}W, u^k)_2.
\]
Since $u_k$ is $F_{t_k}$-measurable and $W(t_{k+1}) - W(t_k)$ is $F_{t_k}$-independent, we have
\[
E(H(u^k)\Delta_{k+1}W, u^{k+1} - u^k)_2 = E E \left[ (H(u^k) \circ Q^{1/2}(W(t_{k+1}) - W(t_k)), u^k) \right] \cdot F_{t_k} = 0.
\]
Using Hölder and Young inequality it follows that for any $\alpha > 0$
\[
E(H(u^k)\Delta_{k+1}W, u^{k+1} - u^k)_2 \leq E(||\Phi_k \Delta_{k+1}W||_2 \cdot \|u^{k+1} - u^k\|_2)
\]
\[
\leq \frac{1}{2} \left( \frac{1}{\alpha} E \left[ \int_{t_k}^{t_{k+1}} |H(u^k)\, dW|_2 \right] \right)^2 + \alpha E\|u^{k+1} - u^k\|_2^2
\]
By Itô isometry and for $\alpha = \frac{1}{2}$ from (29) it follows that
\[
E(H(u^k)\Delta_{k+1}W, u^{k+1} - u^k)
\]
\[
\leq E \int_{t_k}^{t_{k+1}} \|H(u^k)\|_{H^1(D)}^2 \, dt + \frac{1}{4} E\|u^{k+1} - u^k\|_2^2
\]
\[
\leq \tau E\|H(u^k)\|_{H^1(D)}^2 + \frac{1}{4} E\|u^{k+1} - u^k\|_2^2
\]
and therefore we arrive at
\[
\frac{1}{2} E \left( \|u^{k+1}\|_2^2 - \|u^k\|_2^2 + \|u^{k+1} - u^k\|_2^2 \right) + \tau E \int_D |\nabla u^{k+1}|^p \, dx
\]
\[
\leq \tau E\|H(u^k)\|_{H^1(D)}^2 + \frac{1}{4} E\|u^{k+1} - u^k\|_2^2,
\]
hence (25) holds.
Definition 3.1. For $N \in \mathbb{N}$, $\tau > 0$ we introduce the right-continuous step function

$$u_N(t) = \sum_{k=0}^{N-1} u^{k+1} \chi_{[t_k, t_{k+1})}(t), \ t \in [0,T],$$

the left-continuous, $\mathcal{F}_t$-adapted step function

$$u_\tau(t) = \sum_{k=0}^{N-1} u^k \chi_{[t_k, t_{k+1})}(t), \ t \in (0,T], \ u_\tau(0) = u_0,$$

the continuous, square-integrable $\mathcal{F}_t$-martingale

$$B_N(t) = \int_0^t H(u_\tau) \ dW, \ t \in [0,T]$$

and the piecewise affine functions

$$\tilde{u}_N(t) := \sum_{k=0}^{N-1} \left( \frac{u^{k+1} - u^k}{\tau} (t - t_k) + u^k \right) \chi_{[t_k, t_{k+1})}(t), \ t \in [0,T], \ \tilde{u}_N(T) = u^N,$$

$$\tilde{B}_N(t) = \sum_{k=0}^{N-1} \left( \frac{B_N(t_{k+1}) - B_N(t_k)}{\tau} (t - t_k) + B_N(t_k) \right) \chi_{[t_k, t_{k+1})}(t), \ t \in [0,T].$$

Lemma 3.3. There exists a constant $K \geq 0$ not depending on the discretization parameters such that

$$\max_{n=1,\ldots,N} \|u^n\|_2^2 \leq K,$$  \hspace{1cm} (32)

$$\sum_{k=0}^{N-1} E\|u^{k+1} - u^k\|_2^2 \leq 4K + 2\|u_0\|_2^2.$$  \hspace{1cm} (33)

In particular, by (H1) there exists $K(C_1, C_2, \|u_0\|, T) > 0$ such that

$$E\int_0^T \|H(u_\tau)\|_{HS(L^2(D))}^2 \ dt \leq K(C_1, C_2, \|u_0\|, T).$$  \hspace{1cm} (34)

Moreover we have

$$E \sup_{t \in [0,T]} \|\tilde{u}_N\|_2^2 = E \sup_{t \in [0,T]} \|u_N\|_2^2 \leq K,$$  \hspace{1cm} (35)

$$E\int_0^T \int_D |\nabla u_N|^p \ dx \ dt \leq K + \frac{1}{2} \|u_0\|_2^2.$$  \hspace{1cm} (36)

Proof: We fix $n \in \{1,\ldots,N\}$, take the sum over $0,\ldots,n-1$ in (25) to get

$$\frac{1}{2} E\|u^n\|_2^2 - \frac{1}{2} E\|u_0\|_2^2 + \frac{1}{4} \sum_{k=0}^{n-1} E\|u^{k+1} - u^k\|_2^2 \leq \sum_{k=0}^{n-1} \tau E \int_D |\nabla u^{k+1}|^p \ dx$$

$$\leq \sum_{k=0}^{n-1} \tau E\|H(u^k)\|_{HS(L^2(D))}^2.$$  \hspace{1cm} (37)

Discarding nonnegative terms by (H1) it follows that

$$\frac{1}{2} E\|u^n\|_2^2 \leq \frac{1}{2} E\|u_0\|_2^2 + \sum_{k=0}^{n-1} C_1 \tau E\|u^k\|_2^2.$$  \hspace{1cm} (38)
Applying the discrete Gronwall inequality in (38) yields

\[ E\|u^n\|_2^2 \leq \|u_0\|_2^2 e^{2C_1T} \]  

(39)

and (32) follows from (39) with \( K := \|u_0\|_2^2 e^{2C_1T} \). Now (33) follows from (32) and (37) by taking \( n = N \) and keeping the nonnegative term

\[ \frac{1}{4} \sum_{k=0}^{n-1} E\|u^{k+1} - u^k\|_2^2. \]

(34) is a direct consequence of (H1) and (32). Moreover,

\[ E \sup_{t \in [0,T]} \|\tilde{u}_N\|_2^2 = E \sup_{t \in [0,T]} \|u_N\|_2^2 \leq E \max_{k=1,...,N} \|u^k\|_2^2 \leq K. \]  

(40)

Finally, (36) follows now from (37) and (32) by keeping the nonnegative term \( \sum_{k=0}^{n-1} \tau E \int_D |\nabla u^{k+1}|^p \, dx \) and taking \( n = N \).

**Lemma 3.4.** There exists \( C \geq 0 \) not depending on \( N \in \mathbb{N} \) such that

\[ E \int_0^T \frac{d}{dt} (\tilde{u}_N - \tilde{B}_N)^p'_{W^{-1},p'}(D) \, dt \]  

\[ \leq C \left( E \int_0^T \|u_N\|_2^2 + \|\nabla u_N\|_p^p \, dt + 1 \right). \]  

(41)

**Proof:** For all \( t \in (t_k, t_{k+1}) \), and all \( k = 0, \ldots, N-1 \)

\[ \frac{d}{dt} (\tilde{u}_N - \tilde{B}_N) = \frac{u^{k+1} - u^k - H(u^k)\Delta_{k+1}W}{\tau} = \text{div}(\nabla u^{k+1}|^p-2\nabla u^{k+1} + F(u^{k+1})). \]  

(42)

Since \( p \geq 2 \), there exists a constant \( C \geq 0 \) not depending on \( N \in \mathbb{N} \) that may change from line to line such that

\[ \frac{d}{dt} (\tilde{u}_N - \tilde{B}_N)^p'_{W^{-1},p'}(D) \]  

\[ = \sup_{\|\varphi\|_{W^{1,p'}(D)} \leq 1} \int_D \left( |\nabla u^{k+1}|^p-2\nabla u^{k+1} + F(u^{k+1}) \right) \text{div}\varphi \, dx \]  

\[ \leq \sup_{\|\varphi\|_{W^{1,p'}(D)} \leq 1} \left( |\nabla u^{k+1}|_{L^p}^p + |F(u^{k+1})|_{L^2} \right) \text{div}\varphi \]  

\[ \leq \sup_{\|\varphi\|_{W^{1,p'}(D)} \leq 1} \left( |\nabla u^{k+1}|_{L^p}^p + C|F(u^{k+1})|_{L^2} \right) \text{div}\varphi \]  

\[ \leq |\nabla u^{k+1}|_{L^p}^p + F(u^{k+1})_{L^2}. \]  

(43)

Therefore, for \( p' \leq 2, L > 0 \) the Lipschitz constant of \( F \)

\[ \frac{d}{dt} (\tilde{u}_N - \tilde{B}_N)^p'_{W^{-1},p'}(D) \leq \left( |\nabla u^{k+1}|_{L^p}^p + 2(1 + \|u^{k+1}\|_{L^1}^2) \right)^{p'} \]  

\[ \leq 2^{p'} |\nabla u^{k+1}|_{L^p}^p + C \left( |\nabla u^{k+1}|_{L^p}^p + 2(1 + \|u^{k+1}\|_{L^1}^2) \right). \]  

Hence

\[ \int_0^T \frac{d}{dt} (\tilde{u}_N - \tilde{B}_N)^p'_{W^{-1},p'}(D) \, dt \]  

\[ \leq 2^{p'} \tau \sum_{k=0}^{N-1} |\nabla u^{k+1}|_{L^p}^p + C \left( 1 + \tau \sum_{k=0}^{N-1} |u^{k+1}|_{L^1}^2 \right), \]
and

\[
E \int_0^T \left| \frac{d}{dt}(\tilde{u}_N - \tilde{B}_N) \right|_{W^{1, p'}} dt \\
\leq 2p' \tau \sum_{k=0}^{N-1} E\|\nabla u^{k+1}\|_p^p + C(1 + \tau \sum_{k=0}^{N-1} E\|u^{k+1}\|_2^2) \\
\leq C \left( E \int_0^T \|u_N\|_2^2 + \|\nabla u_N\|_p^p dt + 1 \right).
\]

From Lemma 3.3 and Lemma 3.4 we get

Lemma 3.5. There exists a constant \( C \geq 0 \) not depending on \( N \in \mathbb{N} \) such that

\[
E \int_0^T \left| \frac{d}{dt}(\tilde{u}_N - \tilde{B}_N) \right|_{W^{1, p'}} dt \\
\leq C. \quad (44)
\]

Lemma 3.6. For \( T > 0, N \in \mathbb{N} \) we define an equidistant subdivision of \([0, T]\) by

\[
0 = t_0 < t_1 < \ldots < t_N = T
\]

with \( \tau = \frac{T}{N} = t_{k+1} - t_k \) for \( k = 0, \ldots, N - 1 \). Let \( \mathcal{K}, \mathcal{H} \) be separable Hilbert spaces and \( W \) be a Wiener process in \( \mathcal{K} \) with covariance operator \( Q \). For a \( \mathcal{F}_{t_k} \)-measurable random variable \( \Phi_k \) with values in \( HS(Q^{1/2}(\mathcal{K}), \mathcal{H}) \) we define the left-continuous, \( \mathcal{F}_t \)-adapted process

\[
\Phi_t := \sum_{k=0}^{N-1} \Phi_k \chi_{(t_k, t_{k+1}]}.
\]

For any \( p > 2 \), there exists constants \( \gamma > 0 \) and \( C_\gamma \geq 0 \) not depending on \( N \in \mathbb{N} \) and an integrable, real-valued random variable \( \chi \) such that

\[
\sup_{k \in \{0, \ldots, N-1\}} \sup_{s \in [t_k, t_{k+1}]} \left\| \int_{t_k}^s \Phi_t dW \right\|_{\mathcal{H}} \\
\leq C_\gamma \tau^\gamma \left( \sup_{k \in \{0, \ldots, N-1\}} \tau \|\Phi_k\|_{HS(\mathcal{K}, \mathcal{H})} + 1 + X \right).
\]

Moreover, there exists a constant \( C \geq 0 \) such that

\[
E(X) \leq C \text{tr}(Q). \quad (45)
\]

Proof: Let us fix \( s \in [t_k, t_{k+1}] \) and \( k \in \{0, \ldots, N - 1\} \). Then we have

\[
\left\| \int_{t_k}^s \Phi_t dW \right\|_{\mathcal{H}} \leq \|\Phi_k\|_{HS(\mathcal{K}, \mathcal{H})} \|W(s) - W(t_k)\|_{\mathcal{K}}.
\]

Now, from [13], [23, Ex. 2.4.1] (see Lemma 5.6 in the Appendix) for any \( q \geq 1 \) and \( \alpha > 1/4 \) it follows that

\[
\|W(s) - W(t_k)\|_{\mathcal{K}} \\
\leq C_{\alpha,q} \tau^\alpha \left( \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \frac{\|W(t) - W(r)\|_{\mathcal{K}}^q}{|t - r|^{\frac{3}{2}}^{1+q}} dt dr \right)^{1/q} \\
\leq C_{\alpha,q} \tau^\alpha \left( \int_0^T \int_0^T \frac{\|W(t) - W(r)\|_{\mathcal{K}}^q}{|t - r|^{\frac{3}{2} \alpha q + 1}} dt dr \right)^{1/q} \\
= C_{\alpha,q} \tau^\alpha X^{1/q} \quad (46)
\]
Lemma 3.7. There exists a constant $K_1 > 0$ not depending on the discretization parameters such that

$$E \int_0^T \|H(u_t)\|_{L^2(L^2(D), H^1_0(D))}^p \ dt \leq K_1.$$
Lemma 3.8. For any $\tau > 0 \in \mathbb{N}$, since, for all $n \in \mathbb{N}$, from (36), it follows that
\begin{equation}
||H(u_k)(e_n)||^2_{H^1_0(D)} = ||\nabla h_n(u_k)||^2 \leq ||h_n' ||^2_{\infty} ||\nabla h_n(u_k)||^2_2, \tag{49}
\end{equation}
we can use (H2) to estimate
\begin{align}
E \sum_{k=0}^{N-1} \left( \sum_{n=1}^{\infty} ||H(u_k)(e_n)||^2_{H^1_0(D)} \right)^{p/2} & \leq EC_2^{p/2} \tau \sum_{k=0}^{N-1} ||\nabla u_k||^p_2 \\
& \leq C_2^{p/2} EC_p \int_0^T ||\nabla u_k||^p_2 + ||\nabla u_0||^p_2 dt \\
& = C_2^{p/2} C_p E \int_0^T ||\nabla u_k||^p_2 dt + T ||\nabla u_0||^p_2 dt \tag{50}
\end{align}
where $C_p \geq 0$ is a constant not depending on the discretization parameters. According to Lemma 3.3, (36), from (50) it follows that
\begin{equation}
E \int_0^T ||H(u_\tau)||^p_{HS(L^2(D);H^1_0(D))} dt \leq K_1 \tag{51}
\end{equation}
with $K_1 := \max(C_2^{p/2}C_p(K + ||u_0||^2_2), C_2^{p/2}C_p||\nabla u_0||^p_2)$.

**Definition 3.2.** For a Banach space $V$, $T > 0$, $0 < \alpha < 1$ and $1 \leq p < \infty$ we recall the definition of the fractional Sobolev space (see also [1], p.111, [22] for more information):
\begin{equation}
W^{\alpha,p}(0,T;V) := \{f \in L^p(0,T;V) \mid \|f\|_{W^{\alpha,p}(0,T;V)} < +\infty \},
\end{equation}
where
\begin{equation}
\|f\|_{W^{\alpha,p}(0,T;V)} = \left( \int_0^T \int_0^T \frac{||f(r)-f(t)||^p_V}{|t-r|^{\alpha p+1}} dt \right)^{1/p}.
\end{equation}

**Lemma 3.8.** For any $\alpha \in (0, \frac{1}{2})$ there exists a constant $C(\alpha,p) \geq 0$ such that
\begin{equation}
E \int_0^T H(u_\tau) dW \|_{W^{\alpha,p}(0,T;H^1_0(D))} \leq C(\alpha,p) K_1, \tag{52}
\end{equation}
where $K_1 \geq 0$ is defined in Lemma 3.7. In particular, $\int_0^T H(u_\tau) dW$ is bounded in $L^p(\Omega; W^{\alpha,p}(0,T;H^1_0(D)))$.

