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Projectability of stable, partially free $\mathcal{H}$-surfaces in the non-perpendicular case

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Abstract

A projectability result is proved for surfaces of prescribed mean curvature (shortly called $\mathcal{H}$-surfaces) spanned in a partially free boundary configuration. Hereby, the $\mathcal{H}$-surface is allowed to meet the support surface along its free trace non-perpendicularly. The main result generalizes known theorems due to Hildebrandt-Sauvigny and the author himself and is in the spirit of the well known projectability theorems due to Radó and Kneser. A uniqueness and an existence result are included as corollaries.

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1 Introduction

Let us write $B^+ := \{w = (u, v) = u + iv \, : \, |w| < 1, \; v > 0\}$ for the upper unit half disc in the plane. Its boundary is divided into

$$\partial B^+ = I \cup J, \quad I := (-1, 1), \quad J := \partial B^+ \setminus I = \{w \in \overline{B^+} \, : \, |w| = 1\}.$$

In the present paper, a *surface of prescribed mean curvature* $\mathcal{H} = \mathcal{H}(\mathbf{p}) \in C^0(\mathbb{R}^3, \mathbb{R})$ or, shortly, an *$\mathcal{H}$-surface* is a mapping $\mathbf{x} = \mathbf{x}(w) : B^+ \to \mathbb{R}^3 \in C^2(B^+, \mathbb{R}^3)$, which solves the system

$$\begin{aligned} &\Delta \mathbf{x} = 2\mathcal{H}(\mathbf{x})\mathbf{x}_u \wedge \mathbf{x}_v \quad \text{in } B^+, \\ &|\mathbf{x}_u| = |\mathbf{x}_v|, \quad \mathbf{x}_u \cdot \mathbf{x}_v = 0 \quad \text{in } B^+. \end{aligned} \tag{1.1}$$

Here, $\mathbf{y} \wedge \mathbf{z}$ and $\mathbf{y} \cdot \mathbf{z}$ denote the cross product and the standard scalar product in $\mathbb{R}^3$, respectively.

Observe that an $\mathcal{H}$-surface is not supposed to be a regular surface, that means, it may possess *branch points* $w_0 \in B^+$ with $\mathbf{x}_u \wedge \mathbf{x}_v(w_0) = \mathbf{0}$.

We consider $\mathcal{H}$-surfaces spanned in a projectable, partially free boundary configuration, which means the following:

Definition 1. (Projectable boundary configuration)
Let $S = \Sigma \times \mathbb{R} \subset \mathbb{R}^3$ be an embedded cylinder surface over the planar closed Jordan arc $\Sigma = \pi(S)$ of class C^3; here π denotes the orthogonal projection onto

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the x^1, x^2-plane. Furthermore, let $\Gamma \subset \mathbb{R}^3$ be a closed Jordan arc which can be represented as a C^3-graph over the planar closed C^3-Jordan arc $\underline{\Gamma} = \pi(\Gamma)$. Finally, assume $\underline{\Gamma} \cap \Sigma = \{\pi_1, \pi_2\}$, where π_1, π_2 are the distinct end points of $\underline{\Gamma}$ as well as Σ, and Γ and S meet with a positive angle at the respective points $\mathbf{p}_1, \mathbf{p}_2 \in \Gamma \cap S$ correlated by $\pi_j = \pi(\mathbf{p}_j)$, $j = 1, 2$. Then we call $\{\Gamma, S\}$ a projectable (partially free) boundary configuration.

To be precise, in Definition 1, the phrase "Γ and S meet with a positive angle at the respective points $\mathbf{p}_1, \mathbf{p}_2 \in \Gamma \cap S$" means that the tangentential vector of Γ is not an element of the tangential plane of S at these points.

A *partially free $\mathcal{H}$-surface* is a solution $\mathbf{x} \in C^2(B^+, \mathbb{R}^3) \cap C^0(\overline{B^+}, \mathbb{R}^3)$ of (1.1), which satisfies the boundary conditions

$$\begin{aligned} &\mathbf{x}(w) \in S \quad \text{for all } w \in I, \\ &\mathbf{x}|_J : J \to \Gamma \text{ strictly monotonic}, \\ &\mathbf{x}(-1) = \mathbf{p}_1,\ \mathbf{x}(+1) = \mathbf{p}_2 \end{aligned} \tag{1.2}$$

for a given projectable boundary configuration $\{\Gamma, S\}$. Roughly speaking, we aim to show that any such partially free $\mathcal{H}$-surface is itself projectable. This is in the spirit of the famous projectability result for minimal surfaces by Radó and Kneser and will be proved under additional assumptions on the $\mathcal{H}$-surface and the configuration $\{\Gamma, S\}$, namely: The boundary configuration shall be *R-admissible* in the sense of Definition 2 below and the $\mathcal{H}$-surface shall be Hölder-continuous on $\overline{B^+}$, stationary w.r.t. some energy functional $E_{\mathbf{Q}}$ and stable w.r.t. the corresponding generalized area functional $A_{\mathbf{Q}}$. Here $\mathbf{Q}$ is a given vector field which satisfies a natural smallness condition and which possesses a suitable normal component w.r.t. S as well as the divergence $\operatorname{div} \mathbf{Q} = 2\mathcal{H}$; see Section 2 for details.

The first results of this type were given by Hildebrandt-Sauvigny [HS1]-[HS3]. They considered the special case of minimal surfaces; a generalization to F-minimal surfaces can be found in [MW]. Concerning partially free $\mathcal{H}$-surfaces the only projectability result known to the author was proved in [M3]. There, the above mentioned vector field $\mathbf{Q}$ was supposed to be tangential along the *support surface* S, which forces the corresponding stationary $\mathcal{H}$-surface to meet S perpendicularly along its *free trace* $\mathbf{x}|_I$. This condition was essential at many points of the proof in [M3], in particular, while deriving the second variation formula for $A_{\mathbf{Q}}$ and establishing a boundary condition for the third component of the surface normal of our $\mathcal{H}$-surface. One motivation for writing the present paper was to drop this restriction and to study $\mathcal{H}$-surfaces which meet S non-perpendicularly.

Methodically, we orientate on [M3] which in turn is based on the work of Hildebrandt and Sauvigny in [HS3] and on Sauvigny's paper [S1], where a corresponding projectability result for stable $\mathcal{H}$-surfaces subject to Plateau type boundary conditions has been proven.

The paper is organized as follows: In Section 2 we fix notations, specify our assumptions and state the main projectability result, Theorem 1, as well as some preliminary results on the $\mathcal{H}$-surface and its normal. The consequential unique solvability of the studied partially free problem is captured in Corollary 1. In Section 3 we derive the second variation formula for the functional $A_{\mathbf{Q}}$ allowing boundary perturbations on the free trace $\mathbf{x}|_I$. Then, Section 4 contains the crucial boundary condition for the third component of the surface normal and the proof of Theorem 1. We close with an exemplary application of Theorem 1 to the existence question for a mixed boundary value problem for the non-parametric $\mathcal{H}$-surface equation, Corollary 2.

2 Notations and main result

We start by specifying our additional assumptions on the boundary configuration: Let $\{\Gamma, S\}$ be a projectable boundary configuration in the sense of Definition 1. Let $\sigma = \sigma(s)$, $s \in [0, s_0]$, parametrize $\Sigma = \pi(S)$ by arc length, that is,

$$\sigma \in C^3([0, s_0], \mathbb{R}^2), \quad |\sigma'| \equiv 1 \text{ on } [0, s_0], \quad \text{and} \quad s_0 = \text{length}(\Sigma) > 0.$$

Setting $\mathbf{e}_3 := (0, 0, 1)$ we define C^2-unit tangent and normal vector fields $\mathbf{t}, \mathbf{n}$ on S as follows:

$$\mathbf{t}(\mathbf{p}) := (\sigma'(s), 0), \quad \mathbf{n}(\mathbf{p}) := \mathbf{t}(\mathbf{p}) \wedge \mathbf{e}_3 \quad \text{for } \mathbf{p} \in \{\sigma(s)\} \wedge \mathbb{R},\ s \in [0, s_0]. \qquad (2.1)$$

Furthermore, we can write $\Gamma = \{(x^1, x^2, \gamma(x^1, x^2)) \in \mathbb{R}^3 \ :\ (x^1, x^2) \in \underline{\Gamma}\}$, where $\underline{\Gamma} = \pi(\Gamma)$ is a closed C^3-Jordan arc and $\gamma \in C^3(\underline{\Gamma})$ is the *height function*. For the end points $\mathbf{p}_1, \mathbf{p}_2$ of Γ we assume to have representations

$$\mathbf{p}_1 = \big(\sigma(0), \gamma(\sigma(0))\big), \quad \mathbf{p}_2 = \big(\sigma(s_0), \gamma(\sigma(s_0))\big).$$

The set $\underline{\Gamma} \cup \Sigma$ bounds a simply connected domain $G \subset \mathbb{R}^2$, that is, $\partial G = \underline{\Gamma} \cup \Sigma$, and we have $\underline{\Gamma} \cap \Sigma = \{\pi_1, \pi_2\}$ with $\pi_j = \pi(\mathbf{p}_j)$, $j = 1, 2$. With $\alpha_j \in (0, \pi)$ we denote the interior angle between $\underline{\Gamma}$ and Σ at π_j w.r.t. G ($j = 1, 2$). Finally, we assume that Σ is parametrized such that $\nu := \pi(\mathbf{n})$ points to the exterior of G along Σ.

Definition 2. *A projectable boundary configuration* $\{\Gamma, S\}$ *is called* R-admissible*, if the following hold:*

(i) $\Gamma \cup S \subset Z := \{(p^1, p^2, p^3) \in \mathbb{R}^3 \ :\ |(p^1, p^2)| < R\}$ *for some* $R > 0$.

(ii) G *is* $\frac{1}{R}$–convex*, i.e., for any point* $\xi \in \partial G$ *there is an open disc* $D_\xi \subset \mathbb{R}^2$ *of radius* R *such that* $G \subset D_\xi$ *and* $\xi \in \partial D_\xi$.

For a given R-admissible boundary configuration $\{\Gamma, S\}$, we define the class $\mathcal{C}(\Gamma, S; \overline{Z})$ of mappings $\mathbf{x} \in H_2^1(B^+, \overline{Z})$, which satisfy the boundary conditions (1.2) weakly, i.e.,

$$\begin{aligned} &\mathbf{x}(w) \in S \quad \text{for a.a. } w \in I, \\ &\mathbf{x}|_J : J \to \Gamma \text{ continuously and weakly monotonic}, \\ &\mathbf{x}(-1) = \mathbf{p}_1,\ \mathbf{x}(+1) = \mathbf{p}_2. \end{aligned} \qquad (2.2)$$

For arbitrary $\mu \in [0, 1)$, we additionally define its subsets

$$\mathcal{C}_\mu(\Gamma, S; \overline{Z}) := \left\{ \mathbf{x} \in \mathcal{C}(\Gamma, S; \overline{Z}) \ :\ \begin{array}{l} \mathbf{x} \in C^\mu(\overline{B^+}, \overline{Z}), \\ \mathbf{x}|_J : J \to \Gamma \text{ strictly monotonic} \end{array} \right\}. \qquad (2.3)$$

Now let $\mathbf{Q} = \mathbf{Q}(\mathbf{p}) \in C^1(\overline{Z}, \mathbb{R}^3)$ be a vector field satisfying

$$\begin{aligned} &\sup_{p \in \overline{Z}} |\mathbf{Q}(\mathbf{p})| < 1, \\ &\operatorname{div} \mathbf{Q}(\mathbf{p}) = 2\mathcal{H}(\mathbf{p}) \quad \text{for all } \mathbf{p} \in \overline{Z}. \end{aligned} \tag{2.4}$$

Here the function $\mathcal{H} = \mathcal{H}(\mathbf{p})$ belongs to $C^{1,\alpha}(\overline{Z})$ for some $\alpha \in (0,1)$ and fulfills

$$\sup_{\mathbf{p} \in \overline{Z}} |\mathcal{H}(\mathbf{p})| \le \frac{1}{2R}. \tag{2.5}$$

We introduce the functional

$$E_{\mathbf{Q}}(\mathbf{x}) := \iint\limits_{B^+} \Big\{ \frac{1}{2} |\nabla \mathbf{x}(w)|^2 + \mathbf{Q}(\mathbf{x}) \cdot \mathbf{x}_u \wedge \mathbf{x}_v(w) \Big\} \, du\, dv, \quad \mathbf{x} \in H_2^1(B^+, \overline{Z}), \tag{2.6}$$

and consider the variational problem

$$E_{\mathbf{Q}}(\mathbf{x}) \to \min, \quad \mathbf{x} \in \mathcal{C}(\Gamma, S; \overline{Z}). \tag{2.7}$$

The following lemma collects some well known results concerning the existence and regularity of solutions of (2.7) as well as stationary points of $E_{\mathbf{Q}}$.

