Analytical investigation for the finite-strain Cosserat thin plate model with size effects. Bachelor Thesis

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Abstract

We consider the three-dimensional Cosserat model and two-dimensional Cosserat thin plate model provided by Neff for a specific case. In order to find minimizers of the energy function, we derive the corresponding Euler-Lagrange equations. That is done subject to a specific situation treated. After the derivation, we consider the pure bending problem by skipping the membrane factor. In this fashion, we obtain a simplified system of differential equations. Finally, a simple solution is found for the simplified system.

Key words: plates, shells, energy minimization
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1 The underlying three-dimensional Cosserat model

In [?] a finite-strain, fully frame-indifferent, three-dimensional Cosserat micropolar model is introduced. The **two-field** problem has been posed in a variational setting. The task is to find a pair $(\varphi, \overline{R}) : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3 \times \mathrm{SO}(3,\mathbb{R})$ of deformation φ and **independent microrotation** $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$ minimizing the energy functional I,

$$I(\varphi, \overline{R}) = \int_\Omega W_{\mathrm{mp}}(\overline{R}^T \nabla\varphi) + W_{\mathrm{curv}}(\overline{R}^T \mathrm{D_x}\overline{R}) - \Pi_f(\varphi) - \Pi_M(\overline{R})\,\mathrm{dV} \qquad (1.1)$$

$$- \int_{\Gamma_S} \Pi_N(\varphi)\,\mathrm{dS} - \int_{\Gamma_C} \Pi_{M_c}(\overline{R})\,\mathrm{dS} \mapsto \min.\ \text{w.r.t.}\ (\varphi, \overline{R}),$$

together with the Dirichlet boundary condition of place for the deformation φ on Γ: $\varphi_{|\Gamma} = g_{\mathrm{d}}$ and three possible **alternative** boundary conditions for the microrotations $\overline{R}$ on Γ,

$$\overline{R}_{|\Gamma} = \begin{cases} \overline{R}_{\mathrm{d}}\,, & (a)\\ \mathrm{polar}(\nabla\varphi)\,, & (b)\\ \text{no condition for } \overline{R} \text{ on } \Gamma, & (c) \end{cases} \qquad (1.2)$$

(a) : the case of **rigid** prescription

(b) : the case of **strong consistent coupling**

(c) : **induced Neumann-type** relations for $\overline{R}$ on Γ.

The constitutive assumptions on the densities are

$$W_{\mathrm{mp}}(\overline{U}) = \mu\,\|\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c\,\|\mathrm{skew}(\overline{U})\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2, \qquad (1.3)$$

$$\overline{U} = \overline{R}^T F\,,\ F = \nabla\varphi\,,$$

$$W_{\mathrm{curv}}(\mathfrak{K}) = \mu\frac{L_c^{1+p}}{12}\,\left(1 + \alpha_4\,L_c^q\,\|\mathfrak{K}\|^q\right)\,\left(\alpha_5\,\|\mathrm{sym}\,\mathfrak{K}\|^2 + \alpha_6\,\|\mathrm{skew}\,\mathfrak{K}\|^2 + \alpha_7\,\mathrm{tr}\,[\mathfrak{K}]^2\right)^{\frac{1+p}{2}},$$

$$\mathfrak{K} = \overline{R}^T \mathrm{D_x}\overline{R} := \left(\overline{R}^T\nabla(\overline{R}.e_1), \overline{R}^T\nabla(\overline{R}.e_2), \overline{R}^T\nabla(\overline{R}.e_3)\right),$$

the third order **curvature tensor**

under the minimal requirement $p \geq 1$, $q \geq 0$. The total elastically stored energy $W = W_{\mathrm{mp}} + W_{\mathrm{curv}}$ is quadratic in the stretch $\overline{U}$ and possibly super-quadratic in the curvature $\mathfrak{K}$. The strain energy W_{mp} depends on the deformation gradient $F = \nabla\varphi$ and the microrotations $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$, which do not necessarily coincide with the **continuum rotations** $R = \mathrm{polar}(F)$. The curvature energy W_{curv} depends moreover on the space derivatives $\mathrm{D_x}\overline{R}$ which describe the self-interaction of the microstructure.[1] In general, the **micropolar stretch tensor** $\overline{U}$ is **not symmetric** and does not coincide with the **symmetric continuum stretch** tensor $U = R^T F = \sqrt{F^T F}$. By abuse of notation we set $\|\mathrm{sym}\,\mathfrak{K}\|^2 := \sum_{i=1}^3 \|\mathrm{sym}\,\mathfrak{K}^i\|^2$