Proof: We recall that $u_\tau$ is a left-continuous, $F_t^W$-adapted process with values in $W^{1,p}_0(D)$ and $H : W^{1,p}_0(D) \to HS(L^2(D);H^1_0(D))$ is continuous. Thus, $H(u_\tau)$ is a left-continuous, $F_t^W$-adapted process and therefore it is progressively measurable. From [11], Lemma 2.1., p.369 (Lemma 5.7 in the Appendix) it follows that there exists $C(\alpha,p) \geq 0$ such that
\begin{align}
E \int_0^T H(u_\tau) dW \|_{W^{\alpha,p}(0,T;H^1_0(D))} & \leq C(\alpha,p) E \int_0^T ||H(u_\tau)||^p_{HS(L^2(D);H^1_0(D))} dt. \tag{53}
\end{align}
Now, the assertion is a direct consequence of Lemma 3.7.
**Lemma 3.9.** \((\tilde{B}_N)\) is uniformly bounded in \(L^p(\Omega; W^{\alpha,p}(0, T; H^1_0(D)))\) for any \(\alpha \in (0, \gamma)\) and \(\gamma = \frac{1}{2} - \frac{1}{p}\).

Proof: We have to verify the assumptions of Lemma [2], Lemma 3.2 (Lemma 5.8 in the Appendix) for \(G = \tilde{B}_N): For any \(l \in \{0, \ldots, N\}\) we have

\[
\sum_{k=0}^{N-l} \|\tilde{B}_N(t_{k+l}) - \tilde{B}_N(t_k)\|_{L^p(\Omega; H^1_0(D))}^p = \sum_{k=0}^{N-l} E\left( \int_{t_k}^{t_{k+l}} \|H(u_t)\|_{H^{\alpha,p}(L^2(D); H^1_0(D))}^p \, dt \right)^{p/2} \leq E\sum_{k=0}^{N-l} (t_{k+l} - t_k)^{\frac{p}{2} - 1} \left( \int_0^T \|H(u_t)\|_{H^{\alpha,p}(L^2(D); H^1_0(D))}^p \, dt \right). \tag{54}
\]

From (54) and Lemma 3.7 it follows that there exists a constant \(K_1 > 0\) not depending on the discretization parameters such that

\[
\sum_{k=0}^{N-l} \|\tilde{B}_N(t_{k+l}) - \tilde{B}_N(t_k)\|_{L^p(\Omega; H^1_0(D))}^p \leq \tau(N-l)T^{\frac{p}{2} - 1} E\int_0^T \|H(u_t)\|_{H^{\alpha,p}(L^2(D); H^1_0(D))}^p \, dt \leq (T + t_l)^{\frac{p}{2} - 1} K_1 \leq 2TK_1T^{\frac{p}{2} - 2} \tag{56}
\]

For \(\gamma := \frac{1}{2} - \frac{1}{p} > 0\), and \(C := (2TK_1)^{1/p}\) from (56) it follows that

\[
\sum_{k=0}^{N-l} \|\tilde{B}_N(t_{k+l}) - \tilde{B}_N(t_k)\|_{L^p(\Omega; H^1_0(D))}^p \leq C \|H\|^p. \tag{57}
\]

According to Lemma [2], Lemma 3.2 (Lemma 5.8 in the Appendix), from (57) it follows that \((\tilde{B}_N)\) is uniformly bounded in the Nikolskii space \(N^{\gamma,p}(0, T; L^p(\Omega; H^1_0(D)))\) with continuous imbedding for any \(\alpha \in (0, \gamma)\) (see [1], p.111, [22]). Thanks to the Fubini theorem this implies

\[
\|\tilde{B}_N\|_{L^p(\Omega; W^{\alpha,p}(0, T; H^1_0(D)))} \leq C
\]

for all \(N \in \mathbb{N}\) and a constant \(C \geq 0\) not depending on \(N \in \mathbb{N}\).

**Remark 3.2.** It is well-known (see, e.g. [20], Lemma 7.1, p.202 and Lemma 7.7, p.208) that the space

\[
\mathcal{W} := \{v \in L^p(0, T; H^1_0(D)) | \frac{d}{dt}v \in L^p(0, T; W^{-1,p'}(D))\}
\]

is continuously embedded into \(C([0, T]; W^{-1,p'}(D))\) and compactly embedded into \(L^2(0, T; L^2(D))\).
Lemma 3.10. There exists a constant $C \geq 0$ such that
\[
\|\bar{u}_N\|_{L^p(\Omega; L^p(0,T; W^{1,p}(D)))} + \|\bar{u}_N - \tilde{B}_N\|_{L^p(\Omega; W^{1,2p})} \leq C
\]
for all $N \in \mathbb{N}$.

Proof: Elementary calculations yield that there exists a constant $\tilde{C} > 0$ not depending on the discretization parameters such that
\[
E\|\bar{u}_N\|_{L^p(0,T;W^{1,p}_{\alpha}(D))}^p \leq \tilde{C}E_T \sum_{k=0}^{N} \|u_k\|_{W^{1,p}_{\alpha}(D)}^p
\]
\[
\leq \tilde{C} \left( \int_0^T \|
abla u_N\|_p^p \, dt + \|
abla u_0\|_p^p \right)
\]
and by Lemma 3.3 the right-hand side of (59) is bounded. From Lemma 3.9 it follows that $(\tilde{B}_N)$ is bounded in $L^p(\Omega; W^{\alpha,p}(0,T; H^{1}_{0}(D)))$ for $\alpha \in (0, 1 - \frac{1}{p})$. Thus, $(\bar{u}_N - \tilde{B}_N)$ is bounded in $L^p(\Omega; L^p(0,T; H^{1}_{0}(D)))$. Now, the assertion is a direct consequence of Lemma 3.5.

3.4 Tightness

Next, we set
\[
\mathcal{X} := C([0,T]; L^2(D)) \times L^2(0,T; L^2(D)).
\]
For $N \in \mathbb{N}$, we denote the law $P \circ (\bar{u}_N)^{-1}$ of $\bar{u}_N$ on $L^2(0,T; L^2(D))$ by $\mu_{\bar{u}_N}$ and the law $P \circ (B_N)^{-1}$ of $B_N$ on $C([0,T]; L^2(D))$ by $\mu_{B_N}$. Their joint law on $\mathcal{X}$ is denoted by $\mu_N = (\mu_{B_N},\mu_{\bar{u}_N})$.

Proposition 3.11. The sequence $(\mu_{\bar{u}_N})$ is tight on $L^2(0,T; L^2(D))$ and the sequence $(\mu_{B_N})$ is tight on $C([0,T]; L^2(D))$. In particular, the sequence of their joint laws $(\mu_N)$ is tight on $\mathcal{X}$.

Proof: For $\alpha \in (0, \frac{1}{2})$, the linear space
\[
\mathcal{V} := \{u = v + w, \, v \in \mathcal{W}, \, w \in W^{\alpha,p}(0,T; H_{0}^{1}(D))\}
\]
endowed with the norm
\[
\|u\|_{\mathcal{V}} := \inf_{v \in \mathcal{W}} \max_{w \in W^{\alpha,p}(0,T; H_{0}^{1}(D))} (\|v\|_{\mathcal{W}}, \|w\|_{W^{\alpha,p}})
\]
is a Banach space which is compactly embedded into $L^2(0,T; L^2(D))$ (see Lemma 5.9 in the Appendix). Since
\[
\bar{u}_N = (\bar{u}_N - \tilde{B}_N) + \tilde{B}_N
\]
for all $N \in \mathbb{N}$, it follows from Lemma 3.9 and Lemma 3.10 that $(\bar{u}_N)$ is bounded in $L^p(\Omega; \mathcal{V})$. Now, let us fix $\varepsilon > 0$. For any $R > 0$ the set
\[
B_{\mathcal{V}}(R,0) := \{u \in \mathcal{V} \mid \|u\|_{\mathcal{V}} \leq R\}
\]
is compact in $L^2(0,T; L^2(D))$. There exists a constant $C > 0$ not depending $R > 0$, such that for any $R > 0$, and any $N \in \mathbb{N}$
\[
\mu_{\bar{u}_N}(B_{\mathcal{V}}(R,0)) = 1 - \mu_{\bar{u}_N}(B_{\mathcal{V}}(R,0))
\]
\[
= 1 - \int_{\omega \in \Omega \mid \|\bar{u}_N\|_{\mathcal{V}} > R} 1 \, dP
\]
\[
\geq 1 - \frac{1}{R^p} \int_{\omega \in \Omega \mid \|\bar{u}_N\|_{\mathcal{V}} > R} \|\bar{u}_N\|_{\mathcal{V}}^p \, dP
\]
\[
\geq 1 - \frac{1}{R^p} E(\|\bar{u}_N\|_{\mathcal{V}}^p) = 1 - \frac{C}{R^p}
\]
(60)
and from (60) it follows that we can find \( R_\varepsilon > 0 \) such that

\[
\mu_{\bar{\omega}_N}(B_{\psi}(R_\varepsilon, 0)) \geq 1 - \varepsilon
\]

for all \( N \in \mathbb{N} \).

According to [21], p.82, Corollary 2,

\[
W^{\alpha,p}(0,T; H^1_0(D)) \hookrightarrow C([0,T]; L^2(D))
\]

with compact imbedding for all \( \alpha \in (\frac{1}{2}, \frac{3}{2}) \). Thus, for any \( R > 0 \) and any \( \alpha \in (\frac{1}{2}, \frac{3}{2}) \)

\[
B_{W^{\alpha,p}}(R, 0) := \{ u \in W^{\alpha,p}(0,T; H^1_0(D)) \mid \|u\|_{W^{\alpha,p}(0,T; H^1_0(D))} \leq R \}
\]

is compact in \( C([0,T]; L^2(D)) \). By Lemma 3.8 \((B_N)\) is uniformly bounded in \( L^p(\Omega; W^{\alpha,p}(0,T; H^1_0(D))) \) for \( \alpha \in (0, \frac{1}{2}) \), hence there exists a constant \( C > 0 \) not depending \( R > 0 \) such that

\[
\mu_{B_N}(B_{W^{\alpha,p}}(R, 0)) = 1 - \mu_{B_N}(B_{W^{\alpha,p}}^c(R, 0)) \geq 1 - \frac{1}{R^p} E[\|B_N\|_{W^{\alpha,p}(0,T; H^1_0(D))}^p] = 1 - \frac{C}{R^p} \quad (61)
\]

Thanks to (61), for any \( \varepsilon > 0 \) we can find \( R_\varepsilon > 0 \) such that

\[
\mu_{B_N}(B_{W^{\alpha,p}}(R_\varepsilon, 0)) \geq 1 - \varepsilon.
\]

**Remark 3.3.** From Prokhorov theorem (see Theorem 5.1 and 5.3 in the Appendix for references) and Proposition 3.11 it follows that the sequence \((\mu_N) = (\mu_{B_N}, \mu_{\bar{\omega}_N})\) is relatively compact, i.e. there exists a (not relabeled) subsequence of \((\mu_N)\) and a probability measure \( \mu_\infty = (\mu_1^\infty, \mu_2^\infty) \) on \( \mathcal{X} \), such that

\[
\lim_{N \to \infty} \int_{C([0,T]; L^2(D))} \psi \, d\mu_{B_N} = \int_{C([0,T]; L^2(D))} \psi \, d\mu_1^\infty \quad (62)
\]

for all bounded, continuous functions \( \psi : C([0,T]; L^2(D)) \to \mathbb{R} \),

\[
\lim_{N \to \infty} \int_{L^2(0,T; L^2(D))} \varphi \, d\mu_{\bar{\omega}_N} = \int_{L^2(0,T; L^2(D))} \varphi \, d\mu_2^\infty \quad (63)
\]

for all bounded, continuous functions \( \varphi : L^2(0,T; L^2(D)) \to \mathbb{R} \). In particular,

\[
\int_{L^2(0,T; L^2(D))} \varphi \, d\mu_{\bar{\omega}_N} = \int_{\Omega} \varphi(\bar{u}_N) \, dP = E[\varphi(\bar{u}_N)],
\]

\[
\int_{C([0,T]; L^2(D))} \psi \, d\mu_{B_N} = \int_{\Omega} \psi(B_N) \, dP = E[\psi(B_N)],
\]

hence (62) implies \( B_N \mathcal{L}_\alpha \mu_\infty^1 \) and (63) implies \( \bar{u}_N \mathcal{L}_\alpha \mu_\infty^2 \).

### 3.5 Existence of martingale solutions

Now, we use the following version of the theorem of Skorokhod (see [25], Theorem 1.10.4 and Addendum 1.10.5, p.59 and [1], Theorem 2.3, p.119-120), which can be found in the Appendix, to conclude:

There exists a probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})\), a sequence of measurable functions

\[
\phi_N : (\hat{\Omega}, \hat{\mathcal{F}}) \to (\Omega, \mathcal{F}), \quad N \in \mathbb{N}
\]

such that \( P = \hat{P} \circ \phi_N^{-1} \) for all \( N \in \mathbb{N} \) and measurable functions

\[
(B_{\infty}, u_{\infty}) : (\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}) \to \mathcal{X},
\]

having the following properties:
i.) $\tilde{u}_N := \tilde{u}_N \circ \phi_N \to u_\infty$ in $L^2(0, T; L^2(D))$ for $N \to \infty$ a.s. in $\hat{\Omega}$,

ii.) $\hat{B}_N := B_N \circ \phi_N \to B_\infty$ in $C([0, T]; L^2(D))$ for $N \to \infty$ a.s. in $\hat{\Omega}$

iii.) $\mathcal{L}(B_\infty, u_\infty) = \mu_\infty$.

**Definition 3.3.** For $N \in \mathbb{N}$ we define $W_N := W \circ \phi_N$ and

$v^k := u^k \circ \phi_N$, $k = 0, \ldots, N$.

For all $t \in [0, T]$, we introduce the left-continuous function

$v_\tau(t) := \sum_{k=0}^{N-1} v^k \chi_{[t_k, t_{k+1})}(t), \quad t \in (0, T], \quad v_\tau(0) = u_0$,

the right-continuous function

$v_N(t) := \sum_{k=0}^{N-1} v^{k+1} \chi_{[t_k, t_{k+1})}(t), \quad t \in [0, T]

and the piecewise affine function

$b_N(t) := \sum_{k=0}^{N-1} \left( \frac{\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k)}{\tau} \right) \chi_{[t_k, t_{k+1})}(t), \quad t \in [0, T].$

**Lemma 3.12.** For any $N \in \mathbb{N}$, $W_N$ is a $Q$-Wiener process in $U$ with $Q = \text{diag}(\frac{1}{\tau})$, thus a cylindrical Wiener process in $L^2(D) = Q^{1/2}(U)$ adapted to the filtration $(\mathcal{F}_t^W) := \sigma(W \circ \Phi_N(s))_{0 \leq s \leq t}$.

