

Existence Theorems in the Geometrically Non-linear 6-Parameter Theory of Elastic Plates

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Received: 25 April 2012
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Abstract In this paper we show the existence of global minimizers for the geometrically non-linear equations of elastic plates, in the framework of the general 6-parameter shell theory. A characteristic feature of this model for shells is the appearance of two independent kinematic fields: the translation vector field and the rotation tensor field (representing in total 6 independent scalar kinematic variables). For isotropic plates, we prove the existence theorem by applying the direct methods of the calculus of variations. Then, we generalize our existence result to the case of anisotropic plates.

Keywords Elastic plates · Geometrically non-linear plates · Shells · Existence of minimizers · 6-parameter shell theory · Cosserat plate

Mathematics Subject Classification (2010) 74K20 · 74K25 · 74G65 · 74G25

1 Introduction

In this paper we present an existence theorem for the geometrically non-linear equations of elastic plates, in the framework of the 6-parameter shell theory.

The general (6-parameter) non-linear theory of shells, originally proposed by Reissner [37], has been considerably developed in the last 30 years. This theory and most of the

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results in the field have been presented in the books of Libai and Simmonds [24] and recently Chróścielewski, Makowski and Pietraszkiewicz [10]. The model is based on a dimension-reduction procedure of the three-dimensional formulation of the problem to the two-dimensional one, and is expressed through stress resultants and work-averaged deformation fields defined on the shell base surface. Thus, the local equilibrium equations for shells are derived by an exact through-the-thickness integration of the three-dimensional independent balance laws for linear momentum and angular momentum. The deformation of the shell is then characterized by two independent kinematic fields: the translation (displacement) vector and the rotation tensor. The appearance of the rotation tensor as an independent kinematic field variable is one of the most characteristic features of this general shell theory. In this respect, we mention that the kinematical structure of the non-linear 6-parameter shell theory is identical to that of the Cosserat shell model (i.e., the material surface with a triad of rigidly rotating directors attached to any point) proposed initially by the Cosserat brothers [14] and developed subsequently by Zhilin [42], Zubov [43], Altenbach and Zhilin [3], Eremeyev and Zubov [20], Bîrsan and Altenbach [7], among others. A related Cosserat shell-model has been established recently by Neff [27, 29] using the so-called derivation approach.

The subject of derivation and justification of the non-linear, geometrically exact equations for plates and shells has been treated in many works, and the existence of solutions has been investigated using a variety of methods, such as the method of formal asymptotic expansions or the Γ -convergence analysis, see, e.g., [1, 2, 21, 34, 35, 39–41]. For an extensive treatment of this topic, as well as many bibliographic references, we refer to the books of Ciarlet [12, 13]. To the best of our knowledge, one cannot find in the literature any existence theorem for the non-linear 6-parameter theory of plates or shells as developed in [10, 24]. In our work, we describe the geometrically non-linear equations of elastic plates as a two-field minimization problem of the total potential energy and we prove the existence of minimizers by applying the direct methods of the calculus of variations. The first result concerning the existence of minimizers for a geometrically exact (6-parameter) Cosserat plate model has been presented by the second author in [27, 29]. Due to differences in notation, this result has not been much noticed. In this paper, we employ similar techniques as in [27] and adapt the existence proof to the general 6-parameter plate equations.

In the framework of the linearized 6-parameter theory, the existence of weak solutions for micropolar elastic shells has been proved recently in [16]. We mention that the kinematic structure of the general 6-parameter shell theory differs from that of the so-called Cosserat surfaces, i.e., material surfaces with one or more deformable directors [4, 22, 26]. In particular, the kinematics of Cosserat surfaces with only one deformable director [26, 38] leaves indefinite the drilling rotation about the director, while the general 6-parameter shell theory is able to take into account such drilling rotation. For the linear theory of Cosserat surfaces, the existence theorems have been established in [5, 6, 15].

Here is the outline of our paper: In Sect. 2 we briefly review the field equations of the non-linear 6-parameter plate theory. Then, in Sect. 3 we prove the existence theorem for isotropic plates. The generalization of the existence result for anisotropic plates is presented in Sect. 4. We also show that the existence theorem remains valid in the case of some alternative relaxed boundary conditions for the rotation field. We present a comparison between the non-linear 6-parameter plate and the Cosserat plate model proposed and investigated by the second author in [27, 29]. Although this Cosserat model for plates has been obtained independently by a formal dimensional reduction of a finite-strain three-dimensional micropolar model (see also [31]), the kinematical variables and the strain measures of the two models essentially coincide. Moreover, the expressions of the elastic strain energies become identical, provided

we make a suitable identification of the constitutive coefficients for isotropic plates in the two approaches.

The linearized version of this model has also been investigated in [27–30, 32] and its relations to the Reissner–Mindlin, Kirchhoff–Love and other classical plate models have been discussed.

2 Basic Equations of Geometrically Exact Elastic Plates

The governing equations of the general 6-parameter non-linear theory of shells have been derived in [4, 10, 24] by direct integration of the two independent fundamental principles of continuum mechanics: the three-dimensional balance laws of linear momentum and angular momentum. In this section we summarize these equations, specialized here for the case of plates (i.e., initially flat shells). To this aim, we employ mainly the notations introduced in [10, 17, 24].

Consider an elastic plate, which is a three-dimensional body identified in the reference (undeformed) configuration with a region $\Omega = \{(x_1, x_2, z) | (x_1, x_2) \in \omega, z \in (-\frac{h}{2}, \frac{h}{2})\}$ of the Euclidean space. Here $h > 0$ is the thickness of the plate and $\omega \subset \mathbb{R}^2$ is a bounded, open domain with Lipschitz boundary $\partial\omega$. Relative to an inertial frame (O, e_i) , with e_i orthonormal vectors ($i = 1, 2, 3$), the position vector r of any point of Ω can be written as

$$r(x_\alpha, z) = x + ze_3, \quad x = x_\alpha e_\alpha, \quad (x_1, x_2) \in \omega, \quad z \in \left(-\frac{h}{2}, \frac{h}{2}\right). \tag{1}$$

Throughout this paper, we employ the usual convention of summation over repeated indices. The Latin indices i, j, \dots take the values $\{1, 2, 3\}$ and the Greek indices α, β, \dots range over the set $\{1, 2\}$.