[1] Observe that $\overline{R}^T\nabla(\overline{R}.e_i) \neq \overline{R}^T\partial_{x_i}\overline{R} \in \mathfrak{so}(3,\mathbb{R})$.

for third order tensors $\mathfrak{K}$, cf.(??). Here $\Omega \subset \mathbb{R}^3$ is an open domain with boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ is that part of the boundary, where Dirichlet conditions $g_{\rm d}, \overline{R}_{\rm d}$ for deformations and microrotations or coupling conditions for microrotations, are prescribed. $\Gamma_S \subset \partial\Omega$ is a part of the boundary, where traction boundary conditions in the form of the potential of applied surface forces Π_N are given with $\Gamma \cap \Gamma_S = \emptyset$. In addition, $\Gamma_C \subset \partial\Omega$ is the part of the boundary where the potential of external surface couples Π_{M_c} are applied with $\Gamma \cap \Gamma_C = \emptyset$. On the free boundary $\partial\Omega \setminus \{\Gamma \cup \Gamma_S \cup \Gamma_C\}$ corresponding natural boundary conditions for $(\varphi, \overline{R})$ apply. The potential of the external applied volume force is Π_f and Π_M takes on the role of the potential of applied external volume couples. For simplicity we assume

$$
\begin{aligned}
\Pi_f(\varphi) &= \langle f, \varphi \rangle\,, \quad \Pi_M(\overline{R}) = \langle M, \overline{R} \rangle\,,\\
\Pi_N(\varphi) &= \langle N, \varphi \rangle\,, \quad \Pi_{M_c}(\overline{R}) = \langle M_c, \overline{R} \rangle\,,
\end{aligned} \tag{1.4}
$$

for the potentials of applied loads with given functions $f \in L^2(\Omega, \mathbb{R}^3)$, $M \in L^2(\Omega, \mathbb{M}^{3\times3})$, $N \in L^2(\Gamma_S, \mathbb{R}^3)$, $M_c \in L^2(\Gamma_C, \mathbb{M}^{3\times3})$. The parameters μ, $\lambda > 0$ are the Lamé constants of classical isotropic elasticity, the additional parameter $\mu_c \geq 0$ is called the **Cosserat couple modulus**. For $\mu_c > 0$ the elastic strain energy density $W_{\rm mp}(\overline{U})$ is **uniformly convex** in $\overline{U}$. Moreover $\forall\, F \in \mathrm{GL}^+(3, \mathbb{R})$

$$
\begin{aligned}
W_{\rm mp}(\overline{U}) &= W_{\rm mp}(\overline{R}^T F) \geq \min(\mu, \mu_c)\, \|\overline{R}^T F - 1\!\!1\|^2 = \min(\mu, \mu_c)\, \|F - \overline{R}\|^2\\
&\geq \min(\mu, \mu_c) \inf_{R \in \mathrm{O}(3,\mathbb{R})} \|F - R\|^2 = \min(\mu, \mu_c)\, \mathrm{dist}^2(F, \mathrm{O}(3, \mathbb{R}))\\
&= \min(\mu, \mu_c)\, \mathrm{dist}^2(F, \mathrm{SO}(3, \mathbb{R})) = \min(\mu, \mu_c)\, \|F - \mathrm{polar}(F)\|^2\\
&= \min(\mu, \mu_c)\, \|U - 1\!\!1\|^2\,.
\end{aligned} \tag{1.5}
$$