Proof: For all $t \in [0, T]$ and all $N \in \mathbb{N}$, $W_N(t)$ is $\mathcal{F}/\mathcal{B}(L^2(D))$-measurable as the composition of the $\mathcal{F}/\mathcal{F}$-measurable function $\phi_N$ with the $\mathcal{F}/\mathcal{B}(L^2(D))$-measurable function $Q^{1/2} \circ W(t)$. Thus, $W_N : \hat{\Omega} \times [0, T] \to L^2(D)$ is a stochastic process. For an orthonormal basis $(e_n)$ of $L^2(D)$ we have the representation

$W_N(\hat{\omega}, t) = \sum_{n=1}^{\infty} \frac{1}{n} (ne_n) \beta^N_n(t)$

with $\beta^N_n(t) := (\beta_n \circ \Phi_N(t))$ for all $n, N \in \mathbb{N}$ and $t \in [0, T]$, where $(\beta_n(t))$ is a sequence of real-valued $\mathcal{F}_t$-Brownian motions on $(\Omega, \mathcal{F}, P)$. Since

$\hat{P} \circ (\beta^N_n(t) - \beta^N_n(s))^{-1} = P \circ (\beta_n(t) - \beta_n(s))^{-1}$

for all $N, n \in \mathbb{N}$, all $t \in [0, T]$ and all $0 \leq s \leq t$, it follows that $(\beta^N_n(t))$ is a sequence of $\mathcal{F}_t^W$-Brownian motions on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$.

**Lemma 3.13.** For any $N \in \mathbb{N}$ and any $k = 0, \ldots, N - 1$ we have

$v^{k+1} - v^k - \tau \text{div}(|\nabla v^{k+1}|^p - 2|\nabla v^{k+1}|^p + F(v^{k+1})) - H(v^k)\Delta_{k+1}W_N = 0 \quad (64)$

a.s. in $\hat{\Omega}$.

Proof: Since $P = \hat{P} \circ \phi_N^{-1}$, by definition of the image measure for any $\hat{A} \in \hat{\mathcal{F}}$ we have

$\int_{\hat{A}} v^{k+1} - v^k - \tau \text{div}(|\nabla v^{k+1}|^p - 2|\nabla v^{k+1}|^p + F(v^{k+1})) - H(v^k)\Delta_{k+1}W_N d\hat{P}$

$= \int_{\phi_N(\hat{A})} u^{k+1} - u^k - \tau \text{div}(|\nabla u^{k+1}|^p - 2|\nabla u^{k+1}|^p + F(u^{k+1})) - H(u^k)\Delta_{k+1}W dP$

$= 0 \quad (65)$
Lemma 3.14. We have
\[ \hat{B}_N(t) = \int_0^t H(v_\tau) \, dW_N, \]  
for all \( t \in [0, T] \), a.s. in \( \tilde{\Omega} \),
\[ \hat{u}_N(t) = \frac{v^{k+1} - v^k}{\tau} (t - t_k) + v^k, \]  
for all \( t \in [t_k, t_{k+1}) \), \( k = 0, \ldots, N - 1 \) and \( \hat{u}_N(T) = v^N \) a.s. in \( \tilde{\Omega} \). Moreover, there exist constants \( K, K_1 \geq 0 \) such that
\[ E \sup_{t \in [0,T]} \| \hat{u}_N(t) \|_2^2 = E \sup_{t \in [0,T]} \| v_N(t) \|_2^2 \leq \hat{K}, \]  
\[ E \int_0^T \int_D |\nabla v_N|^p \, dx \, dt \leq \hat{K} + \frac{1}{2} \| u_0 \|_2^2. \]  
\[ E \int_0^T \| H(v_\tau) \|_{H^1_0(D)}^p \, dt \leq \hat{K}_1 \]  
for all \( N \in \mathbb{N} \).

Proof: For any \( t \in [t_k, t_{k+1}) \) and \( k = 0, \ldots, N - 1 \) we have
\[ \hat{B}_N(t) = (B_N \circ \phi_N)(t) \]
\[ = \sum_{l=0}^{k-1} H(u^l \circ \phi_N) \circ Q^{1/2}(W(t_{l+1}) \circ \phi_N - W(t_l) \circ \phi_N) + H(u^k \circ \phi_N) \circ Q^{1/2}(W(t) \circ \phi_N - W(t_k) \circ \phi_N) \]
\[ = \sum_{l=0}^{k-1} H(u^l) \circ Q^{1/2}(W_N(t_{l+1}) - W_N(t_l)) + H(u^k) \circ Q^{1/2}(W_N(t) - W_N(t_k)) \]
\[ = \int_0^t H(v_\tau) \, dW_N. \]  
(71)

(67) follows since
\[ \hat{u}_N(\hat{\omega}, t) = \hat{u}_N(\phi_N(\hat{\omega}), t) \]
\[ = \frac{v^{k+1}(\phi_N(\hat{\omega})) - v^k(\phi_N(\hat{\omega}))}{\tau} (t - t_k) + u^k(\phi_N(\hat{\omega})) \]  
for a.e. \( \hat{\omega} \in \hat{\Omega} \) all \( t \in [t_k, t_{k+1}) \), \( k = 0, \ldots, N - 1 \). Moreover
\[ \hat{u}_N(\hat{\omega}, T) = \hat{u}_N(\phi_N(\hat{\omega}), T) = u^N(\phi_N(\hat{\omega})) = v^N. \]

Thanks to (64), (68) and (69) follow repeating the arguments in the proof of Lemma 3.3 with respect to \( u^{k+1} \). Then, (70) follows repeating the arguments in the proof of Lemma 3.7 with respect to \( v_\tau \).

Lemma 3.15. For \( N \to \infty \), we have the following convergence results:
1. \( \hat{B}_N \to B_\infty \) in \( L^q(\hat{\Omega}; C([0,T]; L^2(D))) \) for all \( 1 \leq q < p \),
2. \( \hat{B}_N \to B_\infty \) in \( L^p(\hat{\Omega}; W^{\alpha,p}(0,T; H^1_0(D))) \),
3. \( \hat{u}_N \to u_\infty \) in \( L^q(\hat{\Omega}; L^2(0,T; L^2(D))) \) for all \( 1 \leq q < p \),
4.) \( v_N \to u_\infty \) in \( L^2(\tilde{\Omega} \times Q_T) \),
5.) \( v_\tau \to u_\infty \) in \( L^2(\tilde{\Omega} \times Q_T) \).

6.) \( \tilde{u}_N \overset{\text{a.s.}}{\to} u_\infty \) in \( L^2(\tilde{\Omega}; L^\infty(0, T; L^2(D))) \), where

\[ L^2(\tilde{\Omega}; L^\infty(0, T; L^2(D))) \simeq \left( L^2(\tilde{\Omega}; L^1(0, T; L^2(D))) \right)^* \]

and the space on the left-hand side contains all weak-* measurable mappings

\[ u : \tilde{\Omega} \to L^\infty(0, T; L^2(D)), \quad E\|u\|_{L^\infty(0, T; L^2(D))} < \infty \]

(see [10], Th. 8.20.3, p. 606).

Proof: For \( \alpha \in (\frac{1}{p}, \frac{2}{q}), \)

\[ L^p(\tilde{\Omega}; W^{\alpha, p}(0, T; H^1_0(D))) \hookrightarrow L^p(\tilde{\Omega}; C([0, T]; L^2(D))) \]

with continuous imbedding. Thus, using Lemma 3.14, (70) and [11], Lemma 2.1., p.369 (Lemma 5.7 in the Appendix) it follows that there exists \( C \geq 0 \) such that

\[ \|B_N\|_{L^p(\tilde{\Omega}; C([0, T]; L^2(D)))} + \|\tilde{B}_N\|_{L^p(\tilde{\Omega}; W^{\alpha, p}(0, T; H^1_0(D)))} \leq C \quad (73) \]

for all \( N \in \mathbb{N} \) and therefore \( \tilde{B}_N \) is equi-integrable in \( L^q(\tilde{\Omega}; C([0, T]; L^2(D))) \) for all \( 1 \leq q < p \). Since \( B_N \to B_\infty \) in \( C([0, T]; L^2(D)) \) for \( N \to \infty \) a.s. in \( \tilde{\Omega} \), 1.) follows from the Vitali theorem. Passing to a not relabeled subsequence, from (73) also follows that there exists \( g \in L^p(\tilde{\Omega}; W^{\alpha, p}(0, T; H^1_0(D))) \), such that

\[ \tilde{B}_N \to g \text{ in } L^p(\tilde{\Omega}; W^{\alpha, p}(0, T; H^1_0(D))) \]

for \( N \to \infty \). Since

\[ L^p(\tilde{\Omega}; W^{\alpha, p}(0, T; H^1_0(D))) \hookrightarrow L^2(\tilde{\Omega}; L^2(Q)) \]

with continuous imbedding and \( \tilde{B}_N \to B_\infty \) in \( L^2(\tilde{\Omega} \times Q) \) for \( N \to \infty \), it follows that \( g = B_\infty \) a.s. in \( \tilde{\Omega} \times Q \). Thus, the whole sequence \( (\tilde{B}_N) \) converges weakly to \( B_\infty \) in \( L^p(\tilde{\Omega}; W^{\alpha, p}(0, T; H^1_0(D))) \) and we have shown 2.).

There exists a constant \( \hat{C} \geq 0 \) not depending on \( N \in \mathbb{N} \) such that

\[ E\|\tilde{u}_N\|_{L^p(0, T; W^{\alpha, p}(D))} \leq \hat{C} E \left( \int_0^T \|\nabla v_N\|_p \, dt + \|u_0\|_p^p \right). \quad (74) \]

By Lemma (69) and the Poincaré inequality it follows that \( (\tilde{u}_N) \) is bounded in \( L^p(\tilde{\Omega}; L^2(Q)) \) and therefore equi-integrable in \( L^q(\tilde{\Omega}; L^2(Q)) \) for all \( 1 \leq q < p \). Together with the a.s. convergence of \( (\tilde{u}_N) \) to \( v_\infty \) in \( L^2(Q) \) for \( N \to \infty \), 2.) follows from the Vitali theorem.

From Lemma 3.13 and Lemma 3.14 it follows with similar arguments as in Lemma 3.3 that there exists a constant \( C \geq 0 \) such that

\[ \sum_{k=0}^{N-1} E\|v^{k+1} - v^k\|_2^2 \leq C. \quad (75) \]

For any \( N \in \mathbb{N} \) we have

\[
E \int_0^T \|\tilde{u}_N(t) - v_N(t)\|_2^2 \, dt \\
= E \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \left\| \frac{v^{k+1} - v^k}{\tau} (t - t_k) + v^k - v^{k+1} \right\|_2^2 \, dt \\
= E \sum_{k=0}^{N-1} \|v^{k+1} - v^k\|_2^2 \int_{t_k}^{t_{k+1}} \left( \frac{t - t_k}{\tau} - 1 \right)^2 \, dt \\
= \frac{7}{3} \sum_{k=0}^{N-1} E\|v^{k+1} - v^k\|_2^2 \leq \frac{7}{3} C
\]
therefore 3.) follows. Finally, from (75) we also have
\[
E \int_0^T \| v_T - v_{N} \|_2^2 \, dt = E \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \| v^{k+1} - v^{k} \|_2^2 \, dt
\]
\[
= E T \sum_{k=0}^{N-1} \| v^{k+1} - v^{k} \|_2^2 \leq C T
\]
(77)
and 4.) follows from (77).
Using Lemma 3.14, (68), from the Banach-Alaouglu theorem it follows that there exists \( f \in L^2_a(\Omega; L^\infty(0,T; L^2(D))) \) such that, passing to a not relabeled subsequence, \( \hat{u}_N \xrightarrow{w} f \) in \( L^2_a(\Omega; L^\infty(0,T; L^2(D))) \) for \( N \to \infty \). Now, taking test functions \( \chi_A \psi \) with \( \psi \in \mathcal{D}(Q_T) \) and \( A \in \mathcal{F} \), it follows that \( f = u_\infty \) a.s. in \( \hat{\Omega} \times Q_T \).

**Lemma 3.16.** \( u_\infty \in L^\infty(0,T; L^2(D)) \) a.s. in \( \hat{\Omega} \).

Proof: Since \( u_\infty \in L^2_a(\Omega; L^\infty(0,T; L^2(D))) \), the mapping
\[
\hat{\Omega} \ni \hat{\omega} \mapsto \| u(\hat{\omega}) \|_{L^\infty(0,T; L^2(D))} \in \mathbb{R}
\]
is \( \mathcal{F} \)-measurable and therefore the assertion follows.

The next lemma is a direct consequence of Lemma 3.14, (69):

**Lemma 3.17.** There exists a not relabeled subsequence of \( (v_N) \) such that
\[
\nabla v_N \to \nabla u_\infty \text{ in } L^p(\hat{\Omega} \times Q_T)^d
\]
for \( N \to \infty \). Moreover, there exists \( G \in L^p(\hat{\Omega} \times Q_T)^d \) such that
\[
|\nabla v_N|_{p-2} \nabla v_N \to G \text{ in } L^p(\hat{\Omega} \times Q_T)^d
\]
(79)
for the same subsequence and \( N \to \infty \).

**Lemma 3.18.** There exist constants \( \gamma > 0, C_\gamma \geq 0, C \geq 0 \) such that
\[
E \sup_{t \in [0,T]} \| \hat{B}_N(t) - b_N(t) \|_{H^3(D)}
\]
\[
\leq C \gamma \gamma \left( E \int_0^T \| H(v_T) \|_{H^3(L^2(D) ; H^3(D))}^p \, dt + 1 + C \text{tr}(Q) \right),
\]
(80)
for all \( N \in \mathbb{N} \), where \( Q = \text{diag} \left( \frac{1}{n} \right) \).

Proof: We fix \( N \in \mathbb{N} \). For \( k \in \{0, \ldots, N-1\} \) and \( t \in [t_k, t_{k+1}] \) we have a.s. in \( \hat{\Omega} \)
\[
\| \hat{B}_N(t) - b_N(t) \|_{H^3(D)}
\]
\[
= \left\| \int_0^t H(v_T) \, dW_N - \frac{\hat{B}_N(t_{k+1})}{\tau} - \hat{B}_N(t_k)(t - t_k) - \hat{B}_N(t_k) \right\|_{H^3(D)}
\]
\[
= \left\| \int_{t_k}^t H(v_T) \, dW_N - \frac{t-t_k}{\tau} \int_{t_k}^{t_{k+1}} H(v_T) \, dW_N \right\|_{H^3(D)}
\]
\[
\leq \left\| \int_{t_k}^t H(v_T) \, dW_N \right\|_{H^3(D)} + \left\| \int_{t_k}^{t_{k+1}} H(v_T) \, dW_N \right\|_{H^3(D)}
\]
(81)
Multiplying \((85)\) with \(\chi\) and therefore

Let us write \((86)\) as

that

Proposition 3.20. \(u_\infty : \hat{\Omega} \times [0, T] \to L^2(D)\) is a stochastic process with \(u_\infty(0) = u_0\) such that

holds in \(L^2(D)\) a.s. in \(\hat{\Omega}\) for all \(t \in [0, T]\).