Lemma 1. (Heinz, Hildebrandt, Tomi)
Let $\{\Gamma, S\}$ be an R-admissible boundary configuration $\{\Gamma, S\}$ and assume $\mathbf{Q} \in C^1(\overline{Z}, \mathbb{R}^3)$, $H \in C^{1,\alpha}(\overline{Z})$ to satisfy (2.4) and (2.5). Then there exists a solution $\mathbf{x} = \mathbf{x}(w)$ of (2.7). $\mathbf{x}$ belongs to the class $\mathcal{C}_\mu(\Gamma, S; \overline{Z}) \cap C^{3,\alpha}(B^+, Z)$ for some $\mu \in (0,1)$ and satisfies the system (1.1), i.e., $\mathbf{x}$ is a partially free $\mathcal{H}$-surface.

More generally, any stationary point $\mathbf{x} \in \mathcal{C}_0(\Gamma, S; \overline{Z})$ of $E_{\mathbf{Q}}$ solves (1.1) and belongs to the class $C^{3,\alpha}(B^+, Z)$. Here, stationarity means

$$\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \big\{ E_{\mathbf{Q}}(\mathbf{x}_\varepsilon) - E_{\mathbf{Q}}(\mathbf{x}) \big\} \ge 0$$

for all inner and outer variations $\mathbf{x}_\varepsilon \in \mathcal{C}_0(\Gamma, S; \overline{Z})$, $\varepsilon \in [0, \varepsilon_0)$ with sufficiently small $\varepsilon_0 > 0$; see Definition 2 in [DHT] Section 5.4 for the definition of inner and outer variations.

We also associate the *generalized area functional* to $\mathbf{Q}$:

$$A_{\mathbf{Q}}(\mathbf{x}) := \iint\limits_{B^+} \Big\{ |\mathbf{x}_u \wedge \mathbf{x}_v| + \mathbf{Q}(\mathbf{x}) \cdot \mathbf{x}_u \wedge \mathbf{x}_v(w) \Big\} \, du\, dv, \quad \mathbf{x} \in H_2^1(B^+, \overline{Z}). \tag{2.8}$$

A stationary, partially free $\mathcal{H}$-surface $\mathbf{x} \in \mathcal{C}_0(\Gamma, S; \overline{Z})$ is called *stable*, if it is stable w.r.t. $A_{\mathbf{Q}}$, that means, the second variation $\frac{d^2}{d\varepsilon^2} A_{\mathbf{Q}}\big(\tilde{\mathbf{x}}(\cdot, \varepsilon)\big)|_{\varepsilon=0}$ of $A_{\mathbf{Q}}$ is nonnegative for all outer variations $\tilde{\mathbf{x}}(\cdot, \varepsilon) \in \mathcal{C}_0(\Gamma, S; \overline{Z})$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, for which this quantity exists; note that $\mathbf{x}$ has its image $\mathbf{x}(\overline{B^+})$ in Z, according to Lemma 1. Since the first variation of $A_{\mathbf{Q}}$ w.r.t. such variations $\tilde{\mathbf{x}}$ vanishes for stationary $\mathbf{x}$, any relative minimizer of $A_{\mathbf{Q}}$ in $\mathcal{C}_0(\Gamma, S; \overline{Z})$ is stable. In Definition 4 below, we give an exact definition of stability, which is used in the present paper and which is somewhat less stringent than the above mentioned requirement.

We are now in a position to state our main result:

Theorem 1. *Let $\{\Gamma, S\}$ be an admissible boundary configuration and let $\mathbf{Q} \in C^{1,\alpha}(\overline{Z}, \mathbb{R}^3)$ be chosen such that (2.4) is fullfilled with some $\mathcal{H} \in C^{1,\alpha}(\overline{Z})$, $\alpha \in (0,1)$, satisfying (2.5). In addition, we assume*

$$\frac{\partial}{\partial p^3}\mathcal{H}(\mathbf{p}) \geq 0 \quad \text{for all } \mathbf{p} \in \overline{Z} \tag{2.9}$$

as well as

$$\begin{aligned} &(\mathbf{Q}\cdot\mathbf{n})(\mathbf{p}) = (\mathbf{Q}\cdot\mathbf{n})(p^1, p^2, 0) \quad \text{for all } \mathbf{p} = (p^1, p^2, p^3) \in S, \\ &|(\mathbf{Q}\cdot\mathbf{n})(\mathbf{p}_j)| < \cos\alpha_j, \quad j = 1, 2. \end{aligned} \tag{2.10}$$

Then any stable $\mathcal{H}$-surface $\mathbf{x} \in \mathcal{C}_\mu(\Gamma, S; \overline{Z})$, $\mu \in (0,1)$, possesses a graph representation over $\overline{G}$. More precisely, $\mathbf{x}$ is immersed and can be represented as the graph of some function $\zeta : \overline{G} \to \mathbb{R} \in C^{3,\alpha}(G) \cap C^{2,\alpha}(\overline{G} \setminus \{\pi_1, \pi_2\}) \cap C^0(\overline{G})$, which satisfies the mixed boundary value problem

$$\mathit{div}\Big(\frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}}\Big) = 2\mathcal{H}(\cdot, \zeta) \quad \text{in } G, \tag{2.11}$$

$$\frac{\nabla\zeta\cdot\nu}{\sqrt{1+|\nabla\zeta|^2}} = \psi \quad \text{on } \Sigma \setminus \{\pi_1, \pi_2\}, \qquad \zeta = \gamma \quad \text{on } \underline{\Gamma}. \tag{2.12}$$

Here $\nu = \pi(\mathbf{n})$ denotes the exterior unit normal on Σ w.r.t. G and we defined $\psi := \mathbf{Q}\cdot\mathbf{n}|_\Sigma \in C^1(\Sigma)$.

As a consequence of Theorem 1 we obtain the following

Corollary 1. *Let the assumptions of Theorem 1 be satisfied. Then, apart from reparametrization, there exists exactly one stable $\mathcal{H}$-surface $\mathbf{x} \in \mathcal{C}_\mu(\Gamma, S; \overline{Z})$ with some $\mu \in (0,1)$.*

Proof. The existence of a stable $\mathcal{H}$-surface $\mathbf{x} \in \mathcal{C}_\mu(\Gamma, S; \overline{Z})$ for some $\mu \in (0,1)$ is assured by Lemma 1. Acccording to Theorem 1, we can represent $\mathbf{x}$ as a graph over G, and the height function ζ solves the boundary value problem (2.11), (2.12).

If there would exist another stable $\mathcal{H}$-surface $\tilde{\mathbf{x}} \in \mathcal{C}_{\tilde{\mu}}(\Gamma, S; \overline{Z})$ with some $\tilde{\mu} \in (0,1)$ and if $\tilde{\zeta}$ denotes the height function of its graph representation, which also solves (2.11), (2.12) by Theorem 1, we consider the difference function $f := \zeta_1 - \zeta_2$. As is well known, f solves an elliptic differential equation in G, which is subject to the maximum principle according to assumption (2.9); cf. [S2] Chap. VI, § 2. Consequently, f assumes its maximum and minimum on $\partial G = \Sigma \cup \underline{\Gamma}$.

Assume that f has a positive maximum at $p_0 \in \Sigma \setminus \{\pi_1, \pi_2\}$. Then Hopf's boundary point lemma implies

$$\nabla f(p_0) = (\nabla f(p_0)\cdot\nu(p_0))\nu(p_0) \quad \text{with} \quad \nabla f(p_0)\cdot\nu(p_0) > 0.$$

On the other hand, the first boundary condition in (2.12) yields $(M(p_0)\nabla f(p_0))\cdot \nu(p_0) = 0$, where we have abbreviated

$$M(p) := \int_0^1 Dh\big(t\nabla\zeta_1(p) + (1-t)\nabla\zeta_2(p)\big)\, dt, \quad p \in \Sigma,$$

with $h(z) := \frac{z}{\sqrt{1+|z|^2}}$, $z \in \mathbb{R}^2$. If we finally note

$$\big(Dh(z)\xi\big) \cdot \xi = \frac{|\xi|^2(1+|z|^2) - (\xi \cdot z)^2}{(1+|z|^2)^{\frac{3}{2}}} > 0, \quad \xi \in \mathbb{R}^2 \setminus \{0\}, \quad z \in \mathbb{R}^2,$$

we deduce that M is positive definite on Σ and arrive at the contradiction

$$0 = (M(p_0)\nabla f(p_0)) \cdot \nu(p_0) = (\nabla f(p_0) \cdot \nu(p_0))(M(p_0)\nu(p_0)) \cdot \nu(p_0) > 0.$$

Hence, we conclude $f \le 0$ on $\overline{G}$ and, similarly, one proves $f \ge 0$ on $\overline{G}$. This gives $\zeta \equiv \tilde{\zeta}$ on $\overline{G}$, which yields $\mathbf{x} = \tilde{\mathbf{x}} \circ \omega$ with some positively oriented parameter transformation $\omega : \overline{B^+} \to \overline{B^+}$. This proves the corollary. □

We complete this section with a preparatory lemma, which collects some analytical and geometrical regularity results and first important informations towards the projectability of our $\mathcal{H}$-surfaces:

Lemma 2. *Let the assumptions of Theorem 1 be satisfied and let* $\mathbf{x} = \mathbf{x}(w) \in \mathcal{C}_\mu(\Gamma, S; \overline{Z})$ *be an* $\mathcal{H}$*-surface which is stationary w.r.t.* $E_{\mathbf{Q}}$*. Then there follow:*

(i) $\mathbf{x} \in C^{3,\alpha}(B^+, Z) \cap C^{2,\alpha}(\overline{B^+} \setminus \{-1,+1\}, Z)$*, and there holds*

$$(\mathbf{x}_v + \mathbf{Q}(\mathbf{x}) \wedge \mathbf{x}_u)(w) \perp T_{\mathbf{x}(w)}S \quad \textit{for all } w \in I, \tag{2.13}$$

where $T_{\mathbf{p}}S$ *denotes the tangential plane of* S *at the point* $\mathbf{p} \in S$*.*

(ii) $f(\overline{B^+}) \subset \overline{G}$ *for the projection mapping* $f := \pi(\mathbf{x})$*.*

(iii) $\nabla \mathbf{x}(w) \ne \mathbf{0}$ *for all* $w \in \partial B^+ \setminus \{-1,+1\}$*, and* $\nabla \mathbf{x} = \mathbf{0}$ *for at most finitely many points in* B^+*.*

(iv) Set $W := |\mathbf{x}_u \wedge \mathbf{x}_v|$*,* $B' := \{w \in B^+ : W(w) > 0\}$*, and define the surface normal* $\mathbf{N}(w) := W^{-1}\mathbf{x}_u \wedge \mathbf{x}_v(w)$ *as well the Gaussian curvature* $K = K(w)$ *of* $\mathbf{x}$ *for points* $w \in B'$*. Then* $\mathbf{N}$ *and* KW *can be extended to mappings*

$$\mathbf{N} \in C^{2,\alpha}(B^+, \mathbb{R}^3) \cap C^{1,\alpha}(\overline{B^+} \setminus \{-1,+1\}, \mathbb{R}^3) \cap C^0(\overline{B^+}, \mathbb{R}^3),$$
$$KW \in C^{1,\alpha}(B^+),$$

and $\mathbf{N}$ *satisfies the differential equation*

$$\Delta \mathbf{N} + 2\big(2\mathcal{H}(\mathbf{x})^2 - K - (\nabla\mathcal{H}(\mathbf{x}) \cdot \mathbf{N})\big) W \mathbf{N} = -2W\nabla\mathcal{H}(\mathbf{x}) \quad \textit{in } B^+. \tag{2.14}$$

Proof. (ii) Due to Lemma 1, $\mathbf{x}$ is a stationary, partially free $\mathcal{H}$-surface of class $C^{3,\alpha}(B^+, Z)$. In addition, we have $f(\partial B^+) = \partial G$ due to the geometry of our boundary configuration. An inspection of the proof of Hilfssatz 4 of [S1] shows, that this boundary condition, the smallness condition (2.5) and the $\frac{1}{R}$–convexity of G imply $f(\overline{B^+}) \subset \overline{G}$.