In the deformed configuration, the base surface of the plate (shell) is represented by the position vector $y = \chi(x)$, where $\chi : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is the surface deformation mapping. Let the vector field $u(x)$ represent the translations (displacements) and the proper orthogonal tensor field $Q(x)$ designate the rotations of the shell cross-sections. Then the deformed configuration of the plate is given by (see Fig. 1)

$$y = \chi(x) = x + u(x), \quad d_i = Qe_i, \quad i = 1, 2, 3. \tag{2}$$

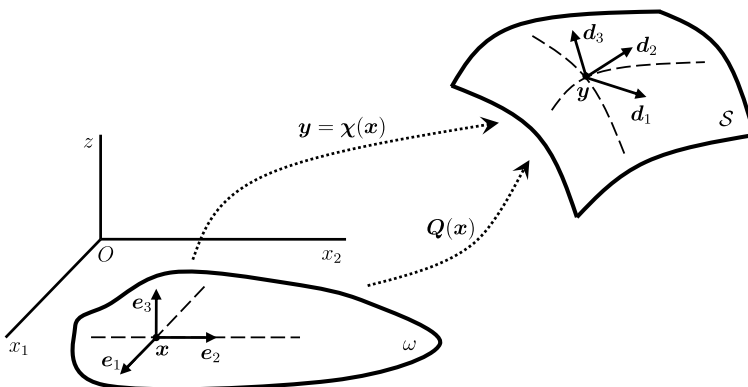


Fig. 1 The reference base surface ω of the plate and the deformed surface S , described by the surface deformation mapping $y = \chi(x)$ and the independent rotation tensor field $Q(x)$

The vectors \mathbf{d}_i introduced in (2) are three orthonormal directors attached to any point of the deformed base surface $\mathcal{S} = \chi(\omega)$. Thus, the rotation field $\mathbf{Q}(\mathbf{x}) \in SO(3)$ can be written as

$$\mathbf{Q} = \mathbf{d}_i \otimes \mathbf{e}_i. \tag{3}$$

According to the Lagrangian description, let \mathbf{f} and \mathbf{c} be the external surface resultant force and couple vectors applied at any point $\mathbf{y} \in \mathcal{S}$, but measured per unit area of ω . Also, let $\mathbf{n}_\nu = \mathbf{N}\mathbf{v}$ and $\mathbf{m}_\nu = \mathbf{M}\mathbf{v}$ be the internal contact stress and couple resultant vectors defined at an arbitrary boundary curve $\partial G \subset \mathcal{S}$, but measured per unit length of the undeformed boundary $\partial\gamma \subset \omega$. (We have denoted here by $G = \chi(\gamma)$ and \mathbf{v} is the external unit normal vector to $\partial\gamma$ lying in the plane of ω .) Here, the tensors $\mathbf{N} = N_{i\alpha}\mathbf{e}_i \otimes \mathbf{e}_\alpha$ and $\mathbf{M} = M_{i\alpha}\mathbf{e}_i \otimes \mathbf{e}_\alpha$ are the internal surface stress resultant and stress couple tensors (of the first Piola–Kirchhoff stress tensor type). Then, the local equilibrium equations are given in the form [10, 25]

$$\text{Div}_s \mathbf{N} + \mathbf{f} = \mathbf{0}, \quad \text{Div}_s \mathbf{M} + \text{axl}(\mathbf{N}\mathbf{F}^T - \mathbf{F}\mathbf{N}^T) + \mathbf{c} = \mathbf{0}, \tag{4}$$

where $\mathbf{F} = \text{Grad}_s \mathbf{y} = \mathbf{y}_{,\alpha} \otimes \mathbf{e}_\alpha$ is the surface gradient of deformation and $\text{Div}_s \mathbf{N} = N_{i\alpha,\alpha}\mathbf{e}_i$, $\text{Div}_s \mathbf{M} = M_{i\alpha,\alpha}\mathbf{e}_i$. As usual, we denote the partial derivative with respect to x_α by $(\cdot)_{,\alpha} = \frac{\partial}{\partial x_\alpha}(\cdot)$. In (4) the superscript $(\cdot)^T$ denotes the transpose and $\text{axl}(\mathbf{A})$ is the axial vector of any three-dimensional skew-symmetric tensor \mathbf{A} , given by

$$\text{axl}(\mathbf{A}) = A_{32}\mathbf{e}_1 + A_{13}\mathbf{e}_2 + A_{21}\mathbf{e}_3, \quad \text{for } \mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \quad \mathbf{A}^T = -\mathbf{A}, \tag{5}$$

such that $\mathbf{A}\mathbf{v} = \text{axl}(\mathbf{A}) \times \mathbf{v}$, for any vector \mathbf{v} . The corresponding weak form of the local balance equations has been presented in [24] Chap. VIII, or in [17].

To formulate the boundary conditions, we take a disjoint partition of the boundary curve $\partial\omega = \partial\omega_d \cup \partial\omega_f$, $\partial\omega_d \cap \partial\omega_f = \emptyset$, with $\text{length}(\partial\omega_d) > 0$. We consider the following boundary conditions [10, 36]

$$\mathbf{u} - \mathbf{u}^* = \mathbf{0}, \quad \mathbf{Q} - \mathbf{Q}^* = \mathbf{0} \quad \text{along } \partial\omega_d, \tag{6}$$

$$\mathbf{N}\mathbf{v} - \mathbf{n}^* = \mathbf{0}, \quad \mathbf{M}\mathbf{v} - \mathbf{m}^* = \mathbf{0} \quad \text{along } \partial\omega_f, \tag{7}$$

where \mathbf{n}^* and \mathbf{m}^* are the external boundary resultant force and couple vectors applied along the part $\partial\omega_f$ of the boundary $\partial\omega$. In general, from a modelling point of view, it is a difficult task to specify boundary conditions for the rotation \mathbf{Q} in (6) at the boundary $\partial\omega_d$. In [17] the following requirement is considered: in relations (6) the functions \mathbf{u}^* and \mathbf{Q}^* defined on $\partial\omega_d$ should be found from the Dirichlet boundary conditions $\mathbf{u}_{3D}(x_\alpha, z) = \mathbf{u}_{3D}^*(x_\alpha, z)$ for the three-dimensional body Ω , at any point $(x_\alpha, z) \in \partial\Omega_d = \partial\omega_d \times (-\frac{h}{2}, \frac{h}{2})$. Thus, the functions \mathbf{u}^* and \mathbf{Q}^* should be determined from the condition that the work done along $\partial\omega_d$ by the resultant stress and couple vectors $\mathbf{n}_\nu, \mathbf{m}_\nu$ on the translation \mathbf{u}^* and rotation \mathbf{Q}^* be the same as the work done along $\partial\Omega_d$ by the nominal three-dimensional stress vector $\mathbf{t}_\nu(x_\alpha, z)$ on the translation $\mathbf{u}_{3D}^*(x_\alpha, z)$.