In contrast, for the interesting case $\mu_c = 0$ the strain energy density is **only convex** w.r.t. F and does not satisfy (1.5).[2] The parameter $L_c > 0$ (with dimension length) introduces an **internal length** which is **characteristic** for the material, e.g. related to the grain size in a polycrystal. The internal length $L_c > 0$ is responsible for **size effects** in the sense that smaller samples are relatively stiffer than larger samples. We assume throughout that $\alpha_4, \alpha_5, \alpha_6 > 0, \alpha_7 \geq 0$. This implies the **coercivity of curvature**

$$
\exists\, c^+ > 0 \quad \forall\, \mathfrak{K} \in \mathfrak{T}(3): \quad W_{\rm curv}(\mathfrak{K}) \geq c^+\, \|\mathfrak{K}\|^{1+p+q}\,, \tag{1.6}
$$

which will be a basic ingredient of the mathematical analysis. The non-standard boundary condition of **strong consistent coupling** ensures that no unwanted non-classical, polar effects may occur at the Dirichlet boundary Γ. It implies for the micropolar stretch that $\overline{U}_{|\Gamma} \in \mathrm{Sym}$ and for the second Piola-Kirchhoff stress tensor $S_2 := F^{-1} D_F W_{\rm mp}(\overline{U}) \in \mathrm{Sym}$ on Γ as in the classical, non-polar case. We refer to the weaker boundary condition $\overline{U}_{|\Gamma} \in \mathrm{Sym}$ as **weak consistent coupling**. It is of prime importance to realize that a linearization of this Cosserat bulk model with $\mu_c = 0$ for small displacement and small microrotations completely decouples the two fields of deformation φ and microrotations $\overline{R}$ and leads

[2] The condition $F \in \mathrm{GL}^+(3, \mathbb{R})$ is necessary, otherwise $\|F - \mathrm{polar}(F)\|^2 = \mathrm{dist}^2(F, \mathrm{O}(3, \mathbb{R})) < \mathrm{dist}^2(F, \mathrm{SO}(3, \mathbb{R}))$, as can be easily seen for the reflection $F = \mathrm{diag}(1, -1, 1)$.

to the classical linear elasticity problem for the deformation.[3] For more details on the modelling of the three-dimensional Cosserat model we refer the reader to the original work of Dr.Neff.

2 The Cosserat thin plate model

Now we introduce a two-dimensional minimization problem for the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the microrotation of the plate (shell) $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathrm{SO}(3,\mathbb{R})$ solving on ω:

$$I = \int_\omega h \cdot W_{\mathrm{mp}}(\overline{U}) + h \cdot W_{\mathrm{curv}}(\mathfrak{K}_{\mathfrak{s}}) + \frac{h^3}{12} \cdot W_{\mathrm{bend}}(\mathfrak{K}_{\mathfrak{b}})\, d\omega \quad \mapsto \min. \tag{2.7}$$

$$\text{w.r.t. } (\mathrm{m}, \overline{\mathrm{R}}), \quad \omega \subset \mathbb{R}^2.$$

$$\overline{U} = \overline{R}^T \hat{F}, \; \hat{F} = (\nabla m | \overline{R}_3), \quad \text{(reconstructed deformation gradient)},$$

$$W_{\mathrm{mp}}(\overline{U}) = \mu \| \operatorname{sym}(\overline{U} - 1\!\!1) \|^2 + \mu_c \| \operatorname{skew}(\overline{U}) \|^2 + \frac{\mu\lambda}{2\mu + \lambda} \cdot \operatorname{tr}(\operatorname{sym}(\overline{U} - 1\!\!1))^2 ,$$

$$W_{\mathrm{curv}}(\mathfrak{K}_{\mathfrak{s}}) = \mu \frac{L_c^{1+p}}{12} (1 + \alpha_4 L_c^q \| \mathfrak{K}_{\mathfrak{s}} \|^q) (\alpha_5 \| \operatorname{sym} \mathfrak{K}_{\mathfrak{s}} \|^2 + \alpha_6 \| \operatorname{skew}(\mathfrak{K}_{\mathfrak{s}}) \|^2 + \alpha_7 \operatorname{tr}(\mathfrak{K}_{\mathfrak{s}})^2)^{\frac{1+p}{2}} ,$$

$$\mathfrak{K}_{\mathfrak{s}} = \left(\overline{R}^T (\nabla(\overline{R} e_1) | 0), \overline{R}^T (\nabla(\overline{R} e_2) | 0), \overline{R}^T (\nabla(\overline{R} e_3) 0) \right) ,$$