(i), (iii) A well known regularity result according to E. Heinz [He] implies $\mathbf{x} \in C^{2,\alpha}(B^+ \cup J, Z)$. And from Theorem 1 in [M6] we obtain $\mathbf{x} \in C^{1,\frac{1}{2}}(B^+ \cup I, Z)$. Setting

$$I' := \big\{ w \in I : f(w) = (\pi \circ \mathbf{x})(w) \notin \{\pi_1, \pi_2\} \big\},$$

the stationarity yields the natural boundary condition (2.13) on I'.

Due to (ii), the arguments from Satz 2 in [S1] yield $\nabla\mathbf{x}(w) \neq \mathbf{0}$ for all $w \in J$. Assume that $w_0 \in I$ is a branch point of $\mathbf{x}$ and set $B_\delta^+(w_0) := \{w \in B^+ : |w - w_0| < \delta\}$. Then the asymptotic expansion from Theorem 2 in [M6] imply that $\mathbf{x}|_{B_\delta^+(w_0)}$, $0 < \delta \ll 1$, looks like a whole perturbed disc. Consequently, the projection $f|_{B_\delta^+(w_0)}$ would meet the complement of $\overline{G}$, in contrast to $f(\overline{B}) \subset \overline{G}$. Indeed, for $w_0 \in I'$ this effects from the natural boundary condition (2.13), which can be rewritten as $(\mathbf{Q}\cdot\mathbf{n})(\mathbf{x}) = -\mathbf{N}\cdot\mathbf{n}(\mathbf{x})$ on I'; see Remark 1 below. And for $w_0 \in I \setminus I'$, i.e. $f(w_0) \in \{\pi_1, \pi_2\}$, this is trivial by geometry. Consequently, we have a contradiction and $\nabla\mathbf{x}(w) \neq \mathbf{0}$ for $w \in I$ follows; this completes the proof of the first part of (iii).

Next we show $I' = I$, i.e. $f(I) = \Sigma \setminus \{\pi_1, \pi_2\}$. From [HJ] or [M5] we then obtain $\mathbf{x} \in C^{2,\alpha}(B^+ \cup I, Z)$ and (2.13) holds on I; this will complete the proof of (i).

Assume there exists $w^* \in I$ with $f(w^*) = \pi_1$. Then there would be a maximal point $w_0 \in I$ with $f(w_0) = \pi_1$ and $f(w) \in \Sigma \setminus \{\pi_1, \pi_2\}$ for $w \in (w_0, w_0 + \varepsilon) \subset I$, $0 < \varepsilon \ll 1$. Consequently, the boundary condition (2.13) holds on $(w_0, w_0 + \varepsilon)$ and, in particular, we get

$$(\mathbf{x}_v + \mathbf{Q}(\mathbf{x}) \wedge \mathbf{x}_u) \cdot \mathbf{t}(\mathbf{x}) = 0 \quad \text{on } (w_0, w_0 + \varepsilon). \tag{2.15}$$

By continuity, (2.15) remains valid for $w = w_0$. In addition, the geometry of S yields $\mathbf{x}_u = \pm|\mathbf{x}_u|\mathbf{e}_3$. This and the relation $\mathbf{n} = \mathbf{t} \wedge \mathbf{e}_3$ on S imply

$$\mathbf{x}_v \cdot \mathbf{t}(\mathbf{x}) = \pm|\mathbf{x}_u|\, \mathbf{Q}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \quad \text{in } w_0. \tag{2.16}$$

According to the conformality relations and $\nabla\mathbf{x} \neq \mathbf{0}$ on I, we have $|\mathbf{x}_u| = |\mathbf{x}_v| \neq 0$ in w_0. Denote the angle between $\mathbf{x}_v(w_0)$ and $\mathbf{t}(\mathbf{x}(w_0))$ by β_1. Then (2.16) and condition (2.10) imply

$$|\cos\beta_1| = |\mathbf{Q}(\mathbf{x}(w_0)) \cdot \mathbf{n}(\mathbf{x}(w_0))| < \cos\alpha_1 \quad \text{or} \quad \beta_1 \in (\alpha_1, \pi - \alpha_1),$$

where $\alpha_1 \in (0, \frac{\pi}{2})$ denotes the interior angle between $\underline{\Gamma}$ and Σ at π_1 w.r.t. G. A simple application of the mean-value theorem then yields a contradiction to the inclusion $f(\overline{B^+}) \subset \overline{G}$. Analogously, one shows that there cannot exist $w^{**} \in I$ with $f(w^{**}) = \pi_2$. In conclusion, we have $I' = I$ and (i) is proved.

We finally show the finiteness of branch points in B^+, completing the proof of (iii): Hildebrandt's asymptotic expansions at interior branch points [Hi] imply the isolated character of these points. By $\nabla\mathbf{x} \neq \mathbf{0}$ on $I \cup J$, the only points where branch points could accumulate are the corner points $w = \pm 1$. But this is impossible, too, according to the asymptotic expansions near these points proven in [M4] Theorem 2.2; see Corollary 7.1 there. We emphasize that the cited result is applicable, since Γ and S meet with positive angles $\gamma_j \in (0, \alpha_j]$ at $\mathbf{p}_j$ by Definition 1, and since we assume

$$|\mathbf{Q}(\mathbf{p}_j) \cdot \mathbf{n}(\mathbf{p}_j)| < \cos\alpha_j \leq \cos\gamma_j, \quad j = 1, 2.$$

(Note that a simple reflection of S can be used to assure $\{\Gamma, S\}$ and $\mathbf{x}$ to fulfill the assumptions of [M4] Corollary 7.1.)

(iv) The interior regularity $\mathbf{N} \in C^{2,\alpha}(B^+,\mathbb{R}^3)$, $KW \in C^{1,\alpha}(B^+)$ as well as equation (2.14) were proven by F. Sauvigny in [S1] Satz 1. The global regularity $\mathbf{N} \in C^{1,\alpha}(\overline{B^+} \setminus \{-1,+1\},\mathbb{R}^3)$ follows from (i) and (iii). Finally, the continuity of $\mathbf{N}$ up to the corner points $w = \pm 1$ was proven in [M4] Theorem 5.4; see the remarks above concerning the applicability of this result. □

Remark 1. *By taking the cross product with $\mathbf{x}_u \in T_{\mathbf{x}}S$, the natural boundary condition (2.13) can be written in the form*

$$\mathbf{Q}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = -\mathbf{N} \cdot \mathbf{n}(\mathbf{x}) \quad \text{on } I. \tag{2.17}$$

This relation describes the well known fact that the normal component of $\mathbf{Q}$ w.r.t. to S prescribes the contact angle between a stationary $\mathcal{H}$-surface and the support surface S.

3 The second variation of $A_{\mathbf{Q}}$, stable $\mathcal{H}$-surfaces

Let us choose an $\mathcal{H}$-surface $\mathbf{x} \in \mathcal{C}_\mu(\Gamma, S; \overline{Z})$, $\mu \in (0,1)$, which is stationary w.r.t. $E_{\mathbf{Q}}$ (and thus belongs to $C^{3,\alpha}(B^+,Z) \cap C^{2,\alpha}(\overline{B^+} \setminus \{-1,+1\}, Z)$ according to Lemma 2 (i)). Consider a one-parameter family $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}(w,\varepsilon)$, which belongs to the class $C^\mu(\Gamma, S; \overline{Z}) \cap C^2(\overline{B^+} \setminus \{-1,+1\},\mathbb{R}^3)$ for any fixed $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and which depends smoothly on ε together with its first and second derivatives w.r.t. u, v. We call $\tilde{\mathbf{x}}$ an *admissible perturbation* of $\mathbf{x}$, if we have:

(i) $\tilde{\mathbf{x}}(w,0) = \mathbf{x}(w)$ for all $w \in \overline{B^+}$,

(ii) $\operatorname{supp}\big(\tilde{\mathbf{x}}(\cdot,\varepsilon) - \mathbf{x}\big) \subset B^+ \cup I$ for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$,

(iii) $\mathbf{y} := \frac{\partial}{\partial\varepsilon}\tilde{\mathbf{x}}(\cdot,\varepsilon)\big|_{\varepsilon=0} \in C^2_c(B^+ \cup I, \mathbb{R}^3)$, $\mathbf{z} := \frac{\partial^2}{\partial\varepsilon^2}\tilde{\mathbf{x}}(\cdot,\varepsilon)\big|_{\varepsilon=0} \in C^1_c(B^+ \cup I, \mathbb{R}^3)$.

The *direction* $\mathbf{y} = \frac{\partial}{\partial\varepsilon}\tilde{\mathbf{x}}(\cdot,\varepsilon)\big|_{\varepsilon=0}$ of an admissible perturbation $\tilde{\mathbf{x}}$ satisfies

$$\mathbf{y}(w) \in T_{\mathbf{x}(w)}S \quad \text{for all } w \in I. \tag{3.1}$$

On the other hand, choosing an arbitrary vector-field $\mathbf{y} \in C^2_c(B^+ \cup I, \mathbb{R}^3)$ with the property (3.1), one may construct an admissible perturbation $\tilde{\mathbf{x}}$ as described above by using a flow argument (compare, e.g., [DHT] pp. 32–33).

In the present section, we compute the second variation $\frac{d^2}{d\varepsilon^2} A_{\mathbf{Q}}(\tilde{\mathbf{x}}(\cdot,\varepsilon))\big|_{\varepsilon=0}$ for admissible perturbations. To this end, we have to examine the quantity

$$\frac{\partial^2}{\partial\varepsilon^2}\big(|\tilde{\mathbf{x}}_u \wedge \tilde{\mathbf{x}}_v| + \mathbf{Q}(\tilde{\mathbf{x}}) \cdot \tilde{\mathbf{x}}_u \wedge \tilde{\mathbf{x}}_v\big)\Big|_{\varepsilon=0} = \frac{\partial^2}{\partial\varepsilon^2}\big(|\tilde{\mathbf{x}}_u \wedge \tilde{\mathbf{x}}_v|\big)\Big|_{\varepsilon=0} + \frac{\partial^2}{\partial\varepsilon^2}\big(\mathbf{Q}(\tilde{\mathbf{x}}) \cdot \tilde{\mathbf{x}}_u \wedge \tilde{\mathbf{x}}_v\big)\Big|_{\varepsilon=0}. \tag{3.2}$$

We first compute (3.2) on $B' \cup I$ with

$$B' = \{w \in B^+ \,:\, W(w) > 0\}, \quad W = |\mathbf{x}_u \wedge \mathbf{x}_v| = |\mathbf{x}_u|^2 = |\mathbf{x}_v|^2,$$

and then observe that the resulting formula can be extended continuously to $B^+ \cup I$. We start with the first addend on the right-hand side of (3.2):