In the general resultant theory of shells, the strain measures are the strain tensor \mathbf{E} and the bending tensor \mathbf{K} , given by [10, 17, 19]

$$\mathbf{E} = \mathbf{Q}^T(\boldsymbol{\varepsilon}_\alpha \otimes \mathbf{e}_\alpha), \quad \boldsymbol{\varepsilon}_\alpha = \mathbf{y}_{,\alpha} - \mathbf{d}_\alpha, \tag{8}$$

$$\mathbf{K} = \mathbf{Q}^T(\boldsymbol{\varkappa}_\alpha \otimes \mathbf{e}_\alpha), \quad \boldsymbol{\varkappa}_\alpha = \text{axl}(\mathbf{Q}_{,\alpha}\mathbf{Q}^T). \tag{9}$$

We mention that the kinematical structure (8), (9) of the general shell theory is identical with that of the classical version of the Cosserat shell [3, 7, 14, 20, 42]. In the case of plates,

these strain tensors can be written in component form relative to the tensor basis $\{e_i \otimes e_\alpha\}$ as

$$E = E_{i\alpha} e_i \otimes e_\alpha = (y_{,\alpha} \cdot d_i - \delta_{i\alpha}) e_i \otimes e_\alpha, \tag{10}$$

$$\begin{aligned} K &= K_{i\alpha} e_i \otimes e_\alpha = \frac{1}{2} e_{ijk} (d_{j,\alpha} \cdot d_k) e_i \otimes e_\alpha \\ &= (d_{2,\alpha} \cdot d_3) e_1 \otimes e_\alpha + (d_{3,\alpha} \cdot d_1) e_2 \otimes e_\alpha + (d_{1,\alpha} \cdot d_2) e_3 \otimes e_\alpha, \end{aligned} \tag{11}$$

where $\delta_{i\alpha}$ is the Kronecker symbol and e_{ijk} is the permutation symbol.

According to the hyperelasticity assumption, the constitutive equations for elastic plates are given in the form [10, 18, 24]

$$N = Q \frac{\partial W}{\partial E}, \quad M = Q \frac{\partial W}{\partial K}, \tag{12}$$

where

$$W = W(E, K) \tag{13}$$

is the strain energy density. Then, (4) are the Euler-Lagrange equations corresponding to the minimization problem of the total potential energy.

In order to characterize the material of the elastic plate, one has to specify the expression of the potential energy function (13). In the paper [18] the conditions for invariance of the strain energy under change of the reference placement are discussed and the local symmetry group is established. The structure of the local symmetry group puts some constraints on the form of W , which allows one to simplify the expression of W . From this representation, the strain energy density corresponding to physically linear isotropic plates is given by (see [18], Sect. 10)

$$\begin{aligned} W(E, K) &= W_{mb}(E) + W_{bend}(K), \\ 2W_{mb}(E) &= \alpha_1 \text{tr}^2 E_{\parallel} + \alpha_2 \text{tr} E_{\parallel}^2 + \alpha_3 \text{tr}(E_{\parallel}^T E_{\parallel}) + \alpha_4 \mathbf{n} E E^T \mathbf{n}, \\ 2W_{bend}(K) &= \beta_1 \text{tr}^2 K_{\parallel} + \beta_2 \text{tr} K_{\parallel}^2 + \beta_3 \text{tr}(K_{\parallel}^T K_{\parallel}) + \beta_4 \mathbf{n} K K^T \mathbf{n}, \\ E_{\parallel} &= E - (\mathbf{n} \otimes \mathbf{n}) E, \quad K_{\parallel} = K - (\mathbf{n} \otimes \mathbf{n}) K, \quad \mathbf{n} = e_3, \end{aligned} \tag{14}$$

where the coefficients α_k, β_k ($k = 1, 2, 3, 4$) are constant material parameters.

Remark 1 In the works [10, 11] the authors have employed a particular form of the expression (14) for the strain energy density in the isotropic homogeneous case, namely

$$\begin{aligned} 2W(E, K) &= C [\nu \text{tr}^2 E_{\parallel} + (1 - \nu) \text{tr}(E_{\parallel}^T E_{\parallel})] + \alpha_s C (1 - \nu) \mathbf{n} E E^T \mathbf{n} \\ &\quad + D [\nu \text{tr}^2 K_{\parallel} + (1 - \nu) \text{tr}(K_{\parallel}^T K_{\parallel})] + \alpha_t D (1 - \nu) \mathbf{n} K K^T \mathbf{n}, \end{aligned} \tag{15}$$

where the coefficients are given by

$$C = \frac{Eh}{1 - \nu^2}, \quad D = \frac{Eh^3}{12(1 - \nu^2)}, \quad \alpha_s = \frac{5}{6}, \quad \alpha_t = \frac{7}{10}. \tag{16}$$

Here E is the Young modulus, ν is the Poisson ratio, C is the stretching (membrane) stiffness of the plate, and D is the bending stiffness. The values of the two shear correction factors α_s ,

and α_t from (16) have been determined in [11] through a numerical treatment of several non-linear shell structures. We observe that the form (15) of the strain energy density W can be obtained from the more general representation (14) by choosing the following coefficients

$$\begin{aligned} \alpha_1 &= Cv = \frac{2\lambda\mu}{\lambda + 2\mu}h, & \alpha_2 &= 0, & \alpha_3 &= C(1 - \nu) = 2\mu h, \\ \alpha_4 &= \alpha_s C(1 - \nu) = 2\mu\alpha_s h, \\ \beta_1 &= Dv = \frac{\lambda\mu}{\lambda + 2\mu} \frac{h^3}{6}, & \beta_2 &= 0, & \beta_3 &= D(1 - \nu) = \frac{\mu h^3}{6}, \\ \beta_4 &= \alpha_t D(1 - \nu) = \frac{\mu\alpha_t h^3}{6}, \end{aligned} \tag{17}$$

where μ and λ are the elastic Lamé moduli of the isotropic and homogeneous material.

3 Existence Theorem for Isotropic Plates

We employ the usual notations for the Lebesgue space $L^2(\omega)$ and the Sobolev space $H^1(\omega)$, endowed with their usual norms $\|\cdot\|_{L^2(\omega)}$ and $\|\cdot\|_{H^1(\omega)}$. We denote the set of proper orthogonal tensors by $SO(3)$ and designate the set of (three-dimensional) translation vectors by \mathbb{R}^3 and the set of second-order tensors by $\mathbb{R}^{3 \times 3}$. The functional spaces of vectorial or tensorial functions will be denoted by $L^2(\omega, \mathbb{R}^3)$, $H^1(\omega, \mathbb{R}^3)$, and respectively $L^2(\omega, \mathbb{R}^{3 \times 3})$, $H^1(\omega, \mathbb{R}^{3 \times 3})$. For tensorial functions with range in $SO(3)$, we employ the notations $L^2(\omega, SO(3))$ and $H^1(\omega, SO(3))$. We also use the classical notations for the norms $\|v\| = (v \cdot v)^{1/2}$, $\forall v \in \mathbb{R}^3$, and $\|X\|^2 = \text{tr}(XX^T)$, $\forall X \in \mathbb{R}^{3 \times 3}$.