(reduced third order curvature tensor),

$$W_{\mathrm{bend}}(\mathfrak{K}_{\mathfrak{b}}) = \mu \| \operatorname{sym}(\mathfrak{K}_{\mathfrak{b}}) \|^2 + \mu_c \| \operatorname{skew}(\mathfrak{K}_{\mathfrak{b}}) \|^2 + \frac{\mu\lambda}{2\mu + \lambda} \cdot \operatorname{tr}(\operatorname{sym}(\mathfrak{K}_{\mathfrak{b}}))^2 ,$$

$$\mathfrak{K}_{\mathfrak{b}} = \overline{R}^T (\nabla \overline{R}_3 | 0) = \mathfrak{K}_3^{\,3}, \quad \text{(second order , non-symmetric bending tensor)}.$$

Here e_i denote the standard Euclidian basis vectors. This problem has been derived by Neff based on (1.7) by a formal asymptotic ansatz. We want to treat this problem mathematically in a simplified setting.

3 Basic theorems of the calculus of variations

In order to construct a system of differential equations with Dirichlet boundary conditions from the minimization problem stated here above, we introduce the fundamental lemma of the calculus of variations and the derivation of the corresponding Euler-Lagrange equation.

Theorem 3.1 (The Euler-Lagrange equations)
Let $F \in C^2(\mathbb{R}^3, \mathbb{R})$ be given and consider the minimization problem,

$$\int_a^b F(x, y, y')\, \mathrm{dx} \quad \mapsto \min. \quad \text{w.r.t.} \quad y, \quad y(a) = \alpha, \quad y(b) = \beta. \tag{3.8}$$

[3]Thinking in the context of an infinitesimal-displacement Cosserat theory one might erroneously believe that $\mu_c > 0$ is strictly necessary also for a "true" finite-strain Cosserat theory.

If y is a solution of (3.8), it is necessary that the following **Euler-Lagrange** *equations hold,*

$$\frac{d}{\mathrm{dx}} F_3(x, y(x), y'(x)) = F_2(x, y(x), y'(x)). \tag{3.9}$$

(Here F_2 and F_3 denote partial derivatives w.r.t. the corresponding slot).

Proof. Let us consider a test function $u \in C^1[a,b]$ with $u(a) = u(b) = 0$. Assume that y is a solution of the minimization problem. For all $\theta \in \mathbb{R}$, the function $y + \theta u$ is a competing function with satisfies the boundary condition. Hence, the stationarity condition is

$$\frac{d}{d\theta}_{|\theta=0} \int_a^b F(x, y(x) + \theta u(x), y'(x) + \theta u'(x))\,\mathrm{dx} = 0\,. \tag{3.10}$$

Since we deal with smooth functions, we may take the derivative $\frac{d}{d\theta}$ inside and obtain

$$\int_a^b \frac{d}{d\theta} F(x, y(x) + \theta u(x), y'(x) + \theta u'(x))\,\mathrm{dx} = 0\,, \tag{3.11}$$

which for $\theta = 0$ turns into

$$\int_a^b (F_2 \cdot u + F_3 \cdot u')\,\mathrm{dx} = 0\,. \tag{3.12}$$

The second term can be integrated by parts. The result is

$$\int_a^b \left(F_2(x, y(x), y'(x)) - \frac{d}{dx} F_3(x, y(x), y'(x)) \right) u(x)\,\mathrm{dx} = 0\,, \tag{3.13}$$

where we used the zero boundary values for u. By invoking the following lemma, the proof is finished.