Proposition 1. *Let* $\tilde{\mathbf{x}}$ *be an admissible perturbation of a stationary* $\mathcal{H}$*-surface* $\mathbf{x} \in \mathcal{C}_\mu(\Gamma, S, \overline{Z})$ *as described above. Define* $\varphi := \mathbf{y} \cdot \mathbf{N} \in C_c^2(B^+ \cup I, \mathbb{R}^3)$*. Then there holds*

$$\begin{aligned}\frac{\partial^2}{\partial \varepsilon^2}\big(|\tilde{\mathbf{x}}_u \wedge \tilde{\mathbf{x}}_v|\big)\Big|_{\varepsilon=0} = {} & |\nabla\varphi|^2 + 2KW\varphi^2 - 2\mathcal{H}(\mathbf{x})\mathbf{y} \cdot (\mathbf{y}_u \wedge \mathbf{x}_v + \mathbf{x}_u \wedge \mathbf{y}_v)\big] \\ & +2\mathcal{H}(\mathbf{x})\big[\varphi(\mathbf{x}_u \cdot \mathbf{y}_u) + \varphi(\mathbf{x}_v \cdot \mathbf{y}_v) + (\mathbf{x}_u \cdot \mathbf{y})\varphi_u + (\mathbf{x}_v \cdot \mathbf{y})\varphi_v\big] \\ & -\big[\varphi\big(\mathbf{N}_u + 2\mathcal{H}(\mathbf{x})\mathbf{x}_u\big) \cdot \mathbf{y}\big]_u - \big[\varphi\big(\mathbf{N}_v + 2\mathcal{H}(\mathbf{x})\mathbf{x}_v\big) \cdot \mathbf{y}\big]_v \\ & +\big[\mathbf{N} \cdot (\mathbf{y} \wedge \mathbf{y}_v)\big]_u + \big[\mathbf{N} \cdot (\mathbf{y}_u \wedge \mathbf{y})\big]_v \\ & -2\mathcal{H}(\mathbf{x})\mathbf{z} \cdot (\mathbf{x}_u \wedge \mathbf{x}_v) + (\mathbf{z} \cdot \mathbf{x}_u)_u + (\mathbf{z} \cdot \mathbf{x}_v)_v \quad \text{on } B',\end{aligned}$$

where K *denotes the Gaussian curvature of* $\mathbf{x}$*.*

Proof. 1. We start by noting the relation

$$\frac{\partial^2}{\partial \varepsilon^2}\big(|\tilde{\mathbf{x}}_u \wedge \tilde{\mathbf{x}}_v|\big)\Big|_{\varepsilon=0} = \frac{1}{2W}\frac{\partial^2}{\partial \varepsilon^2}\big(|\tilde{\mathbf{x}}_u \wedge \tilde{\mathbf{x}}_v|^2\big)\Big|_{\varepsilon=0} - \frac{1}{4W^3}\Big[\frac{\partial}{\partial \varepsilon}\big(|\tilde{\mathbf{x}}_u \wedge \tilde{\mathbf{x}}_v|^2\big)\Big]^2\Big|_{\varepsilon=0} \tag{3.3}$$

on B'. Expanding $\tilde{\mathbf{x}}$ w.r.t. ε, we infer

$$\tilde{\mathbf{x}}(\cdot, \varepsilon) = \mathbf{x} + \varepsilon\mathbf{y} + \frac{\varepsilon^2}{2}\mathbf{z} + o(\varepsilon^2) \quad \text{on } B^+ \tag{3.4}$$

and, consequently,

$$\begin{aligned}\tilde{\mathbf{x}}_u \wedge \tilde{\mathbf{x}}_v = {} & W\mathbf{N} + \varepsilon(\mathbf{x}_u \wedge \mathbf{y}_v + \mathbf{y}_u \wedge \mathbf{x}_v) + \varepsilon^2\mathbf{y}_u \wedge \mathbf{y}_v \\ & + \frac{\varepsilon^2}{2}(\mathbf{x}_u \wedge \mathbf{z}_v + \mathbf{z}_u \wedge \mathbf{x}_v) + o(\varepsilon^2) \quad \text{on } B'\end{aligned} \tag{3.5}$$

as well as

$$\begin{aligned}|\tilde{\mathbf{x}}_u \wedge \tilde{\mathbf{x}}_v|^2 = {} & W^2 + 2\varepsilon W\mathbf{N} \cdot (\mathbf{x}_u \wedge \mathbf{y}_v + \mathbf{y}_u \wedge \mathbf{x}_v) \\ & + \varepsilon^2|\mathbf{x}_u \wedge \mathbf{y}_v + \mathbf{y}_u \wedge \mathbf{x}_v|^2 + 2\varepsilon^2 W\mathbf{N} \cdot (\mathbf{y}_u \wedge \mathbf{y}_v) \\ & + \varepsilon^2 W\mathbf{N} \cdot (\mathbf{x}_u \wedge \mathbf{z}_v + \mathbf{z}_u \wedge \mathbf{x}_v) + o(\varepsilon^2).\end{aligned} \tag{3.6}$$

Combining (3.3) with (3.6) gives

$$\begin{aligned}\frac{\partial^2}{\partial \varepsilon^2}\big(|\tilde{\mathbf{x}}_u \wedge \tilde{\mathbf{x}}_v|\big)\Big|_{\varepsilon=0} = {} & W^{-1}|\mathbf{x}_u \wedge \mathbf{y}_v + \mathbf{y}_u \wedge \mathbf{x}_v|^2 + 2\mathbf{N} \cdot (\mathbf{y}_u \wedge \mathbf{y}_v) \\ & + \mathbf{N} \cdot (\mathbf{x}_u \wedge \mathbf{z}_v + \mathbf{z}_u \wedge \mathbf{x}_v) \\ & - W^{-1}\big[\mathbf{N} \cdot (\mathbf{x}_u \wedge \mathbf{y}_v + \mathbf{y}_u \wedge \mathbf{x}_v)\big]^2 \\ = {} & (\mathbf{y}_u \cdot \mathbf{N})^2 + (\mathbf{y}_v \cdot \mathbf{N})^2 + 2\mathbf{N} \cdot (\mathbf{y}_u \wedge \mathbf{y}_v) \\ & + \mathbf{N} \cdot (\mathbf{x}_u \wedge \mathbf{z}_v + \mathbf{z}_u \wedge \mathbf{x}_v).\end{aligned}$$

And since $\mathbf{x}$ is a conformally parametrized $\mathcal{H}$-surface, we have

$$\begin{aligned}\mathbf{N} \cdot (\mathbf{x}_u \wedge \mathbf{z}_v + \mathbf{z}_u \wedge \mathbf{x}_v) &= \mathbf{z}_v \cdot \mathbf{x}_v + \mathbf{z}_u \cdot \mathbf{x}_u \\ &= (\mathbf{z} \cdot \mathbf{x}_u)_u + (\mathbf{z} \cdot \mathbf{x}_v)_v - 2\mathcal{H}(\mathbf{x})W\mathbf{z} \cdot \mathbf{N} \quad \text{on } B',\end{aligned}$$

arriving at

$$\frac{\partial^2}{\partial\varepsilon^2}\big(|\tilde{\mathbf{x}}_u \wedge \tilde{\mathbf{x}}_v|\big)\Big|_{\varepsilon=0} = (\mathbf{y}_u\cdot\mathbf{N})^2 + (\mathbf{y}_v\cdot\mathbf{N})^2 + 2\mathbf{N}\cdot(\mathbf{y}_u\wedge\mathbf{y}_v) + (\mathbf{z}\cdot\mathbf{x}_u)_u + (\mathbf{z}\cdot\mathbf{x}_v)_v - 2\mathcal{H}(\mathbf{x})W\mathbf{z}\cdot\mathbf{N} \quad \text{on } B'. \tag{3.7}$$

2. In the following, we sometimes write $u^1 := u$, $u^2 := v$ and use Einstein's convention summing up tacitly over sub- and superscript latin indizes from 1 to 2. Furthermore, we set $\lambda^j := W^{-1}\mathbf{x}_{u^j}\cdot\mathbf{y}$ for $j = 1, 2$ obtaining

$$\mathbf{y} = \lambda^j\mathbf{x}_{u^j} + \varphi\mathbf{N} \quad \text{on } B'.$$

Writing $g_{jk} := \mathbf{x}_{u^j}\cdot\mathbf{x}_{u^k}$, g^{jk}, Γ^l_{jk}, and $h_{jk} := \mathbf{x}_{u^ju^k}\cdot\mathbf{N} = -\mathbf{x}_{u^j}\cdot\mathbf{N}_{u^k}$ for the coefficients of the first fundamental form, its inverse and Christoffel symbols, and the coefficients of the second fundamental form, respectively, we then infer

$$\mathbf{y}_{u^k} = \big(\lambda^j_{u^k} + \lambda^l\Gamma^j_{lk} - \varphi h_{kl}g^{lj}\big)\mathbf{x}_{u^j} + \big(\lambda^j h_{jk} + \varphi_{u^k}\big)\mathbf{N} \quad \text{on } B'. \tag{3.8}$$

Due to the conformal parametrization of the $\mathcal{H}$-surface $\mathbf{x}$, we have

$$\begin{aligned} & g_{jk} = W\delta_{jk}, \quad g^{jk} = \frac{\delta^{jk}}{W}, \\ & \Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = \frac{W_u}{2W}, \\ & \Gamma^2_{22} = -\Gamma^2_{11} = \Gamma^1_{21} = \Gamma^1_{12} = \frac{W_v}{2W}, \\ & h_{11} + h_{22} = 2W\mathcal{H}(\mathbf{x}), \quad h_{11}h_{22} - (h_{12})^2 = W^2K \quad \text{on } B', \end{aligned} \tag{3.9}$$

where $\delta_{jk} = \delta^{jk}$ denotes the Kronecker delta.

3. We now evaluate the first line of the right-hand side in (3.7): Using (3.8) and (3.9), the first two terms can be written as

$$\begin{aligned} (\mathbf{y}_u\cdot\mathbf{N})^2 + (\mathbf{y}_v\cdot\mathbf{N})^2 &= (\lambda^1h_{11} + \lambda^2h_{12} + \varphi_u)^2 + (\lambda^1h_{12} + \lambda^2h_{22} + \varphi_v)^2 \\ &= |\nabla\varphi|^2 + \big[(\lambda^1)^2 + (\lambda^2)^2\big](h_{12})^2 + (\lambda^1)^2(h_{11})^2 + (\lambda^2)^2(h_{22})^2 \\ &\quad + 4\lambda^1\lambda^2h_{12}W\mathcal{H}(\mathbf{x}) + 4(\lambda^1\varphi_u + \lambda^2\varphi_v)W\mathcal{H}(\mathbf{x}) \\ &\quad + 2(\lambda^2h_{12} - \lambda^1h_{22})\varphi_u + 2(\lambda^1h_{12} - \lambda^2h_{11})\varphi_v \quad \text{on } B'. \end{aligned} \tag{3.10}$$

We next write the third term on the right-hand side of (3.7) as

$$\begin{aligned} 2\mathbf{N}\cdot(\mathbf{y}_u\wedge\mathbf{y}_v) &= [\mathbf{N}\cdot(\mathbf{y}\wedge\mathbf{y}_v)]_u + [\mathbf{N}\cdot(\mathbf{y}_u\wedge\mathbf{y})]_v \\ &\quad - \mathbf{N}_u\cdot(\mathbf{y}\wedge\mathbf{y}_v) - \mathbf{N}_v\cdot(\mathbf{y}_u\wedge\mathbf{y}) \quad \text{on } B'. \end{aligned} \tag{3.11}$$

Using the relations $\mathbf{N}\wedge\mathbf{x}_u = \mathbf{x}_v$, $\mathbf{N}\wedge\mathbf{x}_v = -\mathbf{x}_u$, we get from (3.8):

$$\begin{aligned} \mathbf{y}\wedge\mathbf{y}_{u^k} &= -\varphi\big(\lambda^2_{u^k} + \lambda^1\Gamma^2_{1k} + \lambda^2\Gamma^2_{2k} - \varphi h_{k2}W^{-1}\big)\mathbf{x}_u \\ &\quad + \varphi\big(\lambda^1_{u^k} + \lambda^1\Gamma^1_{1k} + \lambda^2\Gamma^1_{2k} - \varphi h_{k1}W^{-1}\big)\mathbf{x}_v \\ &\quad + \lambda^2\big(\lambda^1h_{1k} + \lambda^2h_{2k} + \varphi_{u^k}\big)\mathbf{x}_u \\ &\quad - \lambda^1\big(\lambda^1h_{1k} + \lambda^2h_{2k} + \varphi_{u^k}\big)\mathbf{x}_v + (\ldots)\mathbf{N} \quad \text{on } B', \end{aligned}$$

where $(\ldots)\mathbf{N}$ denotes the normal part of $\mathbf{y} \wedge \mathbf{y}_{u^k}$. This identity, formula (3.9), and the Weingarten equations $\mathbf{N}_{u^j} = -h_{jk}g^{kl}\mathbf{x}_{u^l}$ on B' yield

$$
\begin{aligned}
&-\mathbf{N}_u \cdot (\mathbf{y} \wedge \mathbf{y}_v) - \mathbf{N}_v \cdot (\mathbf{y}_u \wedge \mathbf{y}) \\
&\quad = W^{-1}\big[(h_{11}\mathbf{x}_u + h_{12}\mathbf{x}_v) \cdot (\mathbf{y} \wedge \mathbf{y}_v) - (h_{21}\mathbf{x}_u + h_{22}\mathbf{x}_v) \cdot (\mathbf{y} \wedge \mathbf{y}_u)\big] \\
&\quad = 2(\varphi)^2 WK + (\lambda^1 h_{22} - \lambda^2 h_{12})\varphi_u - (\lambda^1 h_{12} - \lambda^2 h_{11})\varphi_v \\
&\qquad +\varphi\big[\lambda^1_v h_{12} - \lambda^1_u h_{22} - \lambda^1 W_u \mathcal{H}(\mathbf{x})\big] - \varphi\big[\lambda^2_v h_{11} - \lambda^2_u h_{12} + \lambda^2 W_v \mathcal{H}(\mathbf{x})\big] \\
&\qquad +\big[(\lambda^1)^2 + (\lambda^2)^2\big]\big[h_{11}h_{22} - (h_{12})^2\big] \quad \text{on } B'.
\end{aligned} \tag{3.12}
$$