Proof: For all \(k = 0, \ldots, N - 1\) from \((64)\) it follows that

Multiplying \((85)\) with \(\chi_A\) for \(A \in \mathcal{F}, \psi \in W^4_0(D), \xi \in \mathcal{D}(0, T),\) integrating over \(\hat{\Omega} \times [t_k, t_{k+1}] \times D\) and summing over \(k = 1, \ldots, N - 1\) it follows that

Let us write \((86)\) as

where

\[
I_1 = \int_A \int_0^T \int_D (\hat{u}_N - \hat{B}_N) \xi_t \psi(x) \, dx \, dt \, d\hat{P},
\]

\[
I_2 = \int_A \int_0^T \int_D (\hat{B}_N - b_N) \xi_t \psi(x) \, dx \, dt \, d\hat{P},
\]

\[
I_3 = \int_A \int_0^T \int_D |\nabla v_N|^2 \nabla v_N \cdot \nabla \psi \xi \, dx \, dt \, d\hat{P},
\]

\[
I_4 = \int_A \int_0^T \int_D F(v_N) \cdot \nabla \psi \xi \, dx \, dt \, d\hat{P}
\]
Since \( \hat{B}_N - \hat{u}_N \to B_\infty - u_\infty \) in \( L^2(\hat{\Omega} \times Q_T) \) for \( N \to \infty \), it follows that

\[
\lim_{N \to \infty} I_1 = \int_A \int_0^T \int_D (u_\infty - B_\infty)\xi_t \psi(x) \, dx \, dt \, d\hat{P}.
\]  

(89)

Moreover, by Hölder inequality,

\[
|I_2| \leq \int_A \int_0^T \|\xi_t\|_2 \|\hat{B}_N - b_N\|_2 \, dt \, d\hat{P}
\]

\[
\leq \int_0^T \sup_{t \in [0,T]} \|\hat{B}(t)_N - b_N(t)\|_2 \, d\hat{P} \int_0^T \|\xi_t\|_2 \, dt
\]

\[
\leq C_D E \sup_{t \in [0,T]} \|\hat{B}_N(t) - b_N(t)\|_{H^1(D)} \int_0^T \|\xi_t\|_2 \, dt
\]

(90)

where \( C_D \geq 0 \) is a constant not depending on \( N \in \mathbb{N} \). From Corollary 3.19 it now follows that

\[
|I_2| \leq C_D C_\gamma \tau^2 (K_1 + 1 + C\text{tr}(Q)) \int_0^T \|\xi_t\|_2 \, dt,
\]

(91)

therefore \( \lim_{N \to \infty} I_2 = 0 \). Since

\[
|\nabla v_N|^{p-2} \nabla v_N \to G \text{ in } L^{p'}(\hat{\Omega} \times Q_T)^d
\]

for \( N \to \infty \) (see Lemma 3.18), we get

\[
\lim_{N \to \infty} I_3 = \int_A \int_0^T \int_D G \cdot \nabla \psi \xi \, dx \, dt \, d\hat{P}
\]

(92)

From Lemma 3.15 it follows that \( v_N \to u_\infty \) in \( L^2(\hat{\Omega} \times Q_T) \) for \( N \to \infty \), thus we can extract a not relabeled subsequence such that

\[
v_N \to u_\infty \text{ a.e. in } \hat{\Omega} \times Q_T
\]

and there exists \( g \in L^2(\hat{\Omega} \times Q_T) \) such that \( |v_N| \leq g \) for all \( N \in \mathbb{N} \) a.e. in \( \hat{\Omega} \times Q_T \). Since \( F \) is Lipschitz continuous, it follows by Lebesgue dominated convergence theorem that

\[
\lim_{N \to \infty} F(v_N) = F(u_\infty).
\]

(93)

in \( L^2(\hat{\Omega} \times Q_T)^d \). Since this argument can be repeated with any arbitrary subsequence of \( (v_N) \), (93) holds for the whole sequence and therefore

\[
\lim_{N \to \infty} I_4 = \int_A \int_0^T \int_D F(u_\infty) \cdot \nabla \psi \xi \, dx \, dt \, d\hat{P}.
\]

(94)

Now from (89)-(94) it follows that

\[
- \int_A \int_0^T \int_D (u_\infty - B_\infty)\xi_t \psi + (G + F(u_\infty)) \cdot \nabla \psi \xi \, dx \, dt \, d\hat{P} = 0
\]

(95)

for all \( A \in \hat{F} \), \( \xi \in D(0,T) \) and all \( \psi \in W_0^{1,p'}(D) \). (95) implies that

\[
\frac{d}{dt}(u_\infty - B_\infty) = \text{div}(G + F(u_\infty))
\]

(96)

in \( L^{p'}(\hat{\Omega}; L^{p'}(0,T;W^{-1,p'}(D))) \). Moreover, from Lemma 3.18, (78) and Lemma 3.15, 2.) it follows that

\[
u_\infty - B_\infty \in L^p(\hat{\Omega}; L^p(0,T;H^1_0(D))),
\]
thus \( u_\infty - B_\infty \in L^p'(\hat{\Omega}; C([0,T]; W^{-1,p'}(D))) \) and, since

\[
B_\infty \in L^2(\hat{\Omega}; C([0,T]; L^2(D)))
\]

(see Lemma 3.15), it follows that \( u_\infty \) is in \( L^p'(\hat{\Omega}; C([0,T]; W^{-1,p'}(D))) \). Thanks to Lemma 3.16 and [24], Lemma 1.4, p.263, it follows that \( u_\infty \) is weakly continuous with values in \( L^2(D) \) a.s. in \( \hat{\Omega} \). Consequently,

\[
(u_\infty - B_\infty)(t) \in L^2(D)
\]

for all \( t \in [0,T] \), a.s. in \( \hat{\Omega} \), hence

\[
\langle (B_\infty - u_\infty)(t), \psi \rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} = \int_D (u_\infty - B_\infty)(t) \psi \, dx \tag{97}
\]

for all \( \psi \in W_0^{1,p}(D) \), a.s. in \( \hat{\Omega} \) for all \( t \in [0,T] \). With this information we may fix \( t \in [0,T] \) and choose a test function \( \xi \in D([t,T]) \) with \( \xi(t) = 1 \). Then, for any \( \psi \in W_0^{1,p}(D) \), a.s. in \( \hat{\Omega} \) using (96) and (97) we get

\[
\int_t^T \int_D (u_\infty - B_\infty) \xi \psi \, dx \, dr = \int_t^T \int_D \xi ((B_\infty - u_\infty)(r), \psi)_{W^{-1,p'}(D), W_0^{1,p}(D)} \, dr
\]

\[
= \int_t^T \xi(t) \langle (B_\infty - u_\infty)(t), \psi \rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} \, dr
\]

\[
= \int_t^T \xi(t) \langle \text{div}(G + F(u_\infty))(s), \psi \rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} \int_s^T \xi(r) \, dr \, ds
\]

and using Fubini theorem we get

\[
\int_t^T \int_D (u_\infty - B_\infty) \xi \psi \, dx \, dr + \int_D (u_\infty - B_\infty)(t) \psi \, dx = \int_t^T \int_D \xi(r) \langle \text{div}(G + F(u_\infty))(s), \psi \rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} \, ds \, dr
\]

\[
= \int_t^T \langle \text{div}(G + F(u_\infty))(s), \psi \rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} \int_s^T \xi(r) \, dr \, ds
\]

\[
= \int_t^T \int_D (G + F(u_\infty)) \cdot \nabla \psi \xi \, dr. \tag{98}
\]

From (98) it follows that

\[
- \int_A \int_t^T \int_D (\hat{u}_N - b_N) \xi \psi \, dx \, dt \, d\hat{P} - \int_A \int_D (\hat{u}_N - b_N)(t) \psi \, dx \, d\hat{P} + \int_A \int_t^T \int_D (|\nabla v_N|^p - 2 \nabla v_N + F(v_N)) \cdot \nabla \psi \xi \, dx \, dt \, d\hat{P} = 0
\]

\[
= - \int_A \int_t^T \int_D (u_\infty - B_\infty) \xi \psi \, dx \, dr \, d\hat{P} - \int_A \int_D (u_\infty - B_\infty)(t) \psi \, dx \, d\hat{P} + \int_A \int_t^T \int_D (G + F(u_\infty)) \cdot \nabla \psi \xi \, dx \, dr \, d\hat{P}.
\tag{99}
\]

From Lemma 3.14, (68) it follows that there exists a subsequence \( (\hat{u}_N(t)) \) of \( (\hat{u}_N(t)) \) converging weakly some \( \chi(t) \) in \( L^2(\hat{\Omega} \times D) \). With respect to this subsequence we have

\[
\int_A \int_D (\hat{u}_N - b_N)(t) \psi \, dx \, d\hat{P} = I_1 + I_2 \tag{100}
\]
where, for $N \to \infty$,
\[ I_1 = \int_A \int_D (\tilde{u}_{N_i} - \tilde{B}_{N_i})(t) \psi \, dx \, d\tilde{P} \to \int_A \int_D (\chi(t) - B_\infty(t)) \psi \, dx \, d\tilde{P} \]
and, using Corollary 3.19,
\[ |I_2| \leq \int_A \| \tilde{B}_{N_i}(t) - b_{N_i}(t) \|_2 \| \psi \|_2 \, d\tilde{P} \]
\[ \leq \| \psi \|_2 CDE \sup_{s \in [0,T]} \| \tilde{B}_{N_i}(s) - b_{N_i}(s) \|_{H_s(\Omega)} \]
\[ \leq CD\gamma \tau^\gamma (\tilde{K}_1 + 1 + C \text{tr}(Q)) \to 0. \]
Passing to the subsequence $N_i$, we can pass to the limit with $N_i \to \infty$ in (99) and it follows that
\[ \int_D \chi(t) \psi \, dx = \int_D u_\infty(t) \psi \, dx \quad (101) \]
a.s. in $\hat{\Omega}$, thus $\chi(t) = u_\infty(t)$ for all $t \in [0,T]$. In particular, for $t = 0$, we get $u_\infty(0) = u_0$ in $L^2(\Omega)$ and equation (84) holds true. Moreover, for any $t \in [0,T]$ the weak convergence to $\chi(t)$ holds for the whole sequence $(u_N(t))$. With this information, using the weak continuity of $u_\infty$ and $u_N$ we can prove that $\chi(T) = u_\infty(T)$ a.s. in $\hat{\Omega} \times D$ and we have

**Corollary 3.21.** For all $t \in [0,T]$, $u_N(t) \to u_\infty(t)$ in $L^2(\hat{\Omega} \times D)$.

With the proof of the following lemma the proof of Proposition 3.20 is completed:

**Lemma 3.22.** $u_\infty$ is a stochastic process with values in $L^2(\Omega)$.

**Proof:** Since $u_\infty$ is weakly continuous with values in $L^2(\Omega)$ a.s. in $\hat{\Omega}$, it follows that
\[ \hat{\Omega} \ni \tilde{\omega} \mapsto u_\infty(\tilde{\omega})(t) \in L^2(\Omega) \]
for all $t \in [0,T]$. We fix $t \in [0,T]$ and prove that $u_\infty(t)$ is a random variable: By Pettis theorem, $u_\infty(t)$ is measurable, if it is weakly measurable, i.e. the mapping
\[ \hat{\Omega} \ni \tilde{\omega} \mapsto (u(t)(\tilde{\omega}), h)_2 \]
is measurable for all $h \in L^2(\Omega)$. Recall that
\[ B_\infty \in L^2(\hat{\Omega}; \mathcal{C}([0,T]; L^2(\Omega))) \]
and
\[ B_\infty - u_\infty \in L^p(\hat{\Omega}; \mathcal{C}([0,T]; W^{-1,p'}(\Omega))), \]
hence it follows that $u_\infty \in L^p(\hat{\Omega}; \mathcal{C}([0,T]; W^{-1,p'}(\Omega)))$, thus for all $h \in W^{1,p}(\Omega)$
\[ \hat{\Omega} \ni \tilde{\omega} \mapsto (u(t)(\tilde{\omega}), h)_2 = \langle u(t)(\tilde{\omega}), h \rangle_{W^{-1,p'}(\Omega), W^{1,p}(\Omega)} \]
is measurable. Now, the assertion follows since any $h \in L^2(\Omega)$ can be approximated by a sequence $(h_n) \subset W^{1,p}(\Omega) \subset L^2(\Omega)$.

**Proposition 3.23.** $B_\infty$ is a $\mathcal{F}_\infty^t$-martingale with respect to the augmentation $(\mathcal{F}_\infty^t)$ of the filtration $\mathcal{F}_\infty^t := \sigma(B_\infty(s), u_\infty(s))_{0 \leq s \leq t}$, $t \in [0,T]$ (i.e. the smallest complete, right-continuous filtration containing $(\mathcal{F}_\infty^t)$) with quadratic variation process
\[ \ll B_\infty \gg_t = \int_0^t H(u_\infty) \circ H^*(u_\infty) \, ds \quad (102) \]
for all $t \in [0,T]$, where we use the formal notation
\[ H(u) \circ H^*(u) := (H(u) \circ Q^{1/2}) \circ (H(u) \circ Q^{1/2})^*, \quad u \in L^2(\Omega). \]
Lemma 3.15, 4.), we may pass to a not relabeled subsequence of \( N \). By definition, \( B_\infty \) is adapted to \( \hat{F}_t^\infty \). Thus we have to prove that
\[
E((B_\infty(t) - B_\infty(s))\chi_A) = 0
\]
(103)
for all \( A \in \hat{F}_s^\infty \) and all \( 0 \leq s \leq t \). (103) is equivalent to
\[
E[[B_\infty(t) - B_\infty(s), h]_2\psi(B_\infty, u_\infty)] = 0
\]
(104)
for all \( t \in [0, T] \), \( 0 \leq s \leq t \), \( \psi \in C_b([0, s]; L^2(D)) \times L^2(0, s; L^2(D)) \) and all \( h \in L^2(D) \). By Lemma 3.15, 4.), we may pass to a not relabeled subsequence of \( (v_\tau) \), such that \( v_\tau \to u_\infty \) for \( N \to \infty \) in \( L^2(\hat{\Omega}; L^2(Q_T)) \) and a.s. in \( L^2(Q_T) \). We will show that
\[
E[\{B_\infty(t) - B_\infty(s), h\}^2\psi(B_\infty, u_\infty)] = 0.
\]
(105)
for all \( t \in [0, T] \), \( 0 \leq s \leq t \), \( \psi \in C_b([0, s]; L^2(D)) \times L^2(0, s; L^2(D)) \) and all \( h \in L^2(D) \). For any \( N \in \mathbb{N} \), for \( t \in [0, T] \) the process
\[
\hat{B}_N(t) = \int_0^t H(v_\tau) \, dW_N
\]
(106)
is a continuous, square-integrable martingale with respect to \( \hat{F}_t^{W_N} \). Moreover, \( \hat{B}_N \) is \( \hat{F}_t^\ast := \sigma(\hat{B}_N(s), v_\tau(s))_{0 \leq s \leq \tau} \in \hat{F}_t^{W_N} \)-adapted and for all \( t \in [0, T] \), for all \( A \in \hat{F}_t^\ast \) we have
\[
E[(\hat{B}_N(t) - \hat{B}_N(s))\chi_A] = E[E((\hat{B}_N(t) - \hat{B}_N(s))\chi_A|\hat{F}_s^\ast)] = E[\chi_A E((\hat{B}_N(t) - \hat{B}_N(s))|\hat{F}_s^\ast)] = E[\chi_A E((\hat{B}_N(t) - \hat{B}_N(s))|\hat{F}_s^\ast)] = 0.
\]
(107)
Thus \( \hat{B}_N \) is also \( \hat{F}_t^\ast \)-martingale with
\[
\ll \hat{B}_N \gg_t = \int_0^t H(v_\tau) \circ H^\ast(v_\tau) \, ds.
\]
(108)
For any \( N \in \mathbb{N} \), (107) is equivalent to
\[
E[(\hat{B}_N(t) - \hat{B}_N(s), h)\psi(\hat{B}_N, v_\tau)] = 0
\]
(109)
for any \( t \in [0, T] \), \( 0 \leq s \leq t \), \( \psi \in C_b([0, s]; L^2(D)) \times L^2(0, s; L^2(D)), h \in L^2(D) \). We fix \( t \in [0, T] \), \( 0 \leq s \leq t \), \( \psi \in C_b([0, s]; L^2(D)) \times L^2(0, s; L^2(D)) \) and all \( h \in L^2(D) \). Our aim is to pass to the limit with \( N \to \infty \) on the left-hand side of
\[
E[(\hat{B}_N(t) - \hat{B}_N(s), h)\psi(\hat{B}_N, v_\tau)] = 0.
\]
(110)
To this end, we will show that
\[\begin{align*}
i.) \quad \hat{B}_N(t) - \hat{B}_N(s), h) \to (B_\infty(t) - B_\infty(s), h) & \text{ in } L^2(\hat{\Omega}), \\
ii.) \quad \psi(\hat{B}_N, v_\tau) \to \psi(B_\infty, u_\infty) & \text{ in } L^2(\hat{\Omega}),
\end{align*}\]
For all \( t \in [0, T] \), \( 0 \leq s \leq t \), \( \delta_{t-s} : L^2(\hat{\Omega}; C([0, T]; L^2(D))) \to L^2(\hat{\Omega} \times D) \) defined by \( \delta_{t-s}(f) = f(t) - f(s) \) is a continuous, linear mapping. We recall that by Lemma 3.15, 1.), \( \hat{B}_N \to B_\infty \) in \( L^2(\hat{\Omega}; C([0, T]; L^2(D))) \) for \( N \to \infty \), thus
\[
\hat{B}_N(t) - \hat{B}_N(s) = \delta_{t-s}(\hat{B}_N) \to \delta_{t-s}(B_\infty) = B_\infty(t) - B_\infty(s)
\]
(110)
for $N \to \infty$ in $L^2(\hat{\Omega} \times D)$ and we have shown $i.$)

To show $ii.$, we recall that $\psi(\hat{B}_N, v_r) \to \psi(B_\infty, u_\infty)$ a.s. in $\hat{\Omega}$ for $N \to \infty$. With Lebesgue’s dominated convergence theorem it follows that

$$\lim_{N \to \infty} \psi(\hat{B}_N, v_r) \to \psi(B_\infty, u_\infty)$$

in $L^2(\hat{\Omega})$. The convergences in $i.$ and $ii.$ are sufficient to pass to the limit with $N \to \infty$ in (110) and we obtain (105). In particular, (105) implies that $B_\infty$ is a martingale with respect to $(\mathcal{F}_t^\infty)$.