Let us define the admissible set \mathcal{A} by

$$\mathcal{A} = \{(y, Q) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, SO(3)) \mid y|_{\partial\omega_d} = y^*, Q|_{\partial\omega_d} = Q^*\}. \tag{18}$$

The boundary conditions in (18) are to be understood in the sense of traces. We assume the existence of a function $\Lambda(u, Q)$ representing the potential of the external surface loads f, c , and boundary loads n^*, m^* [17].

Consider the two-field minimization problem associated to the deformation of elastic plates: find the pair $(\hat{y}, \hat{Q}) \in \mathcal{A}$ which realizes the minimum of the functional

$$I(y, Q) = \int_{\omega} W(E, K) d\omega - \Lambda(u, Q) \quad \text{for } (y, Q) \in \mathcal{A}. \tag{19}$$

Here the strain tensor E and the bending tensor K are expressed in terms of (y, Q) by relations (8) and (9). The variational principle of total potential energy relative to the functional (19) has been presented in [17], Sect. 2.

The external loading potential $\Lambda(u, Q)$ is decomposed additively

$$\Lambda(u, Q) = \Lambda_{\omega}(u, Q) + \Lambda_{\partial\omega_f}(u, Q), \tag{20}$$

where $\Lambda_\omega(\mathbf{u}, \mathbf{Q})$ is the potential of the external surface loads \mathbf{f}, \mathbf{c} , while $\Lambda_{\partial\omega_f}(\mathbf{u}, \mathbf{Q})$ is the potential of the external boundary loads $\mathbf{n}^*, \mathbf{m}^*$, which are taken in the form

$$\begin{aligned} \Lambda_\omega(\mathbf{u}, \mathbf{Q}) &= \int_\omega \mathbf{f} \cdot \mathbf{u} d\omega + \Pi_\omega(\mathbf{Q}), \\ \Lambda_{\partial\omega_f}(\mathbf{u}, \mathbf{Q}) &= \int_{\partial\omega_f} \mathbf{n}^* \cdot \mathbf{u} ds + \Pi_{\partial\omega_f}(\mathbf{Q}). \end{aligned} \tag{21}$$

The load potential functions $\Pi_\omega : L^2(\omega, SO(3)) \rightarrow \mathbb{R}$ and $\Pi_{\partial\omega_f} : L^2(\omega, SO(3)) \rightarrow \mathbb{R}$ are assumed to be continuous and bounded operators, whose expressions are not given explicitly.

We are now able to present the main existence result concerning the deformation of isotropic elastic plates.

Theorem 2 *Assume that the external loads and the boundary data satisfy the regularity conditions*

$$\mathbf{f} \in L^2(\omega, \mathbb{R}^3), \quad \mathbf{n}^* \in L^2(\partial\omega_f, \mathbb{R}^3), \quad \mathbf{y}^* \in H^1(\omega, \mathbb{R}^3), \quad \mathbf{Q}^* \in H^1(\omega, SO(3)). \tag{22}$$

Consider the minimization problem (18), (19) for isotropic plates, i.e., when the strain energy density W is given by the relations (14). If the constitutive coefficients satisfy the conditions

$$\begin{aligned} 2\alpha_1 + \alpha_2 + \alpha_3 > 0, & \quad \alpha_2 + \alpha_3 > 0, & \quad \alpha_3 - \alpha_2 > 0, & \quad \alpha_4 > 0, \\ 2\beta_1 + \beta_2 + \beta_3 > 0, & \quad \beta_2 + \beta_3 > 0, & \quad \beta_3 - \beta_2 > 0, & \quad \beta_4 > 0, \end{aligned} \tag{23}$$

then the problem (18), (19) admits at least one minimizing solution pair $(\hat{\mathbf{y}}, \hat{\mathbf{Q}}) \in \mathcal{A}$.

Proof To prove this assertion, we apply the direct methods of the calculus of variations. First, we observe that for any $\mathbf{Q} \in SO(3)$ we have $\|\mathbf{Q}\|^2 = 3$ and, hence, $\|\mathbf{Q}\|_{L^2(\omega)}$ is bounded independent of \mathbf{Q} . In view of the conditions (22)_{1,2} and the boundedness of Π_ω and $\Pi_{\partial\omega_f}$, we derive from (20), (21) that there exist some positive constants $C_i > 0$ such that

$$\begin{aligned} |\Lambda(\mathbf{u}, \mathbf{Q})| &\leq \|\mathbf{f}\|_{L^2(\omega)} \|\mathbf{u}\|_{L^2(\omega)} + \|\mathbf{n}^*\|_{L^2(\partial\omega_f)} \|\mathbf{u}\|_{L^2(\partial\omega_f)} + |\Pi_\omega(\mathbf{Q})| + |\Pi_{\partial\omega_f}(\mathbf{Q})| \\ &\leq C_1 \|\mathbf{u}\|_{L^2(\omega)} + C_2 \|\mathbf{u}\|_{H^1(\omega)} + C_3 + C_4, \end{aligned}$$

which means that there exists a constant $C > 0$ with

$$|\Lambda(\mathbf{u}, \mathbf{Q})| \leq C(\|\mathbf{y}\|_{H^1(\omega)} + 1), \quad \forall (\mathbf{y}, \mathbf{Q}) \in \mathcal{A}. \tag{24}$$

From (14)₂ we observe that $W_{mb}(\mathbf{E})$ is a quadratic form in the strain variables $E_{i\alpha}$ ($i = 1, 2, 3; \alpha = 1, 2$) given by (10). More precisely, we may write

$$\begin{aligned} W_{mb}(\mathbf{E}) &= \tilde{W}_{str}(E_{i\alpha}) = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3)(E_{11}^2 + E_{22}^2) + \frac{1}{2}\alpha_3(E_{12}^2 + E_{21}^2) \\ &\quad + \frac{1}{2}\alpha_4(E_{31}^2 + E_{32}^2) + \alpha_1 E_{11} E_{22} + \alpha_2 E_{12} E_{21}. \end{aligned} \tag{25}$$