Lemma 3.2 (The fundamental lemma of the calculus of variations)
If v is piecewise continuous on $[a,b]$ and if $\int_a^b u(x)v(x)dx = 0$ for every u in $C^1[a,b]$ that vanishes at the endpoints a and b, then $v = 0$.

Proof. We proceed by contradiction and assume that $v \neq 0$. Then there is a nonempty open interval (α, β) contained in $[a,b]$ in which v is continuous and has no zero. Without loss of generality, we may assume that $v(x) > 0$ on (α, β). We can find easily a function u in $C^1[a,b]$ such that $u(x) > 0$ on (α, β) and $u(x) = 0$ elsewhere in $[a,b]$. Since $\int_a^b uv\,\mathrm{dx} = \int_\alpha^\beta uv\,\mathrm{dx} > 0$, we have a contradiction, hence $v = 0$.

4 Calculation of building blocks of the minimization problem

In the following section all necessary building blocks for the minimization problem will be calculated and the integration involved in (2.7) will be reassembled by those blocks.

First of all we consider the problem (2.7) in a very special setting. We assume a basically one-dimensional situation (bending in one plane) such that we can assume,

$$m : \mathbb{R}^2 \mapsto \mathbb{R}^3, \quad \overline{R} : \mathbb{R}^2 \mapsto \mathrm{SO}(3,\mathbb{R}), \tag{4.14}$$

to be given by

$$m(x,y) = \begin{pmatrix} m_1(x) \\ y \\ m_3(x) \end{pmatrix}, \quad \overline{R}(x,y) = \begin{pmatrix} \cos\overline{\alpha}(x) & 0 & \sin\overline{\alpha}(x) \\ 0 & 1 & 0 \\ -\sin\overline{\alpha}(x) & 0 & \cos\overline{\alpha}(x) \end{pmatrix}, \tag{4.15}$$

where $\overline{\alpha}$ is a continuous differentiable function from $\mathbb{R}$ to $\mathbb{R}$. Looking at W_{bend}, W_{curv} and W_{mp}, we need in the first place the following matrices:

$$\overline{R}_3 = \overline{R}e_3 = \begin{pmatrix} \sin\overline{\alpha}(x) \\ 0 \\ \cos\overline{\alpha}(x) \end{pmatrix}, \quad \nabla\overline{R}_3(x,y) = \begin{pmatrix} \cos\overline{\alpha}(x)\overline{\alpha}'(x) & 0 \\ 0 & 0 \\ -\sin\overline{\alpha}(x)\overline{\alpha}'(x) & 0 \end{pmatrix},$$

$$\overline{R}^T(\nabla\overline{R}_3|0) = \begin{pmatrix} \cos\overline{\alpha}(x) & 0 & -\sin\overline{\alpha}(x) \\ 0 & 1 & 0 \\ \sin\overline{\alpha}(x) & 0 & \cos\overline{\alpha}(x) \end{pmatrix} \cdot \begin{pmatrix} \cos\overline{\alpha}(x)\overline{\alpha}'(x) & 0 & 0 \\ 0 & 0 & 0 \\ -\sin\overline{\alpha}(x)\overline{\alpha}'(x) & 0 & 0 \end{pmatrix} \tag{4.16}$$

$$= \begin{pmatrix} \cos^2\overline{\alpha}(x)\overline{\alpha}'(x) + \sin^2\overline{\alpha}(x)\overline{\alpha}'(x) & 0 & 0 \\ 0 & 0 & 0 \\ \sin\overline{\alpha}(x)\cos\overline{\alpha}(x)\overline{\alpha}'(x) - \sin\overline{\alpha}(x)\cos\overline{\alpha}(x)\overline{\alpha}'(x) & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \overline{\alpha}'(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\nabla m(x,y) = \begin{pmatrix} m_1'(x) & 0 \\ 0 & 1 \\ m_3'(x) & 0 \end{pmatrix}, \quad (\nabla m|\overline{R}_3) = \begin{pmatrix} m_1'(x) & 0 & \sin\overline{\alpha}(x) \\ 0 & 1 & 0 \\ m_3'(x) & 0 & \cos\overline{\alpha}(x) \end{pmatrix},$$