Now let us calculate the quadratic variation process of $B_\infty$: Let $(e_n)$ be an orthonormal basis of $L^2(D)$. To prove (102), we recall that for any $N \in \mathbb{N}$ (108) is equivalent to

$$0 = E[(\hat{B}_N, e_k, e_l)(t) - (\hat{B}_N, e_k, e_l)(s) - A(s, t, v_r, e_k, e_l)]$$

for all $k, j \in \mathbb{N}$, $t, s \in [0, T]$, $\psi \in C_b(C([0, s]; L^2(D))) \times L^2(0, s; L^2(D))$, where

$$(\hat{B}_N, e_k, e_l)(t) := (\hat{B}_N(r), e_k)_2(\hat{B}_N(r), e_l)_2, \quad r \in [0, T]$$

and

$$A(s, t, u, e_k, e_l) := \left( \left[ \int_s^t H(u) \circ H^*(u) \, dr \right] (e_k, e_l) \right)_2$$

for $u \in L^2(D)$. We show that

$$\lim_{N \to \infty} E[(\hat{B}_N, e_k, e_l)(t) - (\hat{B}_N, e_k, e_l)(s) - A(s, t, u_\infty, e_k, e_l)]$$

$$\to E[(B_\infty, e_k, e_l)(t) - (B_\infty, e_k, e_l)(s) - A(s, t, u_\infty, e_k, e_l)]$$

for all $k, j \in \mathbb{N}$, $t, s \in [0, T]$, $\psi \in C_b(C([0, s]; L^2(D))) \times L^2(0, s; L^2(D))$ for a suitable, not relabeled subsequence. To this end, we fix $k, j \in \mathbb{N}$, $t, s \in [0, T]$, $0 \leq s \leq t$, $\psi \in C_b(C([0, s]; L^2(D))) \times L^2(0, s; L^2(D))$ and pass to a not relabeled subsequence of $(v_r)$, such that $v_r \to u_\infty$ for $N \to \infty$ in $L^2(\hat{\Omega}; L^2(Q_T))$ and a.s. in $L^2(Q_T)$. Since $\psi(\hat{B}_N, v_r)$ is uniformly bounded in $L^\infty(\hat{\Omega})$ and $\psi(\hat{B}_N, v_r) \to \psi(B_\infty, u_\infty)$ a.s. in $\hat{\Omega}$,

$$\psi(\hat{B}_N, v_r) \xrightarrow{\ast} \psi(B_\infty, u_\infty)$$

for $N \to \infty$ in $L^\infty(\hat{\Omega})$. We will show

$$(\hat{B}_N(t), e_k)_2(\hat{B}_N(t), e_j)_2 - (\hat{B}_N(s), e_k)_2(\hat{B}_N(s), e_j)_2 - A(s, t, v_r, e_k, e_j)$$

$$\to (B_\infty(t), e_k)_2(B_\infty(t), e_j)_2 - (B_\infty(s), e_k)_2(B_\infty(s), e_j)_2 - A(s, t, u_\infty, e_k, e_j)$$

for $N \to \infty$ in $L^1(\hat{\Omega})$. For any $n \in \mathbb{N}$, the mapping

$$L^2(\hat{\Omega} \times D) \ni u \mapsto (u, e_n)_2 \in L^2(\hat{\Omega})$$

is continuous. Since $\hat{B}_N(r) \to B_\infty(r)$ for $N \to \infty$ in $L^2(\hat{\Omega} \times D)$ for any $r \in [0, T]$, it follows that

$$(\hat{B}_N(r), e_k)_2(\hat{B}_N(r), e_j)_2 \to (B_\infty(r), e_k)_2(B_\infty(r), e_j)_2$$

for $N \to \infty$.
for any \( r \in [0,T] \) and \( N \to \infty \) in \( L^1(\hat{\Omega}) \). Now, the term \( \Lambda(s,t,v_r,e_k,e_j) \) deserves our attention a.s. in \( \hat{\Omega} \):

\[
\Lambda(s,t,v_r,e_k,e_j) = \left( \int_s^t H(v_r) \circ H^*(v_r) \, dr \right)_{(e_k,e_j)}
\]

\[
= \left( \int_s^t H(v_r) \circ H^*(v_r)(e_k) \, dr, e_j \right)_{2}
\]

\[
= \int_s^t (H(v_r) \circ H^*(v_r)(e_k), e_j) \, dr.
\]

(116)

Now from Cauchy inequality it follows that

\[
E|\Lambda(s,t,v_r,e_k,e_j) - \Lambda(s,t,u_\infty,e_k,e_j)|
\]

\[
\leq E \left| \int_s^t [|(H(v_r) \circ H^*(v_r) - H(u_\infty) \circ H^*(u_\infty))(e_k), e_j)|) \, dr \right|
\]

\[
\leq E \left| \int_s^t \|H(v_r) \circ H^*(v_r) - H(u_\infty) \circ H^*(u_\infty)(e_k)|_2 \, dr \right|
\]

\[
\leq CE \left| \int_s^t \|H(v_r) \circ H^*(v_r) - H(u_\infty) \circ H^*(u_\infty)|_{H^2(L^2(D))} \, dr \right|
\]

(117)

for a constant \( C \geq 0 \). Therefore,

\[
E|\Lambda(s,t,v_r,e_k,e_j) - \Lambda(s,t,u_\infty,e_k,e_j)|
\]

\[
\leq CE \left| \int_s^t \|H(v_r) \circ [H^*(v_r) - H^*(u_\infty)]|_{H^2(L^2(D))} \, dr \right|
\]

\[
+ CE \left| \int_s^t \|H(v_r) - H(u_\infty) \circ H^*(u_\infty)|_{H^2(L^2(D))} \, dr \right|
\]

\[
\leq CE \left| \int_s^t \|H(v_r)|_{H^2(L^2(D))}\|H^*(v_r) - H^*(u_\infty)|_{H^2(L^2(D))} \, dr \right|
\]

\[
+ CE \left| \int_s^t \|H(v_r) - H(u_\infty)|_{H^2(L^2(D))}\|H^*(u_\infty)|_{H^2(L^2(D))} \, dr \right|
\]

(118)

Using Hölder inequality, from (117) we get

\[
E|\Lambda(s,t,v_r,e_k,e_j) - \Lambda(s,t,u_\infty,e_k,e_j)|^{1/2}
\]

\[
\leq C \left( E \left| \int_s^t \|H(v_r)|_{H^2(L^2(D))} \, dr \right|^2 \right)^{1/2}
\]

\[
\cdot \left( E \left| \int_s^t \|H^*(v_r) - H^*(u_\infty)|_{H^2(L^2(D))} \, dr \right|^2 \right)^{1/2}
\]

\[
+ C \left( E \left| \int_s^t \|H^*(u_\infty)|_{H^2(L^2(D))} \, dr \right|^2 \right)^{1/2}
\]

\[
\cdot \left( E \left| \int_s^t \|H(v_r) - H(u_\infty)|_{H^2(L^2(D))} \, dr \right|^2 \right)^{1/2}
\]

(119)

By Parseval identity it follows that a.s. in \( \hat{\Omega} \times (0,T) \) we have

\[
\|H^*(v_r) - H^*(u_\infty)|_{H^2(L^2(D))} = \|H(v_r) - H(u_\infty)|_{H^2(L^2(D))}
\]

(120)
and from (120) using (H1) we get

\[
E \int_s^t \|H^*(v_r) - H^*(u_\infty)\|^2_{H \mathcal{S}(L^2(D))} \, dr
\]

\[
= E \int_s^t \|H(v_r) - H(u_\infty)\|^2_{H \mathcal{S}(L^2(D))} \, dr
\]

\[
= E \int_s^t \sum_{n=1}^\infty \|h_n(v_r) - h_n(u_\infty)\|^2_{L^2(D)} \, dr
\]

\[
\leq C_1 E \int_0^\infty \|v_r - u_\infty\|^2_2 \, dr. \quad (121)
\]

Since \(v_r \to u_\infty\) in \(L^2(\Omega \times Q)\) for \(N \to \infty\), from (121) it follows that

\[
H^*(v_r) \to H^*(u_\infty) \quad \text{and} \quad H(v_r) \to H(u_\infty) \quad (122)
\]

in \(L^2(\tilde{\Omega} \times (s,t); H \mathcal{S}(L^2(D)))\) for \(N \to \infty\). In particular, there exists \(C \geq 0\) such that

\[
\|H(v_r)\|^2_{L^2(\tilde{\Omega} \times (s,t); H \mathcal{S}(L^2(D)))} = E \int_s^t \|H(v_r)\|^2_{H \mathcal{S}(L^2(D))} \, dr \leq C \quad (123)
\]

for all \(N \in \mathbb{N}\). Using (119), (122) and (123), it follows that

\[
E|\Lambda(s,t,v_r,e_k,e_j) - \Lambda(s,t,u_\infty,e_k,e_j)| \to 0 \quad (124)
\]

for \(N \to \infty\) and therefore (115) holds true. The convergences in (114) and (115) are enough to conclude (113).

**Proposition 3.24.** There exists a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\), a filtration \((\tilde{\mathcal{F}}_t)\) and a cylindrical Wiener process \(\tilde{W}\) with values in \(L^2(D)\), defined on the probability space

\[
(\tilde{\Omega} := \Omega \times \tilde{\Omega}, \tilde{\mathcal{F}} := \tilde{\mathcal{F}} \times \tilde{\mathcal{F}}, \tilde{P} := \tilde{P} \times \tilde{P})
\]

adapted to \((\mathcal{F}_t) := (\mathcal{F}_t^\infty \times \tilde{\mathcal{F}}_t)\), such that for the extension of \(B_\infty\) to \(\tilde{\Omega}\) defined by

\[
B_\infty(t,\tilde{\omega}) := B_\infty(t,\tilde{\omega}) \quad \text{a.s.} \quad (\tilde{\omega},\tilde{\omega}) \in \tilde{\Omega}
\]

we have the representation

\[
B_\infty(t,\varpi) = \int_0^t H(u_\infty(\omega)) \, d\tilde{W}(s,\varpi)
\]

for all \(t \in [0,T]\) and almost every \(\varpi\) in \(\tilde{\Omega}\).

Proof: According to Proposition 3.23, \(B_\infty\) is a \(\mathcal{F}_t^\infty\)-martingale with quadratic variation process

\[
\ll B_\infty \gg_t = \int_0^t (H(u_\infty) \circ Q^{1/2}) \circ (H^*(u_\infty) \circ Q^{1/2})* \, ds.
\]

Since \(u_\infty\) is a \(\mathcal{F}_t^\infty\)-adapted process with values in \(L^2(D)\) and it is a.s. weakly continuous, for any \(h \in L^2(D)\) the process \((u_\infty, h)_2\) is \(\mathcal{F}_t^\infty\)-adapted with values in \(\mathbb{R}\) and a.s. continuous trajectories. Therefore, \((u_\infty, h)_2\) is a predictable process for all \(h \in L^2(D)\) and by Pettis theorem one gets that \(u_\infty\) is a predictable process with values in \(L^2(D)\). Now, with this measurability and Proposition 3.23 in hand, we can apply the martingale representation theorem of [8], Theorem 8.2, p.220 (see Theorem 5.5 in the Appendix).
Remark 3.4. Without changing notation, we can identify any random variable $X$ in $\hat{\Omega}$ to a random variable in $\Omega$ by setting $X(\hat{\omega}, \hat{\omega}) := X(\hat{\omega})$ a.s. in $\hat{\Omega}$. In particular, all previous estimates and convergences remain true with respect to the probability space $\hat{\Omega}$. Moreover, $u_{\infty} : \Omega \times [0, T] \to L^2(D)$ is a predictable process with a.e. paths

$$u_{\infty}(\omega, \cdot) \in C([0, T]; W^{-1,p'}(D)) \cap L^\infty(0, T; L^2(D)),$$

such that $u_{\infty} \in L^p(\hat{\Omega}; L^p(0, T; W_0^{1,p}(D)))$, $u_{\infty}(0, \cdot) = u_0$ in $L^2(D)$ and

$$u_{\infty}(t) = u_0 + \int_0^t \text{div}(G(s) + F(u_{\infty}(s))) \, ds + \int_0^t H(u_{\infty}) \, d\hat{W} \quad (125)$$
in $L^2(D)$ for all $t \in [0, T]$ a.s. in $\hat{\Omega}$.