The quadratic form $\tilde{W}_{\text{str}}(E_{i\alpha})$ given by (25) is positive definite if and only if the conditions (23)_{1–4} on the coefficients α_k are satisfied. Then, by virtue of the relations (23)_{1–4} we infer that there exists a constant $c_1 > 0$ such that

$$\tilde{W}_{\text{str}}(E_{i\alpha}) \geq c_1 \sum_{i=1}^3 \sum_{\alpha=1}^2 E_{i\alpha}^2, \quad \forall E_{i\alpha} \in \mathbb{R},$$

or, equivalently,

$$W_{\text{mb}}(\mathbf{E}) \geq c_1 \|\mathbf{E}\|^2, \quad \forall \mathbf{E} = E_{i\alpha} \mathbf{e}_i \otimes \mathbf{e}_\alpha, \quad E_{i\alpha} \in \mathbb{R}. \tag{26}$$

Analogously, from the conditions (23)_{5–8} on the coefficients β_k we deduce that there exists a constant $\bar{c}_1 > 0$ such that

$$W_{\text{bend}}(\mathbf{K}) \geq \bar{c}_1 \|\mathbf{K}\|^2, \quad \forall \mathbf{K} = K_{i\alpha} \mathbf{e}_i \otimes \mathbf{e}_\alpha, \quad K_{i\alpha} \in \mathbb{R}. \tag{27}$$

On the other hand, in view of (8) we observe that

$$\|\mathbf{E}\|^2 = \text{tr}(\mathbf{E}\mathbf{E}^T) = \text{tr}[(\boldsymbol{\varepsilon}_\alpha \cdot \boldsymbol{\varepsilon}_\beta) \mathbf{e}_\alpha \otimes \mathbf{e}_\beta] = \boldsymbol{\varepsilon}_\alpha \cdot \boldsymbol{\varepsilon}_\alpha = \mathbf{y}_{,\alpha} \cdot \mathbf{y}_{,\alpha} - 2\mathbf{y}_{,\alpha} \cdot \mathbf{d}_\alpha + 2,$$

since $\|\mathbf{d}_\alpha\| = 1$. Then, the Cauchy-Schwarz inequality yields

$$\int_\omega \|\mathbf{E}\|^2 d\omega \geq \|\mathbf{y}_{,1}\|_{L^2(\omega)}^2 + \|\mathbf{y}_{,2}\|_{L^2(\omega)}^2 - 2\sqrt{2a}(\|\mathbf{y}_{,1}\|_{L^2(\omega)} + \|\mathbf{y}_{,2}\|_{L^2(\omega)})^{1/2} + 2a,$$

or

$$\|\mathbf{E}\|_{L^2(\omega)}^2 \geq \|\mathbf{F}\|_{L^2(\omega)}^2 - 2\sqrt{2a}\|\mathbf{F}\|_{L^2(\omega)} + 2a, \tag{28}$$

where $a = \text{area}(\omega)$ and $\mathbf{F} = \text{Grad}_s \mathbf{y} = \mathbf{y}_{,\alpha} \otimes \mathbf{e}_\alpha$ is the surface gradient of deformation introduced previously.

We show now that the functional $I(\mathbf{y}, \mathbf{Q})$ is bounded from below over \mathcal{A} . Indeed, for any $(\mathbf{y}, \mathbf{Q}) \in \mathcal{A}$ we use consecutively the inequalities (24), (26), (28) to write

$$\begin{aligned} I(\mathbf{y}, \mathbf{Q}) &\geq \int_\omega W_{\text{mb}}(\mathbf{E}) d\omega - \Lambda(\mathbf{u}, \mathbf{Q}) \geq \int_\omega c_1 \|\mathbf{E}\|^2 d\omega - C(\|\mathbf{y}\|_{H^1(\omega)} + 1) \\ &\geq c_1(\|\mathbf{F}\|_{L^2(\omega)}^2 - 2\sqrt{2a}\|\mathbf{F}\|_{L^2(\omega)} + 2a) - C(\|\mathbf{y}\|_{H^1(\omega)} + 1) \end{aligned}$$

so that

$$I(\mathbf{y}, \mathbf{Q}) \geq k_1 \|\mathbf{F}\|_{L^2(\omega)}^2 - k_2 \|\mathbf{y}\|_{H^1(\omega)} - k_3, \quad \forall (\mathbf{y}, \mathbf{Q}) \in \mathcal{A}, \tag{29}$$

for some constants k_1, k_2, k_3 with $k_1 > 0, k_2 > 0$. Using the Poincaré inequality for the field $\mathbf{y} - \mathbf{y}^* \in \mathbf{H}^1(\omega, \mathbb{R}^3)$ (with $\mathbf{y} - \mathbf{y}^* = \mathbf{0}$ on $\partial\omega_d$) in the form

$$\|\text{Grad}_s(\mathbf{y} - \mathbf{y}^*)\|_{L^2(\omega)}^2 = \|(\mathbf{y} - \mathbf{y}^*)_{,1}\|_{L^2(\omega)}^2 + \|(\mathbf{y} - \mathbf{y}^*)_{,2}\|_{L^2(\omega)}^2 \geq c_P^+ \|\mathbf{y} - \mathbf{y}^*\|_{H^1(\omega)}^2,$$

then from (29) we deduce that there exist some constants $K_1 > 0, K_2 > 0$ and K_3 such that

$$I(\mathbf{y}, \mathbf{Q}) \geq K_1 \|\mathbf{y} - \mathbf{y}^*\|_{H^1(\omega)}^2 - K_2 \|\mathbf{y} - \mathbf{y}^*\|_{H^1(\omega)} - K_3, \quad \forall (\mathbf{y}, \mathbf{Q}) \in \mathcal{A}. \tag{30}$$

Relation (30) shows that I is bounded from below over \mathcal{A} , and thus there exists an infimizing sequence $\{y^k, Q^k\}_{k=1}^\infty \subset \mathcal{A}$ with

$$\lim_{k \rightarrow \infty} I(y^k, Q^k) = \inf\{I(y, Q) | (y, Q) \in \mathcal{A}\}. \tag{31}$$

According to the hypotheses (22)_{3,4} we have $(y^*, Q^*) \in \mathcal{A}$ and $I(y^*, Q^*) < \infty$. In view of (30), we may choose the infimizing sequence with the first term $(y^1, Q^1) = (y^*, Q^*)$ and the general term (y^k, Q^k) satisfying

$$\infty > I(y^*, Q^*) \geq I(y^k, Q^k) \geq K_1 \|y^k - y^*\|_{H^1(\omega)}^2 - K_2 \|y^k - y^*\|_{H^1(\omega)} - K_3, \quad \forall k \in \mathbb{N}. \tag{32}$$

Hence, the sequence $\{y^k\}_{k=1}^\infty$ is bounded in $H^1(\omega, \mathbb{R}^3)$. Consequently, we can extract a subsequence of $\{y^k\}_{k=1}^\infty$, not relabeled, which converges weakly to an element \hat{y} in $H^1(\omega, \mathbb{R}^3)$, i.e.