According to the Codazzi-Mainardi equations

$$
h_{21,v} - h_{22,u} + W_u H = 0, \quad h_{11,v} - h_{12,u} - W_v H = 0,
$$

we infer

$$
\begin{aligned}
&\lambda^1_v h_{12} - \lambda^1_u h_{22} - \lambda^1 W_u \mathcal{H}(\mathbf{x}) = (\lambda^1 h_{12})_v - (\lambda^1 h_{22})_u, \\
&\lambda^2_v h_{11} - \lambda^2_u h_{12} + \lambda^2 W_v \mathcal{H}(\mathbf{x}) = (\lambda^2 h_{11})_v - (\lambda^2 h_{12})_u \quad \text{on } B'.
\end{aligned}
$$

Inserting these identities into (3.12) and the resulting relation into (3.11), we arrive at

$$
\begin{aligned}
2\mathbf{N} \cdot (\mathbf{y}_u \wedge \mathbf{y}_v) = {} & [\mathbf{N} \cdot (\mathbf{y} \wedge \mathbf{y}_v)]_u + [\mathbf{N} \cdot (\mathbf{y}_u \wedge \mathbf{y})]_v \\
& +2(\varphi)^2 WK + (\lambda^1 h_{22} - \lambda^2 h_{12})\varphi_u - (\lambda^1 h_{12} - \lambda^2 h_{11})\varphi_v \\
& -\varphi(\lambda^1 h_{22} - \lambda^2 h_{12})_u + \varphi(\lambda^1 h_{12} - \lambda^2 h_{11})_v \\
& +\big[(\lambda^1)^2 + (\lambda^2)^2\big]\big[h_{11}h_{22} - (h_{12})^2\big] \quad \text{on } B'.
\end{aligned} \tag{3.13}
$$

Adding (3.10) and (3.13) we now find

$$
\begin{aligned}
&(\mathbf{y}_u \cdot \mathbf{N})^2 + (\mathbf{y}_v \cdot \mathbf{N})^2 + 2\mathbf{N} \cdot (\mathbf{y}_u \wedge \mathbf{y}_v) \\
&\quad = |\nabla\varphi|^2 + 2(\varphi)^2 KW + [\mathbf{N} \cdot (\mathbf{y} \wedge \mathbf{y}_v)]_u + [\mathbf{N} \cdot (\mathbf{y}_u \wedge \mathbf{y})]_v \\
&\qquad -\big[\varphi(\lambda^1 h_{22} - \lambda^2 h_{12})\big]_u + \big[\varphi(\lambda^1 h_{12} - \lambda^2 h_{11})\big]_v \\
&\qquad +2W\mathcal{H}(\mathbf{x})\big[(\lambda^1)^2 h_{11} + (\lambda^2)^2 h_{22} + 2\lambda^1\lambda^2 h_{12} + 2(\lambda^1\varphi_u + \lambda^2\varphi_v)\big]
\end{aligned} \tag{3.14}
$$

on B'. Finally, we calculate via the Weingarten equations and (3.9)

$$
\begin{aligned}
\lambda^1 h_{22} - \lambda^2 h_{12} &= W^{-1}(h_{22}\mathbf{x}_u - h_{12}\mathbf{x}_v) \cdot \mathbf{y} = (\mathbf{N}_u + 2\mathcal{H}(\mathbf{x})\mathbf{x}_u) \cdot \mathbf{y}, \\
\lambda^1 h_{12} - \lambda^2 h_{11} &= W^{-1}(h_{12}\mathbf{x}_u - h_{11}\mathbf{x}_v) \cdot \mathbf{y} = -(\mathbf{N}_v + 2\mathcal{H}(\mathbf{x})\mathbf{x}_v) \cdot \mathbf{y}
\end{aligned} \tag{3.15}
$$

as well as

$$
\begin{aligned}
(\lambda^1)^2 h_{11} + (\lambda^2)^2 h_{22} + 2\lambda^1\lambda^2 h_{12} + 2(\lambda^1\varphi_u + \lambda^2\varphi_v)& \\
&= \lambda^1(\lambda^1 h_{11} + \lambda^2 h_{12}) + \lambda^2(\lambda^1 h_{12} + \lambda^2 h_{22}) + 2(\lambda^1\varphi_u + \lambda^2\varphi_v) \\
&= -\lambda^1(\mathbf{N}_u \cdot \mathbf{y}) - \lambda^2(\mathbf{N}_v \cdot \mathbf{y}) + 2(\lambda^1\varphi_u + \lambda^2\varphi_v) \\
&= W^{-1}\big[(\mathbf{x}_u \cdot \mathbf{y})(\mathbf{N} \cdot \mathbf{y}_u) + (\mathbf{x}_v \cdot \mathbf{y})(\mathbf{N} \cdot \mathbf{y}_v)\big] + (\lambda^1\varphi_u + \lambda^2\varphi_v) \\
&= W^{-1}\big[\varphi(\mathbf{x}_u \cdot \mathbf{y}_u) + \varphi(\mathbf{x}_v \cdot \mathbf{x}_v) + (\mathbf{x}_u \cdot \mathbf{y})\varphi_u + (\mathbf{x}_v \cdot \mathbf{y})\varphi_v\big] \\
&\quad -W^{-1}\big[\mathbf{y} \cdot (\mathbf{y}_u \wedge \mathbf{x}_v) + \mathbf{y} \cdot (\mathbf{x}_u \wedge \mathbf{y}_v)\big].
\end{aligned}
\tag{3.16}
$$

Inserting (3.15) and (3.16) into (3.14), the asserted identity follows from the resulting relation and formula (3.7). □

Proposition 2. *Under the assumptions of Proposition 1, there holds*

$$
\begin{aligned}
\frac{\partial^2}{\partial\varepsilon^2}\big[\mathbf{Q}(\tilde{\mathbf{x}}) \cdot (\tilde{\mathbf{x}}_u \wedge \tilde{\mathbf{x}}_v)\big]\Big|_{\varepsilon=0}& \\
= 2W\varphi^2\big[\nabla\mathcal{H}(\mathbf{x}) \cdot \mathbf{N} - 2\mathcal{H}(\mathbf{x})^2\big] &+ 2\mathcal{H}(\mathbf{x})\mathbf{y} \cdot (\mathbf{x}_u \wedge \mathbf{y}_v + \mathbf{y}_u \wedge \mathbf{x}_v) \\
-2\mathcal{H}(\mathbf{x})\big[\varphi(\mathbf{x}_u \cdot \mathbf{y}_u) &+ \varphi(\mathbf{x}_v \cdot \mathbf{y}_v) + (\mathbf{x}_u \cdot \mathbf{y})\varphi_u + (\mathbf{x}_v \cdot \mathbf{y})\varphi_v\big] \\
+2\big[\varphi\mathcal{H}(\mathbf{x})(\mathbf{x}_u \cdot \mathbf{y})\big]_u &+ 2\big[\varphi\mathcal{H}(\mathbf{x})(\mathbf{x}_v \cdot \mathbf{y})\big]_v + 2\mathcal{H}(\mathbf{x})\mathbf{z} \cdot (\mathbf{x}_u \wedge \mathbf{x}_v) \\
+\big[\big(D\mathbf{Q}(\mathbf{x})\mathbf{y}\big) \cdot (\mathbf{y} \wedge \mathbf{x}_v)\big]_u &+ \big[\mathbf{Q}(\mathbf{x}) \cdot (\mathbf{z} \wedge \mathbf{x}_v)\big]_u + \big[\mathbf{Q}(\mathbf{x}) \cdot (\mathbf{y} \wedge \mathbf{y}_v)\big]_u \\
+\big[\big(D\mathbf{Q}(\mathbf{x})\mathbf{y}\big) \cdot (\mathbf{x}_u \wedge \mathbf{y})\big]_v &+ \big[\mathbf{Q}(\mathbf{x}) \cdot (\mathbf{x}_u \wedge \mathbf{z})\big]_v + \big[\mathbf{Q}(\mathbf{x}) \cdot (\mathbf{y}_u \wedge \mathbf{y})\big]_v
\end{aligned}
$$

on B'.

Proof. Using (2.4) and the general relation

$$
[\mathbf{M}\mathbf{a}] \cdot (\mathbf{b} \wedge \mathbf{c}) + \mathbf{a} \cdot ([\mathbf{M}\mathbf{b}] \wedge \mathbf{c}) + \mathbf{a} \cdot (\mathbf{b} \wedge [\mathbf{M}\mathbf{c}]) = (\operatorname{tr}\mathbf{M})[\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})] \tag{3.17}
$$

for arbitrary vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ and matrices $\mathbf{M} \in \mathbb{R}^{3\times 3}$ with trace $\operatorname{tr}\mathbf{M}$, we first compute

$$
\begin{aligned}
\frac{\partial}{\partial\varepsilon}\big[\mathbf{Q}(\tilde{\mathbf{x}}) \cdot (\tilde{\mathbf{x}}_u \wedge \tilde{\mathbf{x}}_v)\big]& \\
&= [D\mathbf{Q}(\tilde{\mathbf{x}})\tilde{\mathbf{x}}_\varepsilon] \cdot (\tilde{\mathbf{x}}_u \wedge \tilde{\mathbf{x}}_v) + \mathbf{Q}(\tilde{\mathbf{x}}) \cdot (\tilde{\mathbf{x}}_\varepsilon \wedge \tilde{\mathbf{x}}_v)_u + \mathbf{Q}(\tilde{\mathbf{x}}) \cdot (\tilde{\mathbf{x}}_u \wedge \tilde{\mathbf{x}}_\varepsilon)_v \\
&= 2\mathcal{H}(\tilde{\mathbf{x}})\tilde{\mathbf{x}}_\varepsilon \cdot (\tilde{\mathbf{x}}_u \wedge \tilde{\mathbf{x}}_v) + \big[\mathbf{Q}(\tilde{\mathbf{x}}) \cdot (\tilde{\mathbf{x}}_\varepsilon \wedge \tilde{\mathbf{x}}_v)\big]_u + \big[\mathbf{Q}(\tilde{\mathbf{x}}) \cdot (\tilde{\mathbf{x}}_u \wedge \tilde{\mathbf{x}}_\varepsilon)\big]_v
\end{aligned}
$$

on B^+. Having (3.4) and (3.5) in mind, a second differentiation yields at $\varepsilon = 0$:

$$
\begin{aligned}
\frac{\partial^2}{\partial\varepsilon^2}\big[\mathbf{Q}(\tilde{\mathbf{x}}) \cdot (\tilde{\mathbf{x}}_u \wedge \tilde{\mathbf{x}}_v)\big]\Big|_{\varepsilon=0} &= 2\big[\nabla\mathcal{H}(\mathbf{x}) \cdot \mathbf{y}\big]\mathbf{y} \cdot (\mathbf{x}_u \wedge \mathbf{x}_v) + 2\mathcal{H}(\mathbf{x})\mathbf{z} \cdot (\mathbf{x}_u \wedge \mathbf{x}_v) \\
&\quad +2\mathcal{H}(\mathbf{x})\mathbf{y} \cdot (\mathbf{x}_u \wedge \mathbf{y}_v + \mathbf{y}_u \wedge \mathbf{x}_v) \\
&\quad +\big[\big(D\mathbf{Q}(\mathbf{x})\mathbf{y}\big) \cdot (\mathbf{y} \wedge \mathbf{x}_v)\big]_u + \big[\mathbf{Q}(\mathbf{x}) \cdot (\mathbf{z} \wedge \mathbf{x}_v)\big]_u \\
&\quad +\big[\mathbf{Q}(\mathbf{x}) \cdot (\mathbf{y} \wedge \mathbf{y}_v)\big]_u + \big[\big(D\mathbf{Q}(\mathbf{x})\mathbf{y}\big) \cdot (\mathbf{x}_u \wedge \mathbf{y})\big]_v \\
&\quad +\big[\mathbf{Q}(\mathbf{x}) \cdot (\mathbf{x}_u \wedge \mathbf{z})\big]_v + \big[\mathbf{Q}(\mathbf{x}) \cdot (\mathbf{y}_u \wedge \mathbf{y})\big]_v.
\end{aligned}
\tag{3.18}
$$