Lemma 3.25. $G = |\nabla u_{\infty}|^{p-2} \nabla u_{\infty}$ in $L^p(\hat{\Omega} \times Q_T)^d$

Proof: Taking the scalar product with $v^{k+1}$ in (64), we get

$$(v^{k+1} - v^k, v^{k+1})_2 = (\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k), v^{k+1})_2$$

$$\quad \quad + \quad \tau \int_D (|\nabla v^{k+1}|^{p-2} \nabla v^{k+1} + F(v^{k+1})) \cdot \nabla v^{k+1} \, dx$$

$$\quad \quad \quad = \quad 0.$$  \hspace{1cm} (126)

Using the identity

$$(a - b)a = \frac{1}{2}(|a|^2 - |b|^2 + |a - b|^2), \quad a, b \in \mathbb{R},$$

Gauss-Green theorem on the convection term and taking expectation we get

$$0 = \frac{1}{2} E \left( |v^{k+1}|^2_2 - |v^k|^2_2 + |v^{k+1} - v^k|^2_2 \right) - E(\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k), v^{k+1})_2$$

$$\quad \quad + \quad \tau E \int_D |\nabla v^{k+1}|^{p-2} \nabla v^{k+1} \cdot \nabla v^{k+1} \, dx.$$  \hspace{1cm} (127)

Since $v^k$ is $\mathcal{F}_k^W$-measurable, $E(\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k), v^k)_2 = 0$, thus using

$$-ab = -\frac{1}{2}(|a|^2 + |b|^2) + \frac{1}{2}|a - b|^2, \quad a, b \in \mathbb{R}$$

we can write

$$-E(\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k), v^{k+1})_2 = -E(\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k), v^{k+1} - v^k)_2$$

$$\quad = \frac{1}{2} E\|\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k)\|_2^2 - \frac{1}{2} E\|v^{k+1} - v^k\|_2^2$$

$$\quad + \frac{1}{2} E\|\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k) - (v^{k+1} - v^k)\|_2^2$$  \hspace{1cm} (128)

and therefore

$$0 = \frac{1}{2} E \left( |v^{k+1}|^2_2 - |v^k|^2_2 + |v^{k+1} - v^k|^2_2 - |v^{k+1} - v^k|^2_2 \right)$$

$$\quad = \frac{1}{2} E\|\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k)\|_2^2 + \frac{1}{2} E\|\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k) - (v^{k+1} - v^k)\|_2^2$$

$$\quad + \tau E \int_D |\nabla v^{k+1}|^{p-2} \nabla v^{k+1} \cdot \nabla v^{k+1} \, dx$$

$$\quad \geq \frac{1}{2} E \left( |v^{k+1}|^2_2 - |v^k|^2_2 \right) - \frac{1}{2} E\|\hat{B}_N(t_{k+1}) - \hat{B}_N(t_k)\|_2^2$$

$$\quad + \tau E \int_D |\nabla v^{k+1}|^{p-2} \nabla v^{k+1} \cdot \nabla v^{k+1} \, dx.$$  \hspace{1cm} (129)
Summing over \( k = 0, \ldots, N - 1 \) and using that \( \hat{u}_N(T) = v^N \) a.s. in \( \hat{\Omega} \), from (129) it follows that
\[
\frac{1}{2} \| u_0 \|_2^2 \geq \frac{1}{2} E \| \hat{u}_N(T) \|_2^2 + E \int_0^T \int_D |\nabla v_N|^{p-2} \nabla v_N \cdot \nabla v_N \, dx \, dt
- \frac{1}{2} \sum_{k=0}^{N-1} E \int_{t_k}^{t_{k+1}} H(v_T) \, dW_N \|_2^2
\]
(130)
where, by Itô isometry,
\[
\sum_{k=0}^{N-1} E \int_{t_k}^{t_{k+1}} H(v_T) \, dW_N \|_2^2 = \sum_{k=0}^{N-1} E \int_{t_k}^{t_{k+1}} \| H(v_T) \|_{H^2(L^2(D))}^2 \, dt
= E \int_0^T \| H(v_T) \|_{H^2(L^2(D))}^2 \, dt.
\]
(131)
On the other hand, by Itô formula from (125) it follows that
\[
\frac{1}{2} \| u_\infty(T) \|_2^2 = \frac{1}{2} \| u_0 \|_2^2 - \int_0^T \int_D G \cdot \nabla u_\infty \, dx \, dt
+ \frac{1}{2} \int_0^T \| H(u_\infty) \|_{H^2(L^2(D))}^2 \, dt + \int_0^T (u_\infty, H(u_\infty) \, d\bar{W})_2, \tag{132}
\]
a.s. in \( \bar{\Omega} \), therefore
\[
\frac{1}{2} E \| u_\infty(T) \|_2^2 + E \int_0^T \int_D G \cdot \nabla u_\infty \, dx \, dt - \frac{1}{2} E \int_0^T \| H(u_\infty) \|_{H^2(L^2(D))}^2 \, dt
= \frac{1}{2} \| u_0 \|_2^2
\]
(133)
From (130), (131) and (133) it follows that
\[
\frac{1}{2} E \| u_\infty(T) \|_2^2 + E \int_0^T \int_D G \cdot \nabla u_\infty \, dx \, dt - \frac{1}{2} E \int_0^T \| H(u_\infty) \|_{H^2(L^2(D))}^2 \, dt
\geq E \frac{1}{2} \| \hat{u}_N(T) \|_2^2 + E \int_0^T \int_D |\nabla v_N|^{p-2} \nabla v_N \cdot \nabla v_N \, dx \, dt
- \frac{1}{2} E \int_0^T \| H(v_T) \|_{H^2(L^2(D))}^2 \, dt, \tag{134}
\]
\[\text{hence}
E \int_0^T \int_D G \cdot \nabla u_\infty \, dx \, dt \geq \frac{1}{2} E (\| \hat{u}_N(T) \|_2^2 - \| u_\infty(T) \|_2^2)
- \frac{1}{2} E \int_0^T \| H(v_T) \|_{H^2(L^2(D))}^2 \, dt
- \| H(u_\infty) \|_{H^2(L^2(D))} - \| H(u_\infty) \|_{H^2(L^2(D))}^2 \, dt
+ E \int_0^T \int_D |\nabla v_N|^{p-2} \nabla v_N \cdot \nabla v_N \, dx \, dt. \tag{135}
\]
Since the mapping \( \| \cdot \|_2^2 : L^2(\bar{\Omega} \times D) \to [0, \infty) \) is continuous and convex, it is weakly l.s.c. and from Corollary 3.21 it follows that
\[
0 \leq \liminf_{N \to \infty} E \| \hat{u}_N(T) \|_2^2 - E \| u_\infty(T) \|_2^2.
\]
(136)
Moreover, from (121) in particular it follows that
\[
\lim_{N \to \infty} H(v_r) \to H(u_\infty) \text{ in } L^2(\Omega \times (0,T); HS(L^2(D))),
\]
thus
\[
E \int_0^T \|H(v_r)\|_{HS(L^2(D))}^2 \, dt \to E \int_0^T \|H(u_\infty)\|_{HS(L^2(D))}^2 \, dt
\]
(137)
for \( N \to \infty \). Therefore from (135) and (137) it follows that
\[
E \int_0^T \int_D G \cdot \nabla u_\infty \, dx \, dt \geq 0
\]
\[
E \int_0^T \int_D \sum_{r} \|H(v_r)\|_{HS(L^2(D))}^2 - \|H(u_\infty)\|_{HS(L^2(D))}^2 \, dt
\]
(138)
Since \( p > 2 \), there exists a constant \( C \geq 0 \) not depending on \( N \in \mathbb{N} \) such that
\[
C \limsup_{N \to \infty} E \int_0^T \int_D |\nabla v_N - \nabla u_\infty|^p \, dx \, dt
\]
\[
\leq \limsup_{N \to \infty} E \int_0^T \int_D |\nabla v_N|^p |\nabla v_N| - |\nabla u_\infty|^p |\nabla u_\infty| \, dx \, dt
\]
\[
\leq \limsup_{N \to \infty} E \int_0^T \int_D |\nabla v_N|^{p-2} \nabla v_N \cdot \nabla v_N \, dx \, dt - E \int_0^T \int_D G \cdot \nabla u_\infty \, dx \, dt
\]
\[
\leq 0,
\]
(139)
where the last inequality is a consequence of (138). From (139) it now follows that \( \nabla v_N \to \nabla u_\infty \) in \( L^p(\Omega \times Q_T)^d \) for \( N \to \infty \) and therefore
\[
|\nabla v_N|^{p-2} \nabla v_N \to |\nabla u_\infty|^{p-2} \nabla u_\infty \text{ in } L^p(\Omega \times Q_T)^d.
\]

4 Existence of strong solutions and uniqueness

In this section, we show existence of strong solutions by adapting the argument of [14] in the spirit of [17], [6], [5] (see also [15], [16], [3]), which makes it possible to avoid the application of the martingale representation theorem. To this end we first show uniqueness of two martingale solutions with respect to the same stochastic basis:

**Proposition 4.1.** Assume that \( W \) is a cylindrical Wiener process with values in \( L^2(D) \) on the stochastic basis \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\) and \( u_1, u_2 \) are two martingale solutions to (1) with respect to \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\). Then, \( u_1 = u_2 \).

Proof: For \( \delta > 0 \), let \( \eta_\delta \) be an approximation of the absolute value, i.e.
\[
\eta_\delta(r) = \begin{cases} 
-r & \text{if } r < -2\delta, \\
r^2 & \text{if } -2\delta \leq r \leq 2\delta, \\
\frac{r^2}{2\delta} & \text{if } r > 2\delta.
\end{cases}
\]
Using the Itô formula, it follows that
\[
I_1 = I_2 + I_3 + I_4 + I_5,
\]
(140)
for all \( t \in [0, T] \) a.s. in \( \Omega \), where

\[
I_1 = \int_D \eta_\delta(u_1 - u_2)(t) \, dx,
\]
\[
I_2 = \int_0^t \int_D (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2) \cdot \nabla (u_1 - u_2) \eta_\delta''(u_1 - u_2) \, dx \, ds,
\]
\[
I_3 = \int_0^t \int_D (F(u_1) - F(u_2)) \cdot \nabla (u_1 - u_2) \eta_\delta''(u_1 - u_2) \, dx \, ds,
\]
\[
I_4 = \int_0^t \int_D (\eta_\delta''(u_1 - u_2), H(u_1) - H(u_2)) \, dW(t),
\]
\[
I_5 = \frac{1}{2} \int_0^t \eta_\delta''(u_1 - u_2) \|H(u_1) - H(u_2)\|^2_{H^2(D)} \, ds.
\]

(141)

For all \( t \in [0, T] \), a.s. in \( \Omega \), \( I_2 \geq 0 \). Moreover, \( E[I_4] = 0 \) for all \( t \in [0, T] \). Therefore, from (141) it follows that

\[
E[I_1] \leq E[I_3] + E[I_5].
\]

(142)

Since, for any \( t \in [0, T] \), \( \eta_\delta(u_1 - u_2) \) converges to \( |(u_1 - u_2)(t)| \) for \( \delta \to 0^+ \) a.e. in \( \Omega \times D \), and \( |\eta_\delta(u_1 - u_2)(t)| \leq |(u_1 - u_2)(t)| \) for all \( \delta > 0 \) a.s. in \( \Omega \times D \), it follows that

\[
\lim_{\delta \to 0^+} E[I_1] = E \int_D |(u_1 - u_2)(t)| \, dx
\]

(143)

for any \( t \in [0, T] \). For any \( \delta > 0 \) we have

\[
\eta_\delta''(u_1 - u_2) = \frac{1}{2\delta} \chi_{\{|u_1 - u_2| \leq 2\delta\}}
\]

a.s. on \( \Omega \times Q_T \), thus for \( L \geq 0 \) being the Lipschitz constant of \( F \) we have

\[
|E[I_3]| \leq \frac{1}{2\delta} E \int_{\{|u_1 - u_2| \leq 2\delta\}} |F(u_1) - F(u_2) \cdot \nabla (u_1 - u_2)| \, dx \, ds
\]
\[
\leq \frac{L}{2\delta} E \int_{\{|u_1 - u_2| \leq 2\delta\}} |u_1 - u_2| \|\nabla (u_1 - u_2)| \, dx \, ds
\]
\[
\leq LE \int_{\{|u_1 - u_2| \leq 2\delta\}} |\nabla (u_1 - u_2)| \, dx \, ds.
\]

(144)

Similarly, by \((H1)\)

\[
|E[I_5]| \leq \frac{1}{2\delta} E \int_{\{|u_1 - u_2| \leq 2\delta\}} \sum_{n=1}^\infty |h_n(u_1) - h_n(u_2)|^2 \, dx \, ds
\]
\[
\leq \frac{C_1}{2\delta} E \int_{\{|u_1 - u_2| \leq 2\delta\}} |u_1 - u_2|^2 \, dx \, ds
\]
\[
\leq 2\delta C
\]

(145)

where \( C \geq 0 \) is a constant not depending on \( \delta > 0 \). Thus from (144) it follows that

\[
\lim_{\delta \to 0^+} E[I_3] = E \int_{\{|u_1 = u_2\}} |\nabla (u_1 - u_2)| \, dx \, ds = 0
\]

(146)

and from (145) it follows that

\[
\lim_{\delta \to 0^+} E[I_5] = 0.
\]

(147)
In particular, Proposition 4.1 implies that whenever a strong solution to \((P)\) exists, it is unique. Moreover, if \(\mu^{1,2}\) is the joint law of \((u_1, u_2)\) on \(L^2(0, T; L^2(D))^2\), then Proposition 4.1 implies that

\[
\mu^{1,2}(\{(\xi, \zeta) \in L^2(0, T; L^2(D))^2 \mid \xi = \zeta\}) = \int_{\Omega \times \Omega} \chi_{\{u_1 = u_2\}} \, dP \otimes dP = 1
\]

and we can use the following Lemma (see, e.g. [14] and also [15]) to get existence of strong solutions:

**Lemma 4.2.** Let \(V\) be a Polish space equipped with the Borel \(\sigma\)-algebra. A sequence of \(V\)-valued random variables \((X_n)\) converges in probability if and only if for every pair of subsequences \(X_n\) and \(X_m\) there exists a joint subsequence \((X_{n_k}, X_{m_k})\) which converges for \(k \to \infty\) in law to a probability measure \(\mu\) such that

\[
\mu(\{(w, z) \in V \times V \mid w = z\}) = 1.
\]

Let \((\tilde{u}_M, B_M, W)\) and \((\tilde{u}_L, B_L, W)\) be a pair of subsequences of \((\tilde{u}_N, B_N, W)\). Since

\[
(\tilde{u}_M, \tilde{u}_L, B_M, B_L, W)
\]

is tight on

\[
(L^2(0, T; L^2(D))^2 \times C([0, T]; L^2(D))^2 \times C([0, T]; U),
\]

it is relatively compact, thus we can extract a joint subsequence

\[
\mu^1 := (\tilde{u}_M, j, B_M, j, B_L, j, W)
\]

which converges in law to some probability measure \(\mu\). Applying the theorem of Skorokhod to \((\tilde{u}_M, \tilde{u}_L, B_M, B_L, W)\) we find a probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})\), a sequence of measurable functions

\[
\Phi_j : (\hat{\Omega}, \hat{\mathcal{F}}) \to (\Omega, \mathcal{F}), \quad j \in \mathbb{N}
\]

such that \(P = \hat{P} \circ \Phi_j\) for all \(j \in \mathbb{N}\) and measurable functions \(u_1^j, u_2^j, B_1^j, B_2^j, W^j\) having the following properties:

i.) \(\hat{u}_M := \tilde{u}_M \circ \phi_j \to u_1^j\) in \(L^2(0, T; L^2(D))\) for \(j \to \infty\) a.s. in \(\hat{\Omega}\),

ii.) \(\hat{u}_L := \tilde{u}_L \circ \phi_j \to u_2^j\) in \(L^2(0, T; L^2(D))\) for \(j \to \infty\) a.s. in \(\hat{\Omega}\),

iii.) \(\hat{B}_M := B_M \circ \phi_j \to B_1^j\) in \(C([0, T]; L^2(D))\) for \(j \to \infty\) a.s. in \(\hat{\Omega}\),

iv.) \(\hat{B}_L := B_L \circ \phi_j \to B_2^j\) in \(C([0, T]; L^2(D))\) for \(j \to \infty\) a.s. in \(\hat{\Omega}\),

v.) \(W_j := W \circ \phi_j \to W^j\) in \(C([0, T]; U)\) for \(j \to \infty\) a.s. in \(\hat{\Omega}\).

vi.) \(L(u_1^j, u_2^j, B_1^j, B_2^j, W^j) = \mu\).