$$y^k \rightharpoonup \hat{y} \quad \text{in } H^1(\omega, \mathbb{R}^3), \quad \text{for } k \rightarrow \infty, \tag{33}$$

and moreover it converges strongly in $L^2(\omega, \mathbb{R}^3)$ by Rellich’s selection principle

$$y^k \rightarrow \hat{y} \quad \text{in } L^2(\omega, \mathbb{R}^3), \quad \text{with } \hat{y} \in H^1(\omega, \mathbb{R}^3). \tag{34}$$

On the other hand, from (32), (19), and (24) it follows that the sequence $\int_\omega W_{mb}(E^k) d\omega$ is bounded independent of $k \in \mathbb{N}$. Then, from (26) we deduce that $\{E^k\}_{k=1}^\infty$ is a bounded sequence in $L^2(\omega, \mathbb{R}^{3 \times 3})$. Therefore, there exists a subsequence (not relabeled) and an element $\hat{E} \in L^2(\omega, \mathbb{R}^{3 \times 3})$ such that

$$E^k \rightharpoonup \hat{E} \quad \text{in } L^2(\omega, \mathbb{R}^{3 \times 3}), \quad \text{with } E^k = Q^{k,T} [(y^k_\alpha - Q^k e_\alpha) \otimes e_\alpha]. \tag{35}$$

Similarly, from (32) we deduce that $\int_\omega W_{bend}(K^k) d\omega$ is a bounded sequence and, in view of (27), the sequence $\{K^k\}_{k=1}^\infty$ is bounded in $L^2(\omega, \mathbb{R}^{3 \times 3})$. Taking into account (9) and (5), we observe that $\|K\|^2 = \frac{1}{2}(\|Q_{,1}\|^2 + \|Q_{,2}\|^2)$ and $\|Q\|^2 = 3$. Then, it follows that the sequences $\{Q^k_\alpha\}_{k=1}^\infty$ are bounded in $L^2(\omega, \mathbb{R}^{3 \times 3})$ and, hence, $\{Q^k\}_{k=1}^\infty$ is a bounded sequence in $H^1(\omega, \mathbb{R}^{3 \times 3})$. Consequently, there exists a subsequence (not relabeled) and an element $\hat{Q} \in H^1(\omega, \mathbb{R}^{3 \times 3})$ such that

$$Q^k \rightharpoonup \hat{Q} \quad \text{in } H^1(\omega, \mathbb{R}^{3 \times 3}), \quad \text{and } Q^k \rightarrow \hat{Q} \quad \text{in } L^2(\omega, \mathbb{R}^{3 \times 3}). \tag{36}$$

Moreover, we observe that $L^2(\omega, SO(3))$ is a closed subset of $L^2(\omega, \mathbb{R}^{3 \times 3})$. Indeed, for any sequence $\{R^k\}_{k=1}^\infty \subset L^2(\omega, SO(3))$ with $R^k \rightarrow R$ in $L^2(\omega, \mathbb{R}^{3 \times 3})$, we can write $\mathbb{1} = R^k R^{k,T} \rightarrow R R^T$ in $L^1(\omega, \mathbb{R}^{3 \times 3})$. Thus, $R R^T = \mathbb{1}$ and $R \in L^2(\omega, SO(3))$.

Consequently, from (36) and $Q^k \in L^2(\omega, SO(3))$ we obtain that $\hat{Q} \in H^1(\omega, SO(3))$.

By virtue of the boundedness of $\{K^k\}_{k=1}^\infty$ in $L^2(\omega, \mathbb{R}^{3 \times 3})$, there exists a subsequence (not relabeled) and an element $\hat{K} \in L^2(\omega, \mathbb{R}^{3 \times 3})$ such that

$$K^k \rightharpoonup \hat{K} \quad \text{in } L^2(\omega, \mathbb{R}^{3 \times 3}), \quad \text{with } K^k = Q^{k,T} [axl(Q^k_\alpha Q^{k,T}) \otimes e_\alpha]. \tag{37}$$

Concerning the (weak) limits \hat{y} , \hat{Q} , \hat{E} and \hat{K} specified by relations (33)–(37), it remains to show that they satisfy the equations

$$\hat{E} = \hat{Q}^T [(\hat{y}_\alpha - \hat{Q} e_\alpha) \otimes e_\alpha], \quad \hat{K} = \hat{Q}^T [axl(\hat{Q}_\alpha \hat{Q}^T) \otimes e_\alpha]. \tag{38}$$

Indeed, from (33) and (36)₂ it follows that

$$\mathbf{Q}^{k,T} \mathbf{y}_{,\alpha}^k \rightharpoonup \hat{\mathbf{Q}}^T \hat{\mathbf{y}}_{,\alpha} \quad \text{in } L^1(\omega, \mathbb{R}^3), \alpha = 1, 2,$$

or, equivalently,

$$\mathbf{Q}^{k,T} [(\mathbf{y}_{,\alpha}^k - \mathbf{Q}^k \mathbf{e}_\alpha) \otimes \mathbf{e}_\alpha] \rightharpoonup \hat{\mathbf{Q}}^T [(\hat{\mathbf{y}}_{,\alpha} - \hat{\mathbf{Q}} \mathbf{e}_\alpha) \otimes \mathbf{e}_\alpha] \quad \text{in } L^1(\omega, \mathbb{R}^{3 \times 3}). \tag{39}$$

Since the weak limits of $\{\mathbf{E}^k\}_{k=1}^\infty$ in the spaces $L^1(\omega, \mathbb{R}^{3 \times 3})$ and $L^2(\omega, \mathbb{R}^{3 \times 3})$ must coincide, we deduce from (35) and (39) that the relation (38)₁ holds true.

In order to prove (38)₂ we proceed analogously: from (36) we deduce that

$$\mathbf{Q}^k \mathbf{Q}_{,\alpha}^{k,T} \rightharpoonup \hat{\mathbf{Q}} \hat{\mathbf{Q}}_{,\alpha}^T \quad \text{in } L^1(\omega, \mathbb{R}^{3 \times 3}). \tag{40}$$

On the other hand, by virtue of the relations (5) and (37)₂ we can write

$$\begin{aligned} \|\mathbf{K}^k\|_{L^2(\omega)}^2 &= \|\text{axl}(\mathbf{Q}_{,\alpha}^k \mathbf{Q}^{k,T}) \otimes \mathbf{e}_\alpha\|_{L^2(\omega)}^2 = \sum_{\alpha=1}^2 \|\text{axl}(\mathbf{Q}_{,\alpha}^k \mathbf{Q}^{k,T})\|_{L^2(\omega)}^2 \\ &= \frac{1}{2} \sum_{\alpha=1}^2 \|\mathbf{Q}^k \mathbf{Q}_{,\alpha}^{k,T}\|_{L^2(\omega)}^2. \end{aligned}$$