Writing again $\mathbf{y} = \lambda^j \mathbf{x}_{u^j} + \varphi \mathbf{N}$ on B' with $\lambda^j = W^{-1} \mathbf{x}_{u^j} \cdot \mathbf{y}$ and employing (1.1), the assertion follows from (3.18) and the identity

$$\begin{aligned}
&2\big[\nabla\mathcal{H}(\mathbf{x})\cdot\mathbf{y}\big]\mathbf{y}\cdot(\mathbf{x}_u\wedge\mathbf{x}_v)\\
&\qquad= 2W\varphi^2\nabla\mathcal{H}(\mathbf{x})\cdot\mathbf{N} + 2\varphi\lambda^j W\mathcal{H}(\mathbf{x})_{u^j}\\
&\qquad= 2W\varphi^2\nabla\mathcal{H}(\mathbf{x})\cdot\mathbf{N} + 2\big[\varphi\mathcal{H}(\mathbf{x})(\mathbf{x}_u\cdot\mathbf{y})\big]_u + 2\big[\varphi\mathcal{H}(\mathbf{x})(\mathbf{x}_v\cdot\mathbf{y})\big]_v\\
&\qquad\quad -2\mathcal{H}(\mathbf{x})\big[\varphi(\mathbf{x}_u\cdot\mathbf{y}_u)+\varphi(\mathbf{x}_v\cdot\mathbf{y}_v)+(\mathbf{x}_u\cdot\mathbf{y})\varphi_u+(\mathbf{x}_v\cdot\mathbf{y})\varphi_v\big]\\
&\qquad\quad -4W\varphi^2\mathcal{H}(\mathbf{x})^2.
\end{aligned}$$

□

As already announced, the right-hand sides in the results of Propositions 1 and 2 can be extended continuously onto $B^+ \cup I$, according to Lemma 2. Hence we can compute the second variation via the divergence theorem for *any* admissible one-parameter family $\tilde{\mathbf{x}}(\cdot,\varepsilon)$ with direction $\mathbf{y} \in C_c^2(B^+ \cup I, \mathbb{R}^3)$ satisfying (3.1). Nevertheless, we concentrate on directions of the form

$$\mathbf{y}(w) := \frac{\varphi(w)}{1+\mathbf{Q}(\mathbf{x}(w))\cdot\mathbf{N}(w)}\big[\mathbf{Q}(\mathbf{x}(w))+\mathbf{N}(w)\big], \tag{3.19}$$

with some function $\varphi \in C_c^2(B^+ \cup I)$. Note that $\mathbf{y}$ is well-defined according to assumption (2.4), belongs to $C_c^2(B^+ \cup I, \mathbb{R}^3)$, and satisfies $\mathbf{y}\cdot\mathbf{N} \equiv \varphi$ as well as (3.1); for the latter, see Remark 1.

Definition 3. *For given $\varphi \in C_c^2(B^+ \cup I)$ we define $\mathbf{y} \in C_c^2(B^+ \cup I, \mathbb{R}^3)$ by (3.19) and consider the admissible perturbation $\tilde{\mathbf{x}}(\cdot,\varepsilon)$ with direction $\mathbf{y}$. Then we set*

$$\delta^2 A_{\mathbf{Q}}(\mathbf{x},\varphi) := \frac{d^2}{d\varepsilon^2} A_{\mathbf{Q}}\big(\tilde{\mathbf{x}}(\cdot,\varepsilon))\big)\Big|_{\varepsilon=0}$$

for the second variation of $A_{\mathbf{Q}}(\mathbf{x})$ with dilation φ.

In order to compute $\delta^2 A_{\mathbf{Q}}(\mathbf{x},\varphi)$, we introduce the curvature of the cylindrical support surface S defined by

$$\kappa(\mathbf{p}) := -\big(\sigma''(s),0\big)\cdot\mathbf{n}(\mathbf{p}) \quad \text{for } \mathbf{p}\in\{\sigma(s)\}\times\mathbb{R},\ s\in[0,s_0], \tag{3.20}$$

compare Section 2. Note that, due to the cylindrical structure of S, we have the relation

$$\big[D\mathbf{n}(\mathbf{p})\boldsymbol{\zeta}_1\big]\cdot\boldsymbol{\zeta}_2 = \kappa(\mathbf{p})\big[\boldsymbol{\zeta}_1\cdot\mathbf{t}(\mathbf{p})\big]\big[\boldsymbol{\zeta}_2\cdot\mathbf{t}(\mathbf{p})\big] \quad \text{for all } \boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2\in T_{\mathbf{p}}S,\ \mathbf{p}\in S, \tag{3.21}$$

interpreting $D\mathbf{n}$ as the Weingarten map of S.

Lemma 3. *Let $\mathbf{x} \in \mathcal{C}_\mu(\Gamma, S; \overline{Z})$, $\mu \in (0,1)$, be a stationary $\mathcal{H}$-surface w.r.t. $E_{\mathbf{Q}}$ and let $\varphi \in C_c^2(B^+ \cup I)$ be chosen. Setting*

$$q(w) := \big[2\mathcal{H}(\mathbf{x}(w))^2 - K(w) - \nabla\mathcal{H}(\mathbf{x}(w))\cdot\mathbf{N}(w)\big]W(w), \quad w\in B^+\cup I, \tag{3.22}$$

we then have

$$\delta^2 A_{\mathbf{Q}}(\mathbf{x},\varphi) = \iint\limits_{B^+} \Big\{|\nabla\varphi|^2 - 2q\varphi^2\Big\}\,du\,dv + \int\limits_I \varphi^2 \frac{\mathbf{N}_v\cdot\mathbf{Q}(\mathbf{x})}{1+\mathbf{Q}(\mathbf{x})\cdot\mathbf{N}}\,du$$
$$+\int\limits_I \varphi^2\bigg\{\frac{\big[D\mathbf{Q}(\mathbf{x})\big(\mathbf{Q}(\mathbf{x})+\mathbf{N}\big)\big]\cdot\big[\mathbf{x}_v+\mathbf{Q}(\mathbf{x})\wedge\mathbf{x}_u\big]}{(1+\mathbf{Q}(\mathbf{x})\cdot\mathbf{N})^2}$$
$$+\frac{\kappa(\mathbf{x})\big[\big(\mathbf{x}_v+\mathbf{Q}(\mathbf{x})\wedge\mathbf{x}_u\big)\cdot\mathbf{n}(\mathbf{x})\big]\big[\big(\mathbf{Q}(\mathbf{x})+\mathbf{N}\big)\cdot\mathbf{t}(\mathbf{x})\big]^2}{(1+\mathbf{Q}(\mathbf{x})\cdot\mathbf{N})^2}\bigg\}du. \tag{3.23}$$

Proof. We add the results of Propositions 1 and 2 obtaining

$$\frac{\partial^2}{\partial\varepsilon^2}\big(|\tilde{\mathbf{x}}_u\wedge\tilde{\mathbf{x}}_v|+\mathbf{Q}(\tilde{\mathbf{x}})\cdot\tilde{\mathbf{x}}_u\wedge\tilde{\mathbf{x}}_v\big)\Big|_{\varepsilon=0}$$
$$= |\nabla\varphi|^2 - 2q\varphi^2 - \big[\varphi(\mathbf{N}_u\cdot\mathbf{y})\big]_u - \big[\varphi(\mathbf{N}_v\cdot\mathbf{y})\big]_v$$
$$+\big[\big(D\mathbf{Q}(\mathbf{x})\mathbf{y}\big)\cdot(\mathbf{y}\wedge\mathbf{x}_v)\big]_u + \big[\big(D\mathbf{Q}(\mathbf{x})\mathbf{y}\big)\cdot(\mathbf{x}_u\wedge\mathbf{y})\big]_v$$
$$+\big[\big(\mathbf{Q}(\mathbf{x})+\mathbf{N}\big)\cdot(\mathbf{y}\wedge\mathbf{y}_v)\big]_u + \big[\big(\mathbf{Q}(\mathbf{x})+\mathbf{N}\big)\cdot(\mathbf{y}_u\wedge\mathbf{y})\big]_v$$
$$+\big[\mathbf{z}\cdot\big(\mathbf{x}_u+\mathbf{x}_v\wedge\mathbf{Q}(\mathbf{x})\big)\big]_u + \big[\mathbf{z}\cdot\big(\mathbf{x}_v+\mathbf{Q}(\mathbf{x})\wedge\mathbf{x}_u\big)\big]_v.$$

Having $\mathbf{y}\parallel(\mathbf{Q}(\mathbf{x})+\mathbf{N})$ on I in mind, the divergence theorem yields

$$\delta^2 A_{\mathbf{Q}}(\mathbf{x},\varphi) = \iint\limits_{B^+}\frac{\partial^2}{\partial\varepsilon^2}\big(|\tilde{\mathbf{x}}_u\wedge\tilde{\mathbf{x}}_v|+\mathbf{Q}(\tilde{\mathbf{x}})\cdot\tilde{\mathbf{x}}_u\wedge\tilde{\mathbf{x}}_v\big)\Big|_{\varepsilon=0}$$
$$= \iint\limits_{B^+}\Big\{|\nabla\varphi|^2-2q\varphi^2\Big\}\,du\,dv + \int\limits_I \varphi(\mathbf{N}_v\cdot\mathbf{y})\,du$$
$$-\int\limits_I\Big\{\big(D\mathbf{Q}(\mathbf{x})\mathbf{y}\big)\cdot(\mathbf{x}_u\wedge\mathbf{y}) + \mathbf{z}\cdot\big(\mathbf{x}_v+\mathbf{Q}(\mathbf{x})\wedge\mathbf{x}_u\big)\Big\}\,du. \tag{3.24}$$

Due to the special choice (3.19) of $\mathbf{y}$, the first three terms on the right-hand side of (3.24) are identical with those in the announced relation (3.23). In order to identify the fourth terms of (3.23) and (3.24), we recall Lemma 2 (i) and deduce

$$\mathbf{z}\cdot\big(\mathbf{x}_v+\mathbf{Q}(\mathbf{x})\wedge\mathbf{x}_u\big) = \big(\mathbf{z}\cdot\mathbf{n}(\mathbf{x})\big)\big[\big(\mathbf{x}_v+\mathbf{Q}(\mathbf{x})\wedge\mathbf{x}_u\big)\cdot\mathbf{n}(\mathbf{x})\big] \quad \text{on } I. \tag{3.25}$$

Similar to [HS3] p. 431, we compute $\mathbf{z}\cdot\mathbf{n}(\mathbf{x})$ on I: Since $\tilde{\mathbf{x}}(w,\varepsilon)\in S$ holds for all $w\in I$ and $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$, we have $\frac{\partial}{\partial\varepsilon}\tilde{\mathbf{x}}(w,\varepsilon)\cdot\mathbf{n}(\tilde{\mathbf{x}}(w,\varepsilon))=0$ and, consequently,

$$\frac{\partial^2}{\partial\varepsilon^2}\tilde{\mathbf{x}}(w,\varepsilon)\cdot\mathbf{n}(\tilde{\mathbf{x}}(w,\varepsilon)) + \frac{\partial}{\partial\varepsilon}\tilde{\mathbf{x}}(w,\varepsilon)\cdot\Big[D\mathbf{n}(\tilde{\mathbf{x}}(w,\varepsilon))\frac{\partial}{\partial\varepsilon}\tilde{\mathbf{x}}(w,\varepsilon)\Big] = 0$$

for $w\in I$ and $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$. For $\varepsilon=0$ we employ (3.21) and infer

$$\mathbf{z}\cdot\mathbf{n}(\mathbf{x}) = -\kappa(\mathbf{x})\big[\mathbf{y}\cdot\mathbf{t}(\mathbf{x})\big]^2 \quad \text{on } I.$$

Together with (3.25), we arrive at

$$\mathbf{z}\cdot\big(\mathbf{x}_v+\mathbf{Q}(\mathbf{x})\wedge\mathbf{x}_u\big) = -\kappa(\mathbf{x})\big[\big(\mathbf{x}_v+\mathbf{Q}(\mathbf{x})\wedge\mathbf{x}_u\big)\cdot\mathbf{n}(\mathbf{x})\big]\big[\mathbf{y}\cdot\mathbf{t}(\mathbf{x})\big]^2 \quad \text{on } I.$$

Putting this relation into (3.24), proves the assertion. □

Remark 2. *By a standard approximation argument, dilations* $\varphi \in H_2^1(B^+) \cap C_c^0(B^+ \cup I)$ *are admissible in the second variation* $\delta^2 A_{\mathbf{Q}}(\mathbf{x}, \varphi)$ *due to formula (3.23).*

Definition 4. *A partially free* $\mathcal{H}$*-surface* $\mathbf{x} \in C_\mu(\Gamma, S; \overline{Z})$ *with* $\delta^2 A_{\mathbf{Q}}(\mathbf{x}, \varphi) \geq 0$ *for any dilation* $\varphi \in H_2^1(B^+) \cap C_c^0(\overline{B^+})$ *is called* stable.