**Definition 4.1.** Let us denote the augmentation of the filtration

\[
\sigma(u_M(j)(s), u_L(j)(s), W(j)(s))_{0 \leq s \leq t}, \quad t \in [0, T]
\]

by \((\mathcal{F}^j_t)\) and the augmentation of the filtration

\[
\sigma(u_1^j(s), u_2^j(s), W^j(s))_{0 \leq s \leq t}, \quad t \in [0, T]
\]

by \((\mathcal{F}^W, j)\).
As in the previous section, we can now recover the structure of the equation on the new probability space \((\hat{\Omega}, \hat{F}, \hat{P})\) using the functions \(\phi_j\): If we define \(\tau_{L_j} := \frac{T}{T_j}, \tau_{M_j} := \frac{T}{M_j},\)

\[
v_i^{k} = u^k \circ \phi_j, \quad k = 0, \ldots, M_j, \\
v_i^{k} = u^k \circ \phi_j, \quad k = 0, \ldots, L_j,
\]

then \(\hat{u}_{M_j}(T) = v^{M_j}, \hat{u}_{L_j}(T) = v^{L_j}\) and

\[
\hat{u}_{M_j}(t) = \frac{v^{k+1}_M - v^k_M}{\tau}(t - t_k) + v^k_M
\]

for \(t \in [t_k, t_{k+1})\) and \(k = 0, \ldots, M_j,\)

\[
\hat{u}_{L_j}(t) = \frac{v^{k+1}_L - v^k_L}{\tau}(t - t_k) + v^k_L
\]

for \(t \in [t_k, t_{k+1})\) and \(k = 0, \ldots, L_j.\) Moreover, we introduce the left-continuous, \(\mathcal{F}_t\)-adapted functions

\[
v_{\tau_{M_j}}(t) := \sum_{k=0}^{M_j-1} v^k x(t_k, t_{k+1})(t), \quad t \in (0, T], \quad v_{\tau_{M_j}}(0) = u_0,
\]

\[
v_{\tau_{L_j}}(t) := \sum_{k=0}^{L_j-1} v^k x(t_k, t_{k+1})(t), \quad t \in (0, T], \quad v_{\tau_{L_j}}(0) = u_0
\]

and find that

\[
\hat{B}_{M_j}(t) = \int_0^t H(v_{\tau_{M_j}}) \, dW_j, \quad t \in [0, T] \\
\hat{B}_{L_j}(t) = \int_0^t H(v_{\tau_{L_j}}) \, dW_j, \quad t \in [0, T].
\]

Now we repeat the arguments of Lemma 3.13-3.18, Corollary 3.19, Proposition 3.20, Corollary 3.21 and Lemma 3.22 to obtain that

\[
\lim_{j \to \infty} v_{\tau_{M_j}} = u_1^i, \quad \lim_{j \to \infty} v_{\tau_{L_j}} = u_2^i
\]

in \(L^2(\hat{\Omega} \times Q_T)\) and, for \(i = 1, 2,\) the function \(u_i^1 : \hat{\Omega} \times [0, T] \to L^2(D)\) is a stochastic process with \(u_i^1(0) = u_0\) and there exists \(G^i \in L^p(\hat{\Omega} \times Q_T)^2\) such that

\[
u_i^1(t) = B_i^1(t) + u_0 + \int_0^t \text{div}(G^i + F(u_i^1)) \, ds
\]  

(148)

holds in \(L^2(D)\) a.s. in \(\hat{\Omega}\) for all \(t \in [0, T].\)

**Lemma 4.3.** We have the following convergence results for \(j \to \infty: \)

i.) \(\hat{B}_{M_j} \to B_1^\infty\) and \(\hat{B}_{L_j} \to B_2^\infty\) in \(L^2(\hat{\Omega}; \mathcal{C}([0, T]; L^2(D)))\)

ii.) \(\hat{B}_{M_j}(t) - \hat{B}_{M_j}(s) \to B_1^\infty(t) - B_1^\infty(s)\) and \(\hat{B}_{L_j}(t) - \hat{B}_{L_j}(s) \to B_2^\infty(t) - B_2^\infty(s)\) in \(L^2(\hat{\Omega} \times D)\)

for all \(t \in [0, T], 0 \leq s \leq t\)

iii.) \(W_j \to W_\infty\) in \(L^2(\hat{\Omega}; \mathcal{C}([0, T]; U))\)

iv.) \(W_j(t) - W_j(s) \to W_\infty(t) - W_\infty(s)\) in \(L^2(\hat{\Omega}; U)\) for all \(t \in [0, T], 0 \leq s \leq t\)
Moreover, passing to a (not relabeled) subsequence if necessary,

\[
\lim_{j \to \infty} \psi(v_{\tau_{M_j}}, v_{\tau_{L_j}}, W_j) = \psi(u_\infty^1, u_\infty^2, W_\infty) \tag{149}
\]

in \(L^2(\hat{\Omega})\) for all \(t \in [0, T]\), \(0 \leq s \leq t\) and all \(\psi \in \mathcal{C}_b(L^2(0, s; L^2(D))^2 \times \mathcal{C}([0, s]; U))\).

Proof: Since the sequences \((\hat{B}_{M_j})\) and \((\hat{B}_{L_j})\) respectively converge a.s. in \(\mathcal{C}([0, T]; L^2(D))\) and are uniformly bounded in \(L^p(\hat{\Omega}; \mathcal{C}([0, T]; L^2(D)))\) (with the same arguments as used in the proof of Lemma 3.15), ii.) follows from i.) since

\[
\hat{B}_{M_j}(t) - \hat{B}_{M_j}(s) = \delta_{t-s}(\hat{B}_{M_j}) \quad \text{and} \quad \hat{B}_{L_j}(t) - \hat{B}_{L_j}(s) = \delta_{t-s}(\hat{B}_{L_j})
\]

for all \(t \in [0, T]\), \(0 \leq s \leq t\). By equality of law and Burkholder inequality we have

\[
E \sup_{t \in [0, T]} \|W_j(t)\|_U^p = E \sup_{t \in [0, T]} \|W(t)\|_U^p \leq CE \left( T \sum_{n=1}^\infty \frac{1}{n^2} \right)^{p/2}, \tag{150}
\]

where \(C \geq 0\) is the constant from Burkholder inequality which is not depending on \(j \in \mathbb{N}\). Together with the a.s. convergence of \(W_j\) to \(W_\infty\) for \(j \to \infty\), it follows from Vitali theorem that \(W_j \to W_\infty\) in \(L^2(\hat{\Omega}; \mathcal{C}([0, T]; U))\) for \(j \to \infty\), thus in particular

\[
W_j(t) - W_j(s) \to \delta_{t-s}(W_\infty) = W_\infty(t) - W_\infty(s)
\]

for \(j \to \infty\), all \(t \in [0, T]\) and all \(0 \leq t \leq s\) in \(L^2(\hat{\Omega}; U)\). We fix \(t \in [0, T]\), \(0 \leq s \leq t\) and \(\psi \in \mathcal{C}_b(L^2(0, s; L^2(D))^2 \times \mathcal{C}([0, s]; U))\). Since \((v_{\tau_{M_j}})\) converges to \(u_\infty^1\) and \((v_{\tau_{L_j}})\) converges to \(u_\infty^2\) in \(L^2(\hat{\Omega} \times QT)\) for \(j \to \infty\), we may pass to a (not relabeled) subsequence which converges in \(L^2(0, T; L^2(D))\) for a.e. \(\omega \in \hat{\Omega}\). Then,

\[
\psi(v_{\tau_{M_j}}, v_{\tau_{L_j}}, W_j) \to \psi(u_\infty^1, u_\infty^2, W_\infty) \tag{151}
\]

a.s. in \(\hat{\Omega}\) for \(j \to \infty\). Thus, with Lebesgue’s dominated convergence theorem we get the convergence of (149) in \(L^2(\hat{\Omega})\) for \(j \to \infty\).

**Lemma 4.4.** For \(i = 1, 2\), \(B^i_\infty\) is a \(F^{W_\infty}_t\)-martingale with quadratic variation process

\[
\ll B^i_\infty \gg_t = \int_0^t H(u_\infty^i) \circ H(u_\infty^i)^* \, ds \tag{152}
\]

for all \(t \in [0, T]\), where we use the formal notation

\[
H(u) \circ H^*(u) := (H(u) \circ Q^{1/2}) \circ (H(u) \circ Q^{1/2})^*, \quad u \in L^2(D).
\]

Proof: Let \((e_i)\) be an orthonormal basis of \(L^2(D)\). We choose (not relabeled) subsequences of \((v_{\tau_{M_j}})\) and \((v_{\tau_{L_j}})\) respectively that converge a.s. in \(\hat{\Omega}\) and fix \(t \in [0, T]\), \(0 \leq s \leq t\) and \(\psi \in \mathcal{C}_b(L^2(0, s; L^2(D))^2 \times \mathcal{C}([0, s]; U))\), \(n, m \in \mathbb{N}\). Moreover, for \(u \in L^2(D)\), and \(B(r) \in L^2(D), r \in [0, T]\) we define

\[
(B, e_n, e_m)(r) := (B(r), e_n)_2(B(r), e_m)_2,
\]

\[
\Lambda(s, t, u, e_n, e_m) := \left[ \int_s^t H(u) \circ H^*(u) \, dr \right] (e_n, e_m)_2
\]

With the convergence results of Lemma 4.3 we are able to show

\[
0 = \lim_{j \to \infty} E[ (\hat{B}_{M_j}(t) - \hat{B}_{M_j}(s), e_n)_2 \psi(v_{\tau_{M_j}}, v_{\tau_{L_j}}, W_j) ] \quad = \quad E[ (B^1_\infty(t) - B^1_\infty(s), e_n)_2 \psi(u_\infty^1, u_\infty^2, W_\infty) ], \tag{153}
\]

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\[
0 = \lim_{j \to \infty} E[(\hat{B}_{L_j}(t) - \hat{B}_{L_j}(s), e_n)_{\tau_{M_j}, v_{T_{L_j}}, W_j}]
= E[(B^2_{\infty}(t) - B^2_{\infty}(s), e_n)_{\tau_{M_j}, v_{T_{L_j}}, W_\infty}],
\]

(154)

\[
0 = E[((\hat{B}_{M_j}, e_n, e_m)(t) - (\hat{B}_{M_j}, e_n, e_m)(s) - \Lambda(s, t, v_{\tau_{M_j}}, e_n, e_m))
\psi(v_{\tau_{M_j}, v_{T_{L_j}}, W_j})]
\to E[((B^1_{\infty}, e_n, e_m)(t) - (B^1_{\infty}, e_n, e_m)(s) - \Lambda(s, t, v_{\tau_{M_j}}, e_n, e_m))\psi(u_{1\infty}^1, u_{2\infty}^1, W_\infty)]
\]

(155)

for \(j \to \infty\) and

\[
0 = E[((\hat{B}_{L_j}, e_n, e_m)(t) - (\hat{B}_{L_j}, e_n, e_m)(s) - \Lambda(s, t, v_{T_{L_j}}, e_n, e_m))
\psi(v_{\tau_{M_j}, v_{T_{L_j}}, W_j})]
\to E[((B^2_{\infty}, e_n, e_m)(t) - (B^2_{\infty}, e_n, e_m)(s) - \Lambda(s, t, u_{\infty}^1, e_n, e_m))\psi(u_{1\infty}^1, u_{2\infty}^1, W_\infty)].
\]

(156)

for \(j \to \infty\).

**Lemma 4.5.** \(W_\infty\) is a \((F_t^{W,\infty})\)-martingale.

Proof: By definition of \((F_t^{W,\infty})\), \(W_\infty\) is adapted to \((F_t^{W,\infty})\). We choose (not relabeled) subsequences of \((v_{\tau_{M_j}})\) and \((v_{T_{L_j}})\) respectively that converge a.s. in \(\Omega\) and fix \(t \in [0, T]\), \(0 \leq s \leq t\) and \(\psi \in \mathcal{C}(L^2(0, s; L^2(D)) \times \mathcal{C}([0, s]; U))\) and \(h \in U\). Since \(v_{\tau_{M_j}}\) and \(v_{T_{L_j}}\) are \(F_t^{W_j}\)-adapted for all \(j \in \mathbb{N}\), we have

\[
E[(W_j(t) - W_j(s), h)_U \psi(v_{\tau_{M_j}, v_{T_{L_j}}, W_j})] = 0
\]

(157)

for all \(j \in \mathbb{N}\). Using the convergence results of Lemma 4.3, we may pass to the limit with \(j \to \infty\) in (157) and find that

\[
E[(W_\infty(t) - W_\infty(s), h)_U \psi(u_{1\infty}^1, u_{2\infty}^1, W_\infty)] = 0.
\]

(158)

**Lemma 4.6.** \(W_\infty\) is a \(Q\)-Wiener process in \(U\), adapted to \((F_t^{W,\infty})\), with increments \(W(t) - W(s), 0 \leq s \leq t \leq T\), independent of \(F_s^{W,\infty}\).

Proof: Since we have already know that \(W_\infty\) is a \((F_t^{W,\infty})\)-martingale with \(W_\infty(0) = 0\), according to [8], Theorem 4.4, p. 89 it is left to show that

\[
\ll W_\infty \gg_t = tQ \text{ for all } t \in [0, T].
\]

(159)

Recall that \(\ll W_j \gg_t = tQ\) for all \(t \in [0, T]\) and all \(j \in \mathbb{N}\). Let \((g_t)\) be an orthonormal basis of \(U\). We choose (not relabeled) subsequences of \((v_{\tau_{M_j}})\) and \((v_{T_{L_j}})\) respectively that converge a.s. in \(\Omega\) and fix \(t \in [0, T]\), \(0 \leq s \leq t\) and \(\psi \in \mathcal{C}(L^2(0, s; L^2(D)) \times \mathcal{C}([0, s]; U))\), \(n, m \in \mathbb{N}\). Using the convergence results of Lemma 4.3 we show that

\[
0 = E[((W_j, g_n, g_m)(t) - (W_j, g_n, g_m)(s) - ((t - s)Q(g_n), g_m)_U)
\psi(v_{\tau_{M_j}, v_{T_{L_j}}, W_j})]
\to E[((W_\infty, g_n, g_m)(t) - (W_\infty, g_n, g_m)(s) - ((t - s)Q(g_n), g_m)_U)
\psi(u_{1\infty}^1, u_{2\infty}^1, W_\infty)]
\]

(160)

for \(j \to \infty\), where

\[(W, g_n, g_m)(r) := (W(r), g_n)_U(W(r), g_m)_U\]

for \(W(r) \in U, r \in [0, T]\), thus (159) holds true.
Corollary 4.7. For $i = 1, 2$, the processes

$$M_i(t) := \int_0^t H(u^i_\infty) \, dW_\infty, \; t \in [0,T]$$

are $\mathcal{F}^W_t$-martingales with quadratic variation process

$$\ll M_i \gg_t = \int_0^t (H(u^i_\infty) \circ Q^{1/2}) \circ (H(u^i_\infty) \circ Q^{1/2})^* \, ds$$

for all $t \in [0,T]$.