Since $\{\mathbf{K}^k\}_{k=1}^\infty$ is bounded in $L^2(\omega, \mathbb{R}^{3 \times 3})$, we deduce from the above relations that the sequences $\{\mathbf{Q}^k \mathbf{Q}_{,\alpha}^{k,T}\}_{k=1}^\infty$ are also bounded (for $\alpha = 1, 2$) and, hence, they admit subsequences (not relabeled) weakly convergent in $L^2(\omega, \mathbb{R}^{3 \times 3})$. Taking into account (40), we deduce that these weak limits must coincide with $\hat{\mathbf{Q}} \hat{\mathbf{Q}}_{,\alpha}^T$, i.e., we have

$$\mathbf{Q}^k \mathbf{Q}_{,\alpha}^{k,T} \rightharpoonup \hat{\mathbf{Q}} \hat{\mathbf{Q}}_{,\alpha}^T \quad \text{in } L^2(\omega, \mathbb{R}^{3 \times 3}), \alpha = 1, 2,$$

and

$$\text{axl}(\mathbf{Q}_{,\alpha}^k \mathbf{Q}^{k,T}) \otimes \mathbf{e}_\alpha \rightharpoonup \text{axl}(\hat{\mathbf{Q}}_{,\alpha} \hat{\mathbf{Q}}^T) \otimes \mathbf{e}_\alpha \quad \text{in } L^2(\omega, \mathbb{R}^{3 \times 3}). \tag{41}$$

Since $\mathbf{Q}^{k,T} \rightarrow \hat{\mathbf{Q}}^T$ in $L^2(\omega, \mathbb{R}^{3 \times 3})$ according to (36)₂, we get from (41)

$$\mathbf{Q}^{k,T} [\text{axl}(\mathbf{Q}_{,\alpha}^k \mathbf{Q}^{k,T}) \otimes \mathbf{e}_\alpha] \rightharpoonup \hat{\mathbf{Q}}^T [\text{axl}(\hat{\mathbf{Q}}_{,\alpha} \hat{\mathbf{Q}}^T) \otimes \mathbf{e}_\alpha] \quad \text{in } L^1(\omega, \mathbb{R}^{3 \times 3}).$$

If we compare the last relation with (37), then we obtain that (38)₂ holds.

In the next step of the proof, we show the convexity of the strain energy density function. By virtue of the conditions on the constitutive coefficients (23), the Hessian matrix of the quadratic form $W_{\text{mb}}(\mathbf{E})$ in (25) is positive definite. Similar arguments hold for $W_{\text{bend}}(\mathbf{K})$, and altogether we obtain

$$W(\mathbf{E}, \mathbf{K}) \quad \text{is convex in } (\mathbf{E}, \mathbf{K}). \tag{42}$$

By (35), (37) and (38) we deduce $\mathbf{E}^k \rightharpoonup \hat{\mathbf{E}}$ and $\mathbf{K}^k \rightharpoonup \hat{\mathbf{K}}$ in $L^2(\omega, \mathbb{R}^{3 \times 3})$, and from (42) we find that

$$\int_\omega W(\hat{\mathbf{E}}, \hat{\mathbf{K}}) d\omega \leq \liminf_{k \rightarrow \infty} \int_\omega W(\mathbf{E}^k, \mathbf{K}^k) d\omega. \tag{43}$$

If we denote by $\mathbf{u}^k = \mathbf{y}^k - \mathbf{x}$, $\hat{\mathbf{u}} = \hat{\mathbf{y}} - \mathbf{x}$, then from (20)–(22), (34), (36)₂ and the continuity of Π_ω , $\Pi_{\partial\omega_f}$ it follows

$$\lim_{k \rightarrow \infty} \Lambda(\mathbf{u}^k, \mathbf{Q}^k) = \Lambda(\hat{\mathbf{u}}, \hat{\mathbf{Q}}). \quad (44)$$

Since the pairs $(\mathbf{y}^k, \mathbf{Q}^k)$ satisfy the boundary conditions on $\partial\omega_d$, we deduce in view of the convergence relations (33), (34), (36) and the compact embedding in the sense of traces, that $\hat{\mathbf{y}} = \mathbf{y}^*$, $\hat{\mathbf{Q}} = \mathbf{Q}^*$ on $\partial\omega_d$. Hence, we have $(\hat{\mathbf{y}}, \hat{\mathbf{Q}}) \in \mathcal{A}$.

Finally, from (31), (43) and (44) we obtain $I(\hat{\mathbf{y}}, \hat{\mathbf{Q}}) \leq \inf\{I(\mathbf{y}, \mathbf{Q}) \mid (\mathbf{y}, \mathbf{Q}) \in \mathcal{A}\}$, which means that $(\hat{\mathbf{y}}, \hat{\mathbf{Q}})$ is a minimizer of the functional I over \mathcal{A} . The proof is complete. \square

Remark 3 We observe that the minimizing solution pair $(\hat{\mathbf{y}}, \hat{\mathbf{Q}})$ belongs to $\mathbf{H}^1(\omega)$ and moreover $\hat{\mathbf{Q}} \in L^\infty(\omega, SO(3))$. Thus, the rotation field $\hat{\mathbf{Q}}$ may fail to be continuous, according to the limit case of the Sobolev embedding.

Remark 4 We notice that the conditions on the constitutive coefficients (23) are satisfied for the particular model of isotropic plates presented in Remark 1. Indeed, taking into account the identification (17), we find that the inequalities (23) reduce to

$$\mu > 0, \quad 2\mu + 3\lambda > 0.$$

These conditions are satisfied in view of the positive definiteness of the three-dimensional quadratic elastic strain energy density for isotropic materials. Thus, the existence result given by Theorem 2 applies to the particular plate model presented in [10, 11].

4 Generalization of Existence Result and Discussions

In this section we present some variants and generalizations of Theorem 2, as well as the comparison with a Cosserat model for plates obtained by a different approach [27, 29].

We observe that the boundary conditions imposed on the rotation \mathbf{Q} can be relaxed or even omitted in the definition of the admissible set (18). For a discussion of some possible alternative boundary conditions for the rotation field \mathbf{Q} on $\partial\omega_d$ we refer to the works [27, 29]. In this line of thought, we present next the existence result corresponding to a larger admissible set.