4 Boundary condition for the surface normal and proof of the theorem

In order to deduce the crucial relation $N^3 > 0$ on $\overline{B^+}$ for the third component of the surface normal of our stable $\mathcal{H}$-surface, we will combine formula (3.23) with the following boundary condition:

Lemma 4. *Let the assumptions of Theorem 1 be satisfied and let a stationary* $\mathcal{H}$*-surface* $\mathbf{x} \in \mathcal{C}_\mu(\Gamma, S; \overline{Z})$*,* $\mu \in (0,1)$*, be given. Then, the third component* N^3 *of the surface normal of* $\mathbf{x}$ *fulfills the boundary condition*

$$
\begin{aligned}
N_v^3 \;=\;& \left\{ \frac{\mathbf{N}_v \cdot \mathbf{Q}(\mathbf{x})}{1 + \mathbf{Q}(\mathbf{x}) \cdot \mathbf{N}} + \frac{\big[D\mathbf{Q}(\mathbf{x})\big(\mathbf{Q}(\mathbf{x}) + \mathbf{N}\big)\big] \cdot \big[\mathbf{x}_v + \mathbf{Q}(\mathbf{x}) \wedge \mathbf{x}_u\big]}{(1 + \mathbf{Q}(\mathbf{x}) \cdot \mathbf{N})^2} \right. \\
&\left. + \frac{\kappa(\mathbf{x})\big[\big(\mathbf{x}_v + \mathbf{Q}(\mathbf{x}) \wedge \mathbf{x}_u\big) \cdot \mathbf{n}(\mathbf{x})\big]\big[\big(\mathbf{Q}(\mathbf{x}) + \mathbf{N}\big) \cdot \mathbf{t}(\mathbf{x})\big]^2}{(1 + \mathbf{Q}(\mathbf{x}) \cdot \mathbf{N})^2} \right\} N^3 \quad \text{on } I,
\end{aligned} \tag{4.1}
$$

where $\mathbf{t}$*,* $\mathbf{n}$*, and* κ *were defined in (2.1), (3.20).*

Proof.

1. From (1.1) and Lemma 2 (iv) we get the well known relations

$$
\mathbf{N}_u = \mathbf{N} \wedge \mathbf{N}_v - 2\mathcal{H}(\mathbf{x})\mathbf{x}_u, \quad \mathbf{N}_v = -\mathbf{N} \wedge \mathbf{N}_u - 2\mathcal{H}(\mathbf{x})\mathbf{x}_v \quad \text{on } B^+ \cup I. \tag{4.2}
$$

Writing $\mathcal{H} = \mathcal{H}(\mathbf{x})$, $\mathbf{Q} = \mathbf{Q}(\mathbf{x})$, $\kappa = \kappa(\mathbf{x})$ etc. and employing (4.2) as well as (2.17), we compute

$$
\begin{aligned}
(\mathbf{N}_v \cdot \mathbf{Q})N^3 \;&=\; \big\{[(\mathbf{Q} + \mathbf{N}) \cdot \mathbf{N}_v]\mathbf{N}\big\} \cdot \mathbf{e}_3 \\
&=\; -\Big\{(\mathbf{N} \wedge \mathbf{N}_v) \wedge (\mathbf{Q} + \mathbf{N}) - [\mathbf{N} \cdot (\mathbf{Q} + \mathbf{N})]\mathbf{N}_v\Big\} \cdot \mathbf{e}_3 \\
&=\; -\Big\{\mathbf{N}_u \wedge (\mathbf{Q} + \mathbf{N}) + 2\mathcal{H}\mathbf{x}_u \wedge (\mathbf{Q} + \mathbf{N}) - [1 + (\mathbf{Q} \cdot \mathbf{N})]\mathbf{N}_v\Big\} \cdot \mathbf{e}_3 \\
&=\; (\mathbf{N} \wedge \mathbf{e}_3)_u \cdot (\mathbf{Q} + \mathbf{N}) + [1 + (\mathbf{Q} \cdot \mathbf{N})]N_v^3 \quad \text{on } I.
\end{aligned}
$$

Consequently, the asserted relation (4.1) is equivalent to the identity

$$
\begin{aligned}
(\mathbf{N} \wedge \mathbf{e}_3)_u \cdot (\mathbf{Q} + \mathbf{N}) \;=\;& -\Big\{[D\mathbf{Q}(\mathbf{Q} + \mathbf{N})] \cdot (\mathbf{x}_v + \mathbf{Q} \wedge \mathbf{x}_u) \\
&+ \kappa[(\mathbf{x}_v + \mathbf{Q} \wedge \mathbf{x}_u) \cdot \mathbf{n}][(\mathbf{Q} + \mathbf{N}) \cdot \mathbf{t}]^2\Big\} \frac{N^3}{1 + \mathbf{Q} \cdot \mathbf{N}}
\end{aligned} \tag{4.3}
$$

on I.

2. Next, we manipulate the left-hand side of (4.3): Having (2.17) in mind, we find

$$(\mathbf{Q}+\mathbf{N})\wedge\mathbf{e}_3 = (\mathbf{Q}+\mathbf{N})\wedge(\mathbf{n}\wedge\mathbf{t}) = [(\mathbf{Q}+\mathbf{N})\cdot\mathbf{t}]\mathbf{n} \quad \text{on } I.$$

Together with (3.21), we infer

$$\begin{aligned}[(\mathbf{Q}+\mathbf{N})\wedge\mathbf{e}_3]_u\cdot(\mathbf{Q}+\mathbf{N}) &= [(\mathbf{Q}+\mathbf{N})\cdot\mathbf{t}]\big\{[(D\mathbf{n})\mathbf{x}_u]\cdot(\mathbf{Q}+\mathbf{N})\big\}\\ &= \kappa[(\mathbf{Q}+\mathbf{N})\cdot\mathbf{t}]^2(\mathbf{x}_u\cdot\mathbf{t}) \quad \text{on } I.\end{aligned} \tag{4.4}$$

On the other hand, we calculate

$$\begin{aligned}(\mathbf{x}_u\cdot\mathbf{t})(1+\mathbf{Q}\cdot\mathbf{N}) &= (\mathbf{x}_u\cdot\mathbf{t})[\mathbf{N}\cdot(\mathbf{Q}+\mathbf{N})]\\ &= [\mathbf{x}_u\wedge(\mathbf{Q}+\mathbf{N})]\cdot(\mathbf{t}\wedge\mathbf{N}) - (\mathbf{x}_u\cdot\mathbf{N})[\mathbf{t}\cdot(\mathbf{Q}+\mathbf{N})]\\ &= \big\{[\mathbf{x}_u\wedge(\mathbf{Q}+\mathbf{N})]\cdot\mathbf{n}\big\}[\mathbf{n}\cdot(\mathbf{t}\wedge\mathbf{N})]\\ &= -[(\mathbf{x}_v+\mathbf{Q}\wedge\mathbf{x}_u)\cdot\mathbf{n}]N^3 \quad \text{on } I\end{aligned}$$

or, equivalently,

$$\mathbf{x}_u\cdot\mathbf{t} = -\frac{N^3}{1+\mathbf{Q}\cdot\mathbf{N}}[(\mathbf{x}_v+\mathbf{Q}\wedge\mathbf{x}_u)\cdot\mathbf{n}] \quad \text{on } I. \tag{4.5}$$

From (4.4) and (4.5) we now deduce

$$\begin{aligned}(\mathbf{N}\wedge\mathbf{e}_3)_u\cdot(\mathbf{Q}+\mathbf{N}) &= [(\mathbf{Q}+\mathbf{N})\wedge\mathbf{e}_3]_u\cdot(\mathbf{Q}+\mathbf{N}) - (\mathbf{Q}\wedge\mathbf{e}_3)_u\cdot(\mathbf{Q}+\mathbf{N})\\ &= -\kappa[(\mathbf{x}_v+\mathbf{Q}\wedge\mathbf{x}_u)\cdot\mathbf{n}][(\mathbf{Q}+\mathbf{N})\cdot\mathbf{t}]^2\frac{N^3}{1+\mathbf{Q}\cdot\mathbf{N}}\\ &\quad -(\mathbf{Q}\wedge\mathbf{e}_3)_u\cdot(\mathbf{Q}+\mathbf{N}) \quad \text{on } I.\end{aligned} \tag{4.6}$$

By inserting (4.6) into (4.3), the claimed relation (4.1) becomes equivalent to

$$(\mathbf{Q}\wedge\mathbf{e}_3)_u\cdot(\mathbf{Q}+\mathbf{N}) = [D\mathbf{Q}(\mathbf{Q}+\mathbf{N})]\cdot(\mathbf{x}_v+\mathbf{Q}\wedge\mathbf{x}_u)\frac{N^3}{1+\mathbf{Q}\cdot\mathbf{N}} \quad \text{on } I. \tag{4.7}$$

3. In the next step, we observe that (4.7) is equivalent to the identity

$$[(D\mathbf{Q})\mathbf{x}_u]\cdot[\mathbf{e}_3\wedge(\mathbf{Q}+\mathbf{N})] + \mathbf{x}_u\cdot\big\{\mathbf{e}_3\wedge[(D\mathbf{Q})(\mathbf{Q}+\mathbf{N})]\big\} = 0 \quad \text{on } I. \tag{4.8}$$

Indeed, the left hand side of (4.7) can be written as

$$(\mathbf{Q}\wedge\mathbf{e}_3)_u\cdot(\mathbf{Q}+\mathbf{N}) = \big\{[(D\mathbf{Q})\mathbf{x}_u]\wedge\mathbf{e}_3\big\}\cdot(\mathbf{Q}+\mathbf{N}) = [(D\mathbf{Q})\mathbf{x}_u]\cdot[\mathbf{e}_3\wedge(\mathbf{Q}+\mathbf{N})],$$

whereas we compute in the right hand side

$$\begin{aligned}&[D\mathbf{Q}(\mathbf{Q}+\mathbf{N})]\cdot(\mathbf{x}_v+\mathbf{Q}\wedge\mathbf{x}_u)N^3\\ &\qquad = [(\mathbf{x}_v+\mathbf{Q}\wedge\mathbf{x}_u)\wedge\mathbf{N}]\cdot\big\{[D\mathbf{Q}(\mathbf{Q}+\mathbf{N})]\wedge\mathbf{e}_3\big\}\\ &\qquad = (1+\mathbf{Q}\cdot\mathbf{N})\,\mathbf{x}_u\cdot\big\{[D\mathbf{Q}(\mathbf{Q}+\mathbf{N})]\wedge\mathbf{e}_3\big\} \quad \text{on } I.\end{aligned}$$

This proves the claimed equivalence.