Lemma 4.8. For $i = 1, 2$ we have the cross quadratic variation process

$$\ll B^i_\infty, W_\infty \gg_t = \int_0^t H(u^i_\infty) \circ Q \, ds. \quad (161)$$

Proof: For all $j \in \mathbb{N}$ and $t \in [0,T]$ we have

$$\ll \hat{B}_{M_j}, W_j \gg_t = \ll \int_0^t H(v_{\tau_{M_j}}) \, dW_j, W_j \gg_t = \int_0^t H(v_{\tau_{M_j}}) \, ds \ll W_j \gg_t$$

$$= \int_0^t H(v_{\tau_{M_j}}) \circ Q \, ds, \quad (162)$$

and

$$\ll \hat{B}_{L_j}, W_j \gg_t = \ll \int_0^t H(v_{\tau_{L_j}}) \, dW_j, W_j \gg_t = \int_0^t H(v_{\tau_{L_j}}) \, ds \ll W_j \gg_t$$

$$= \int_0^t H(v_{\tau_{L_j}}) \circ Q \, ds. \quad (163)$$

We choose orthonormal bases $(g_l)$ of $U$ and $(e_l)$ of $L^2(D)$, (not relabeled) subsequences of $(v_{\tau_{M_j}})$ and $(v_{\tau_{L_j}})$ respectively that converge a.s. in $\Omega$ and fix $t \in [0,T]$, $0 \leq s \leq t$ and $\psi \in C_b(L^2(0,s;L^2(D))^2 \times C([0,s];U))$ and $n,m \in \mathbb{N}$. Using the convergence results of Lemma 4.3 we show that

$$0 = E[((\hat{B}_{M_j}, W_j, e_n, g_m)(t) - (\hat{B}_{M_j}, W_j, e_n, g_m)(s)$$

$$- \int_s^t (H(v_{\tau_{M_j}}) \circ Q(g_m), e_n) \, ds \psi(v_{\tau_{M_j}}, v_{\tau_{L_j}}, W_j)]$$

$$\to E[((B^1_\infty, W_\infty, e_n, g_m)(t) - (B^1_\infty, W_\infty, e_n, g_m)(s)$$

$$- \int_s^t (H(u^1_\infty) \circ Q(g_m), e_n) \, ds \psi(u^1_\infty, u^2_\infty, W_\infty)] \quad (164)$$

and

$$0 = E[((\hat{B}_{L_j}, W_j, e_n, g_m)(t) - (\hat{B}_{L_j}, W_j, e_n, g_m)(s)$$

$$- \int_s^t (H(v_{\tau_{L_j}}) \circ Q(g_m), e_n, u) \, ds \psi(v_{\tau_{M_j}}, v_{\tau_{L_j}}, W_j)]$$

$$\to E[((B^2_\infty, W_\infty, e_n, g_m)(t) - (B^2_\infty, W_\infty, e_n, g_m)(s)$$

$$- \int_s^t (H(u^2_\infty) \circ Q(g_m), e_n) \, ds \psi(u^1_\infty, u^2_\infty, W_\infty)] \quad (165)$$

for $j \to \infty$, where

$$(B, W, e_n, g_m)(r) := (B(r), e_n) \circ (W(r), g_m) \circ r$$

for $r \in [0,T]$ and $W(r) \in U$, $B(r) \in L^2(D)$. 37
Lemma 4.9. For $i = 1, 2$ and all $t \in [0, T]$ we have

$$
\langle B_i^\infty - \int_0^t H(u_i^\infty) dW_\infty \rangle_t = 0,
$$

(166)

Proof: For $i = 1, 2$ from (152), Corollary 4.7 and Lemma 4.9 it follows that

$$
\langle B_i^\infty - \int_0^t H(u_i^\infty) dW_\infty \rangle_t
= \langle B_i^\infty \rangle_t - 2 \langle B_i^\infty, \int_0^t H(u_i^\infty) dW_\infty \rangle_t + \langle \int_0^t H(u_i^\infty) dW_\infty \rangle_t
= 2 \int_0^t (H(u_i^\infty) \circ Q^{1/2}) \circ (H(u_i^\infty) \circ Q^{1/2})^* ds - 2 \int_0^t H(u_i^\infty) \ d \langle B_i^\infty, W_\infty \rangle_t
$$

(167)

where, according to Lemma 4.8

$$
\int_0^t H(u_i^\infty) \ d \langle B_i^\infty, W_\infty \rangle_t = \int_0^t H(u_i^\infty) \circ H(u_i^\infty) \circ Q ds
= \int_0^t H(u_i^\infty) \circ (H(u_i^\infty) \circ Q^{1/2}) \circ Q^{1/2} ds
= \int_0^t H(u_i^\infty) \circ (H(u_i^\infty) \circ Q^{1/2}) \circ (Q^{1/2})^* ds
= \int_0^t (H(u_i^\infty) \circ Q^{1/2}) \circ (H(u_i^\infty) \circ Q^{1/2})^* ds
$$

(168)

Now, (166) follows from (167) and (168).

Corollary 4.10. We have $u_1^\infty = u_2^\infty$ and therefore convergence of $(\tilde{u}_N)$ in probability on the initial probability space $(\Omega, F, P)$, hence existence and uniqueness of strong solutions to (1).

Proof: From Lemma 4.9 it follows that for $i = 1, 2$

$$
B_i^\infty(t) = \int_0^t H(u_i^\infty) \ dW, \ t \in [0, T]
$$

thus $u_1^\infty, u_2^\infty$ are martingale solutions to (1) with respect to $(\hat{\Omega}, \hat{F}, (F^{W, \infty})_t, W_\infty)$. Now, from Proposition 4.1 it follows that $u_1^\infty = u_2^\infty$ and thus $B_1^\infty = B_2^\infty$ and, by Lemma 4.2 and equality of laws this implies convergence in probability of the initial approximating sequence $(\tilde{u}_N)$ in probability on the initial probability space $(\Omega, F, P)$ to a strong solution of (1), which is, again by Proposition 4.1, unique.

5 Appendix

5.1 On Prokhorov compactness theorem

Definition 5.1 (see [4], p.59). Let $\Pi$ be a family of probability measures on the metric space $V$ with the Borel $\sigma$-algebra $B(V)$. The family $\Pi$ is tight iff, for every $\epsilon > 0$, there exists a compact set $K_{\epsilon}$ such that

$$
P(K_{\epsilon}) > 1 - \epsilon
$$

for every $P \in \Pi$. 

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Tightness can be used as a compactness criterion in the narrow topology, this is the direct half of Prokhorov theorem:

**Theorem 5.1.** [see [4], Theorem 5.1., p.59] If \( \Pi \) is tight, then it is relatively compact with respect to the narrow topology \( \sigma(C_b(V)',C_b(V)) \), i.e. for any subsequence \( (P_n) \subset \Pi \) there exists a subsequence \( (P_{n_k}) \) and a probability measure \( \mu \) such that

\[
\lim_{k \to \infty} \int_V f \, dP_{n_k} = \int_V f \, d\mu
\]

for all \( f \in C_b(V) \).

We have the following subsequence principle:

**Corollary 5.2.** If the sequence of probability measures \( (P_n)_{n \in \mathbb{N}} \) is tight, and if each subsequence that converges narrowly at all in fact converges narrowly to \( \mu \), then the entire sequence converges narrowly to \( \mu \).

If, in addition, \( V \) is a Polish space, then the converse part of Prokhorov theorem also holds true:

**Theorem 5.3.** [see [4], Theorem 5.2, p.60] Suppose that \( V \) is separable and complete. If \( \Pi \) is relatively compact with respect to the narrow topology \( \sigma(C_b(V)',C_b(V)) \), then it is tight.

### 5.2 On Skorokhod representation theorem

**Definition 5.2** ([25] p.17). For \( n \in \mathbb{N} \), let \( X_n : (\Omega,F,P) \to (V,B(V)) \) be a random variable with values in a metric space \( V \). We say that \( (X_n) \) converges to a Borel measure \( \mu \) in law, (or distribution), and write \( X_n \overset{L}{\to} \mu \), iff

\[
Ef(X_n) \to \int_V f \, d\mu
\]

for any bounded, continuous function \( f \) on \( V \).

**Remark 5.1.** Note that \( X_n \overset{L}{\to} \mu \) is equivalent to \( P \circ X_n^{-1} \overset{\text{a.s.}}{\to} \mu \) with respect to the narrow topology on the bounded Borel measures where \( P \circ X_n^{-1} \) is the image measure of \( X_n \) for all \( n \in \mathbb{N} \).

**Theorem 5.4** (see [25], Theorem 1.10.4, p.59). Let \( (\Omega,F,P) \) be a probability space, \( V \) a separable metric space and \( X_n : \Omega \to V \) be a sequence of random variables such that \( X_n \overset{L}{\to} X_\infty \). Then there exists a sequence of random variables \( \hat{X}_n : \bar{\Omega} \to V, n \in \mathbb{N} \cup \{\infty\} \), on some probability space \( (\bar{\Omega},\bar{F},\bar{P}) \) with the following properties:

i.) \( \hat{X}_n \to \hat{X}_\infty \) in \( V \) for \( N \to \infty \) a.s. in \( \bar{\Omega} \)

ii.) The laws of \( X_n \) and \( \hat{X}_n \) are the same for all \( n \in \mathbb{N} \cup \{\infty\} \). In particular, for any bounded measurable function \( f : V \to \mathbb{R} \), \( Ef(X_n) = Ef(\hat{X}_n) \) for all \( n \in \mathbb{N} \).

**Remark 5.2.** According to [25], Addendum 1.10.5, p.59, there exist random variables \( \phi_n : \bar{\Omega} \to \Omega \) such that \( \hat{X}_n = X_n \circ \phi_n \) and \( P = P \circ \phi_n^{-1} \).

---

3i.e. a measure on the Borel sets, finite on the compact ones.
5.3 Martingale representation theorem

Theorem 5.5 (see [8], Theorem 8.2, p.220). Assume $\mathcal{U}, \mathcal{H}$ are separable Hilbert spaces, $M$ is a square-integrable martingale with

$$\ll M \gg_t = \int_0^t (\Phi \circ Q^{1/2}) \circ (\Phi \circ Q^{1/2})^* \, ds, \ t \in [0,T],$$

where $\mathcal{U}_0 = Q^{1/2}(\mathcal{U})$, $\Phi$ is a predictable, $HS(\mathcal{U}_0, \mathcal{H})$-valued process and $Q$ a given, bounded, symmetric nonnegative operator in $\mathcal{U}$. Then there exists a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, a filtration $(\mathcal{F}_t)$ and a $Q$-Wiener process $\bar{W}$ with values in $\mathcal{U}$, defined on $(\Omega \times \bar{\Omega}, \mathcal{F} \times \mathcal{F}_t, \mathbb{P} \times \mathcal{F}_t)$ adapted to $(\mathcal{F}_t \times \mathcal{F}_t)$, such that

$$M(t, \omega, \varpi) = \int_0^t \Phi(s, \omega, \varpi) \, d\bar{W}(s, \omega, \varpi), \ t \in [0,T],$$

for a.e. $(\omega, \varpi) \in (\Omega \times \bar{\Omega})$ where

$$M(t, \omega, \varpi) = M(t, \omega), \ \Phi(t, \omega, \varpi) = \Phi(t, \omega)$$

for all $t \in [0,T]$, a.s. in $\Omega \times \bar{\Omega}$.

5.4 Technical lemmas

5.4.1 On the Garsia-Rodemich-Rumsey inequality

Lemma 5.6. (Garsia-Rodemich-Rumsey inequality, see [13], 23, Ex. 2.4.1) Let $q \geq 1$, $\alpha > 1/q$ and $f : [a, b] \to V$ be continuous, then

$$\|f(s) - f(s')\|_V^q \leq C_{\alpha,q}|s - s'|^{\alpha q - 1} \int_a^b \frac{|f(t) - f(r)|^q}{|t - r|^{\alpha q + 1}} \, dt \, dr.$$

5.4.2 $W^{\alpha,p}$-regularity

Lemma 5.7 ([11], Lemma 2.1, p.369). Let $\mathcal{K}, \mathcal{H}$ be separable Hilbert spaces and $W$ be a cylindrical Wiener process in $\mathcal{K}$. Assume $p \geq 2$, $\alpha \in (0, \frac{1}{2})$. Then, for any progressively measurable process $f \in L^p(\Omega \times [0,T); HS(\mathcal{K}; \mathcal{H}))$ we have

$$\int_0^T f \, dW \in L^p(\Omega; W^{\alpha,p}(0,T; \mathcal{H}))$$

and there exists a constant $C(p, \alpha) > 0$ such that

$$E \|\int_0^T f \, dW\|_{W^{\alpha,p}(0,T; \mathcal{H})}^p \leq C(\alpha, p) E \|f(t)\|^p_{HS(\mathcal{K}; \mathcal{H})} \, dt.$$

Lemma 5.8 ([2], Lemma 3.2). Let $V$ be a Banach space. Assume that $\tau > 0$ and that $I_\tau = \{t_k\}_{k=0}^N$ is an equidistant mesh of size $\tau > 0$ covering $[0,T]$. Assume that $G \in C([0,T]; V)$ is such that, for every $k \in \{0, \ldots, N - 1\}$ the function

$$[t_k, t_{k+1}] \ni t \mapsto G(t)$$

is affine. Assume that, for some $p \geq 1$, $\alpha > 0$ and $C > 0$ and every $l \in \{1, \ldots, N\}$,

$$\tau \sum_{k=0}^{N-1} \|G(t_{k+l}) - G(t_k)\|_V^p \leq C p^{al}.$$

Then, $G$ is uniformly bounded in the Nikolskii space $N^{\alpha,p}(0,T; V)$ and there exists a constant $C = C(T) > 0$, which does not depend on $\tau > 0$ such that

$$\|G\|_{N^{\alpha,p}(0,T; V)} = \sup_{s > 0} s^{-\alpha} \|G(\cdot + s) - G(\cdot)\|_{L^p(-s,T-s; V)} \leq C.$$
5.4.3 Further results

Lemma 5.9. Let $W$ be a Banach space which is compactly embedded into $L^2(0,T;L^2(D))$ and $p \geq 2$. For $\alpha \in (0, \frac{1}{2})$, the linear space

$$V := \{u = v + w, \ v \in W, \ w \in W^\alpha,p(0,T;H^1_0(D))\} \subset L^2(0,T;L^2(D))$$

endowed with the norm

$$\|u\|_V := \inf_{v \in W, \ w \in W^\alpha,p(0,T;L^2(D)), \ u = v + w} \max(\|v\|_W, \|w\|_{W^\alpha,p})$$

is a Banach space which is compactly embedded into $L^2(0,T;L^2(D))$.

Proof: It follows from [12], Remark 5.13, p.12-13 that $(V, \|\cdot\|_V)$ is a Banach space. There exists $C \geq 0$ such that for any $u \in V$ and any $v \in W$, $w \in W^\alpha,p(0,T;H^1_0(D))$ with $u = v + w$

$$\|u\|_{L^2(0,T;L^2(D))} \leq C \max(\|v\|_W, \|w\|_{W^\alpha,p}) \quad (171)$$

and therefore the imbedding $V \rightarrow L^2(0,T;L^2(D))$ is continuous. Let $(u_n)$ be a bounded sequence in $V$, i.e. there exists $R > 0$ such that $\|u_n\|_V \leq R$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be fixed. According to the definition of the norm in $V$, for any $k \in \mathbb{N}$, there exist $v^n_k \in W$, $w^n_k \in W^\alpha,p(0,T;H^1_0(D))$ such that $u_n = v^n_k + w^n_k$ and

$$\|v^n_k\|_W \leq R + \frac{1}{k}, \ |w^n_k|_{W^\alpha,p} \leq R + \frac{1}{k}.$$ 

Consequently, choosing $k = n$ we can construct $(v^n_n) \subset W$, $(w^n_n) \subset W^\alpha,p(0,T;H^1_0(D))$ such that $u_n = v^n_n + w^n_n$ and

$$\|v^n_n\|_W \leq R + 1, \ |w^n_n|_{W^\alpha,p} \leq R + 1$$

for all $n \in \mathbb{N}$. Passing to a not relabeled subsequence if necessary, there exists $v \in L^2(0,T;L^2(D))$ such that $v^n_n \rightarrow v$ in $L^2(0,T;L^2(D))$. Following [21], Corollary 2, p.82,

$$W^\alpha,p(0,T;H^1_0(D)) \hookrightarrow L^2(0,T;L^2(D))$$

with compact imbedding. Therefore passing to a not relabeled subsequence if necessary, there exists $w \in L^2(0,T;L^2(D))$ such that $w^n_n \rightarrow w$ in $L^2(0,T;L^2(D))$. Therefore, passing to a not relabeled subsequence if necessary,

$$u_n = v^n_n + w^n_n \rightarrow v + w$$

in $L^2(0,T;L^2(D))$ and therefore the imbedding $V \rightarrow L^2(0,T;L^2(D))$ is compact.

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