Theorem 5 Consider the minimization problem (19), over the admissible set

$$\tilde{\mathcal{A}} = \{(\mathbf{y}, \mathbf{Q}) \in \mathbf{H}^1(\omega, \mathbb{R}^3) \times \mathbf{H}^1(\omega, SO(3)) \mid \mathbf{y}|_{\partial\omega_d} = \mathbf{y}^*\}. \quad (45)$$

If the external loads \mathbf{f} , \mathbf{n}^* and the boundary data \mathbf{y}^* satisfy the conditions (22)_{1–3} and the constitutive coefficients α_k, β_k verify the inequalities (23), then the minimization problem (19), (45) admits at least one minimizing solution pair $(\hat{\mathbf{y}}, \hat{\mathbf{Q}}) \in \tilde{\mathcal{A}}$.

Proof The proof can be achieved in a similar manner as the proof of Theorem 2, where the boundary condition $\mathbf{Q} = \mathbf{Q}^*$ on $\partial\omega_d$ has not played an important role. \square

The Theorem 2 is concerned with isotropic plates for which the strain energy density $W(\mathbf{E}, \mathbf{K})$ is given by relations (14). We can generalize this existence result to the case of anisotropic non-linear plates, provided the function W satisfies the conditions of convexity and coercivity:

Theorem 6 (Anisotropic plates) *Consider the minimization problem (18), (19) associated to the deformation of anisotropic plates, and assume that the external loads and boundary data satisfy the conditions (22). Assume that the strain energy density $W(\mathbf{E}, \mathbf{K})$ is an arbitrary quadratic convex function in (\mathbf{E}, \mathbf{K}) , and moreover W is coercive, in the sense that*

$$W(\mathbf{E}, \mathbf{K}) \geq k(\|\mathbf{E}\|^2 + \|\mathbf{K}\|^2), \quad \forall \mathbf{E} = E_{i\alpha} \mathbf{e}_i \otimes \mathbf{e}_\alpha, \mathbf{K} = K_{i\alpha} \mathbf{e}_i \otimes \mathbf{e}_\alpha, E_{i\alpha}, K_{i\alpha} \in \mathbb{R}, \quad (46)$$

for some constant $k > 0$. Then, the minimization problem (18), (19) admits at least one minimizing solution pair $(\hat{\mathbf{y}}, \hat{\mathbf{Q}}) \in \mathcal{A}$.

Proof We follow the same steps as in the proof of Theorem 2. In view of (46), we show first that the estimate (30) holds also in our case. Then, there exists an infimizing sequence $\{\mathbf{y}^k, \mathbf{Q}^k\}_{k=1}^\infty \subset \mathcal{A}$, and we can prove similarly that it verifies the relations (31)–(38). By virtue of our hypothesis, W is convex in (\mathbf{E}, \mathbf{K}) , so that the properties (42)–(44) are satisfied, and we can reach the conclusion of the theorem. \square

The Theorem 6 remains valid also for the minimization problem written over the larger admissible set $\tilde{\mathcal{A}}$ given by (45), instead of the admissible set (18).

Remark 7 The 6-parameter theory of shells can be used to model also composite thin elastic structures. In this case, the internal energy density has a more complicated structure, and exhibits multiplicative coupling of the strain tensor \mathbf{E} with the bending tensor \mathbf{K} , see, e.g., [9, 23]. Nevertheless, the Theorem 6 can be applied to deduce the existence of minimizers for layered composite plates, under appropriate conditions on the material/geometrical parameters [8].

The Cosserat model for plates (planar shells) proposed and investigated by the second author in [27, 29] is obtained by a consistent formal dimensional reduction of a finite-strain three-dimensional Cosserat (micropolar) model. Apart from the differences in notations, there are essential similarities between this Cosserat plate model and the 6-parameter plate theory presented in Sect. 1. Firstly, in both approaches the primary independent kinematical variables are the deformation field $\mathbf{y} \in \mathbb{R}^3$ and the rotation tensor field $\mathbf{Q} \in SO(3)$. The rotation field \mathbf{Q} in the derivation approach is inherited from the parent Cosserat bulk model, which already includes a triad of rigidly rotating directors [33]. Moreover, the measures of strain are essentially the same in the two approaches. In particular, the so-called *stretch tensor* $\bar{\mathbf{U}}$ is introduced in [27] through the relation $\bar{\mathbf{U}} = \mathbf{Q}^T (\mathbf{y}_{,\alpha} \otimes \mathbf{e}_\alpha + \mathbf{Q} \mathbf{e}_3 \otimes \mathbf{e}_3)$. Comparing $\bar{\mathbf{U}}$ with the strain measure \mathbf{E} defined in (8), we find that $\mathbf{E} = \bar{\mathbf{U}} - \mathbf{1}$.

Let us compare now the expressions of the strain energy density for isotropic plates. With a suitable identification of parameters, the expressions of the strain energy density in the two approaches coincide, in terms of the independent kinematical variables (\mathbf{y}, \mathbf{Q}) . The membrane part of the strain energy density is assumed in the Cosserat plate model in the form

$$\begin{aligned} W_{\text{mb}}(\mathbf{E}) &= W_{\text{mb}}(\bar{\mathbf{U}} - \mathbf{1}) \\ &= h \left[\mu \|\text{sym}(\bar{\mathbf{U}} - \mathbf{1})\|^2 + \mu_c \|\text{skew}(\bar{\mathbf{U}} - \mathbf{1})\|^2 + \frac{\lambda\mu}{\lambda + 2\mu} (\text{tr}(\bar{\mathbf{U}} - \mathbf{1}))^2 \right], \quad (47) \end{aligned}$$

where $\mu_c \geq 0$ is the Cosserat couple modulus. To realize the coincidence of the two strain energy functions $(14)_2$ and (47), we need to identify the set of constitutive coefficients

$(\alpha_1, \dots, \alpha_4)$ from (14)₂ with the parameters (λ, μ, μ_c) in the following way

$$\alpha_1 = h \frac{2\lambda\mu}{\lambda + 2\mu}, \quad \alpha_2 = h(\mu - \mu_c), \quad \alpha_3 = h(\mu + \mu_c), \quad \alpha_4 = \kappa h(\mu + \mu_c), \quad (48)$$

where κ is the formal shear correction factor. We observe that $\alpha_3 - \alpha_2 = 2h\mu_c$. Thus, the requirement $\alpha_3 - \alpha_2 > 0$ assumed by the hypotheses (23) of Theorem 2 corresponds to the condition $\mu_c > 0$. In the interesting degenerate case when $\mu_c = 0$ (i.e., $\alpha_3 - \alpha_2 = 0$) the energy function W is only positive semi-definite, and the proof of the existence results is more delicate, see [29].

In a future contribution we will extend our results to the general case of 6-parameter shells.

Acknowledgements The first author (M.B.) is supported by the German state grant: “Programm des Bundes und der Länder für bessere Studienbedingungen und mehr Qualität in der Lehre”. Useful discussions with Professor V.A. Eremeyev are gratefully acknowledged.

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