4. It remains to prove (4.8). Applying the relation (3.17) with $\mathbf{a} = \mathbf{x}_u$, $\mathbf{b} = \mathbf{e}_3$, $\mathbf{c} = \mathbf{Q} + \mathbf{N}$, and $\mathbf{M} = D\mathbf{Q}$, we obtain

$$\begin{aligned}
&[(D\mathbf{Q})\mathbf{x}_u] \cdot [\mathbf{e}_3 \wedge (\mathbf{Q}+\mathbf{N})] + \mathbf{x}_u \cdot \big\{\mathbf{e}_3 \wedge [(D\mathbf{Q})(\mathbf{Q}+\mathbf{N})]\big\} \\
&\quad = -\mathbf{x}_u \cdot \big\{[(D\mathbf{Q})\mathbf{e}_3] \wedge (\mathbf{Q}+\mathbf{N})\big\} + (\operatorname{tr} D\mathbf{Q})\big\{\mathbf{x}_u \cdot [\mathbf{e}_3 \wedge (\mathbf{Q}+\mathbf{N})]\big\} \\
&\quad = [(D\mathbf{Q})\mathbf{e}_3] \cdot [\mathbf{x}_u \wedge (\mathbf{Q}+\mathbf{N})] \quad \text{on } I,
\end{aligned}$$

where we also used $\mathbf{Q} + \mathbf{N} \parallel T_{\mathbf{x}}S$. For the same reason, $\mathbf{x}_u \wedge (\mathbf{Q} + \mathbf{N})$ is normal to S along I and, as a consequence, the right hand side of the above identity vanishes. Indeed, we have

$$[D\mathbf{Q}(\mathbf{p})\mathbf{e}_3] \cdot \mathbf{n}(\mathbf{p}) = \Big[\frac{\partial}{\partial p^3}\mathbf{Q}(\mathbf{p})\Big] \cdot \mathbf{n}(\mathbf{p}) = \frac{\partial}{\partial p^3}\big[\mathbf{Q}(\mathbf{p}) \cdot \mathbf{n}(\mathbf{p})\big] = 0 \quad \text{on } S,$$

by assumption. This completes the proof of (4.8), and (4.1) is confirmed.

q.e.d.

We are now able to give the

Proof of Theorem 1. 1. According to Lemma 2 (iv), the surface normal $\mathbf{N} = (N^1, N^2, N^3)$ of $\mathbf{x}$ belongs to $C^{2,\alpha}(B^+) \cap C^{1,\alpha}(\overline{B^+} \setminus \{-1,+1\}) \cap C^0(\overline{B^+})$. In addition, the inclusion $f(\overline{B}) \subset \overline{G}$ and the $\frac{1}{R}$-convexity of G imply $N^3 > 0$ on $J \setminus \{-1,+1\}$ as was shown in [S1] Satz 2. The behaviour of the surface normal near the corner points ± 1 was studied in [M4] Theorem 5.4; the applicability of the cited result follows – after reflecting S and rotating appropriately in $\mathbb{R}^3$ – from the assumption $|(\mathbf{Q} \cdot \mathbf{n})(\mathbf{p}_j)| < \cos\alpha_j \le \cos\gamma_j$ for $j = 1, 2$, where γ_j denote the angles between Γ and S at $\mathbf{p}_j$ ($j = 1, 2$). In particular, $N^3(\pm 1)$ cannot vanish and, by continuity, we infer $N^3(\pm 1) > 0$. Consequently, the dilation $\omega := (N^3)^- = \max\{0, -N^3\} \in C^0_c(B^+ \cup I) \cap H^1_2(B^+)$ is admissible in the second variation of $A_{\mathbf{Q}}(\mathbf{x})$. Writing $\omega^2 = -\omega\, N^3$ and $|\nabla\omega|^2 = -\nabla\omega \cdot \nabla N^3$, we obtain from Lemmas 3 and 4:

$$\begin{aligned}
\delta^2 A_{\mathbf{Q}}(\mathbf{x}, \omega) &= \iint_{B^+} \{|\nabla\omega|^2 - 2q\omega^2\}\, du\, dv - \int_I \omega N^3_v\, du \\
&= -\iint_{B^+} \big\{\operatorname{div}(\omega\nabla N^3) + \omega(\Delta N^3 + 2qN^3)\big\}\, du\, dv - \int_I \omega N^3_v\, du \\
&= \iint_{B^+} \omega(\Delta N^3 + 2qN^3)\, du\, dv = -2\iint_{B^+} \omega\mathcal{H}_{p^3}(\mathbf{x})W\, du\, dv \le 0,
\end{aligned}$$

where we have applied Gauss' theorem, equation (2.14), and assumption (2.9) in the last line. The stability of $\mathbf{x}$ thus yields $\delta^2 A_{\mathbf{Q}}(\mathbf{x}, \omega) = 0$.

2. Now we choose $\xi \in C^\infty_c(B^+)$ arbitrarily. Then also $\omega + \varepsilon\xi$ is admissible in $\delta^2 A_{\mathbf{Q}}(\mathbf{x}, \cdot)$ for any $\varepsilon \in \mathbb{R}$. The function $\Xi(\varepsilon) := \delta^2 A_{\mathbf{Q}}(\mathbf{x}, \omega + \varepsilon\xi)$ depends

smoothly on $\varepsilon \in \mathbb{R}$ and satisfies $\Xi \geq 0$ as well as $\Xi(0) = 0$. Consequently, we have $\Xi'(0) = 0$, which means

$$\iint\limits_{B^+} \{\nabla\omega \cdot \nabla\xi - 2q\omega\xi\}\, du\, dv = 0 \quad \text{for any } \xi \in C_c^\infty(B^+),$$

according to formula (3.23). From $\omega = 0$ near J, we conclude $\omega \equiv 0$ by means of the weak Harnack inequality. Hence, we have $N^3 \geq 0$ in $\overline{B^+}$. Due to assumption (2.9) and equation (2.14), we further have $\Delta N^3 + 2qN^3 \leq 0$ in B^+. Therefore, Harnack's inequality, in conjunction with $N^3 > 0$ near J, yields $N^3 > 0$ in $B^+ \cup J$. Finally, we have $N^3 > 0$ on I and hence everywhere on the closed half disc $\overline{B^+}$. Indeed, if $N^3(w_0) = 0$ would be true for some point $w_0 \in I$, relation (4.1) would imply $N_v^3(w_0) = 0$. But this is impossible due to Hopf's boundary point lemma.

3. Since we have no branch points on $\partial B^+ \setminus \{-1, +1\}$ according to Lemma 2 (iii), the relation $N^3 > 0$ on ∂B^+ implies $x_u^1 x_v^2 - x_u^2 x_v^1 > 0$ on $\partial B^+ \setminus \{-1, +1\}$. Consequently, the projection $f = \pi(\mathbf{x}) = (x^1, x^2) : \overline{B^+} \to \mathbb{R}^2$ maps ∂B^+ topologically and positively oriented onto ∂G. As in [S1] Hilfssatz 7, an index argument now shows that $f : \overline{B^+} \to \overline{G}$ is a homeomorphism, $\mathbf{x}$ has no branch points in $\overline{B^+}$, and $J_f > 0$ is satisfied in $\overline{B^+} \setminus \{-1, +1\}$. By the inverse mapping theorem and the regularity of $\mathbf{x}$, the mapping $f : \overline{G} \to \overline{B^+}$ belongs to $C^2(\overline{G} \setminus \{p_1, p_2\}) \cap C^0(\overline{G})$, where we abbreviated $p_j = \pi(\mathbf{p}_j)$, $j = 1, 2$.

   Now we consider $\zeta := x^3 \circ f^{-1} \in C^2(\overline{G} \setminus \{p_1, p_2\}) \cap C^0(\overline{G})$. Since we have $(x^1, x^2, \zeta(x^1, x^2)) = \mathbf{x} \circ f^{-1}(x^1, x^2)$, ζ is the desired graph representation over $\overline{G}$ satisfying the differential equation (2.11) and the second boundary condition in (2.12). In addition, we compute

$$\begin{aligned} \psi(\mathbf{x}) &= \mathbf{Q}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \overset{(2.17)}{=} -\mathbf{N} \cdot \mathbf{n}(\mathbf{x}) \\ &= \frac{1}{\sqrt{1 + |\nabla\zeta|^2}} (\zeta_{x^1}, \zeta_{x^2}, -1) \cdot (\nu(\mathbf{x}), 0) \\ &= \frac{\nabla\zeta \cdot \nu(\mathbf{x})}{\sqrt{1 + |\nabla\zeta|^2}}, \quad \mathbf{x} = (x^1, x^2, \zeta(x^1, x^2)), \quad (x^1, x^2) \in \Sigma. \end{aligned}$$

   Hence, ζ is a solution of the boundary value problem (2.11), (2.12), and standard elliptic theory yields $\zeta \in C^{3,\alpha}(G) \cap C^{2,\alpha}(\overline{G} \setminus \{p_1, p_2\})$ according to the regularity assumptions on $\mathbf{Q}$, $\mathcal{H}$, S, and Γ. This completes the proof.

□

We finally give an example of how to apply Theorem 1 to the existence question for the mixed boundary value problem (2.11), (2.12).

Corollary 2. *Let $G \subset B_R := \{(x^1, x^2) \in \mathbb{R}^2 : |(x^1, x^2)| < R\}$ be a $\frac{1}{R}$-convex domain with boundary $\partial G = \underline{\Gamma} \cup \Sigma$, where $\underline{\Gamma}, \Sigma \in C^3$ are closed Jordan arcs, which satisfy $\underline{\Gamma} \cap \Sigma = \{\pi_1, \pi_2\}$ and which meet with interior angles $\alpha_j \in (0, \frac{\pi}{2}]$ w.r.t. G at the distinct points π_j $(j = 1, 2)$. In addition, assume that Σ can be written as a graph*

$$\Sigma = \big\{(x^1, x^2)) \in \mathbb{R}^2 : x^2 = g(x^1),\ a \leq x^1 \leq b\big\}, \qquad -R < a < b < R,$$

with some function $g \in C^3([-R,R])$. Moreover, let $\mathcal{H} \in C^{1,\alpha}(\overline{B_R})$, $\psi \in C^{1,\alpha}(\Sigma)$ and $\gamma \in C^3(\underline{\Gamma})$ be given functions and abbreviate $h_0 := \sup_{B_R} |H|$, $\psi_0 := \sup_\Sigma |\psi|$, $g_0 := \sup_{[-R,R]} |g'|$. Finally, suppose the conditions

$$4Rh_0 + \psi_0\sqrt{1+g_0^2} < 1, \qquad |\psi(\pi_j)| < \cos\alpha_j, \quad j = 1,2, \tag{4.9}$$

to be satisfied. Then, the boundary value problem (2.11), (2.12) has a unique solution $\zeta \in C^{3,\alpha}(G) \cap C^{2,\alpha}(\overline{G} \setminus \{\pi_1, \pi_2\}) \cap C^0(\overline{G})$.

Remark 3. *Note that the prescribed mean curvature function $\mathcal{H}$ in Corollary 2 does not depend on the hight p^3. If one wants to allow such a dependence, one has to use estimates for the length of the free trace as given in [M2]; see [M3] sec. 6 for a description of the required arguments.*

Proof of Corollary 2. We assume w.l.o.g. that the exterior normal ν w.r.t. G is given by $\nu = (1+(g')^2)^{-\frac{1}{2}}(g', -1)$ along Σ and set

$$Q_2(p^1,p^2) := 2\int_{g(p^1)}^{p^2} H(p^1,\eta)\,d\eta - \psi(p^1, g(p^1))\sqrt{1+g'(p^1)}, \quad (p^1,p^2) \in \overline{B_R}.$$

We use the notations $Z = B_R \times \mathbb{R}$, $\Gamma = \operatorname{graph}\varphi$, $S = \Sigma \times \mathbb{R}, \mathbf{n} = (\nu, 0)$, ... from above and set $\mathbf{Q}(\mathbf{p}) := (0, Q_2(p^1,p^2), 0)$ for $\mathbf{p} = (p^1,p^2,p^3) \in \overline{Z}$. Then, $\mathbf{Q}$ belongs to $C^{1,\alpha}(\overline{Z}, \mathbb{R}^3)$ and satisfies

$$\operatorname{div}\mathbf{Q} = Q_{2,p^2} = 2\mathcal{H} \quad \text{in } \overline{Z}, \qquad \mathbf{Q}\cdot\mathbf{n} = \psi \quad \text{on } \Sigma.$$

In addition, $\mathbf{Q}$ fulfills relations (2.10) and $\sup_Z |Q| < 1$, according to our assumtions (4.9). Consequently, the preconditions of Theorem 1 and Corollary 1 are satisfied. The graph representation of the existing (and unique) stable $\mathcal{H}$-surface $\mathbf{x} \in \mathcal{C}_\mu(\Gamma, S, \overline{Z})$ yields the desired solution of (2.11), (2.12). □

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