Moreover,

$$\overline{R}^T(\nabla m|\overline{R}_3) = \begin{pmatrix} \cos\overline{\alpha}(x) & 0 & -\sin\overline{\alpha}(x) \\ 0 & 1 & 0 \\ \sin\overline{\alpha}(x) & 0 & \cos\overline{\alpha}(x) \end{pmatrix} \cdot \begin{pmatrix} m_1'(x) & 0 & \sin\overline{\alpha}(x) \\ 0 & 1 & 0 \\ m_3'(x) & 0 & \cos\overline{\alpha}(x) \end{pmatrix} \tag{4.17}$$

$$= \begin{pmatrix} m_1'(x)\cdot\cos\overline{\alpha}(x) - m_3'(x)\cdot\sin\overline{\alpha}(x) & 0 & \sin\overline{\alpha}(x)\cdot\cos\overline{\alpha}(x) - \sin\overline{\alpha}(x)\cdot\cos\overline{\alpha}(x) \\ 0 & 1 & 0 \\ m_1'(x)\cdot\sin\overline{\alpha}(x) + m_3'(x)\cdot\cos\overline{\alpha}(x) & 0 & \sin^2\overline{\alpha}(x) + \cos^2\overline{\alpha}(x) \end{pmatrix}$$

$$= \begin{pmatrix} m_1'(x)\cdot\cos\overline{\alpha}(x) - m_3'(x)\cdot\sin\overline{\alpha}(x) & 0 & 0 \\ 0 & 1 & 0 \\ m_1'(x)\cdot\sin\overline{\alpha}(x) + m_3'(x)\cdot\cos\overline{\alpha}(x) & 0 & 1 \end{pmatrix}.$$

By definition $\operatorname{sym}\overline{U} = \frac{\overline{U}+\overline{U}^T}{2}$, hence,

$$\operatorname{sym}\overline{R}^T(\nabla m|\overline{R}_3) = \frac{1}{2}\begin{pmatrix} m_1'(x)\cdot\cos\overline{\alpha}(x) - m_3'(x)\cdot\sin\overline{\alpha}(x) & 0 & 0 \\ 0 & 1 & 0 \\ m_1'(x)\cdot\sin\overline{\alpha}(x) + m_3'(x)\cdot\cos\overline{\alpha}(x) & 0 & 1 \end{pmatrix} \tag{4.18}$$

$$+ \frac{1}{2}\begin{pmatrix} m_1'(x)\cdot\cos\overline{\alpha}(x) - m_3'(x)\cdot\sin\overline{\alpha}(x) & 0 & m_1'(x)\cdot\sin\overline{\alpha}(x) + m_3'(x)\cdot\cos\overline{\alpha}(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2m_1'(x)\cdot\cos\overline{\alpha}(x) - 2m_3'(x)\cdot\sin\overline{\alpha}(x) & 0 & m_1'(x)\cdot\sin\overline{\alpha}(x) + m_3'(x)\cdot\cos\overline{\alpha}(x) \\ 0 & 2 & 0 \\ m_1'(x)\cdot\sin\overline{\alpha}(x) + m_3'(x)\cdot\cos\overline{\alpha}(x) & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} m_1'(x)\cdot\cos\overline{\alpha}(x) - m_3'(x)\cdot\sin\overline{\alpha}(x) & 0 & \frac{1}{2}(m_1'(x)\cdot\sin\overline{\alpha}(x) + m_3'(x)\cdot\cos\overline{\alpha}(x)) \\ 0 & 1 & 0 \\ \frac{1}{2}(m_1'(x)\cdot\sin\overline{\alpha}(x) + m_3'(x)\cdot\cos\overline{\alpha}(x)) & 0 & 1 \end{pmatrix}.$$

Therefore, with $\operatorname{sym}(\overline{U}-1\!\!1) = \frac{(\overline{U}-1\!\!1)+(\overline{U}-1\!\!1)^T}{2} = \frac{1}{2}(\overline{U}-1\!\!1+\overline{U}^T-1\!\!1) = \frac{1}{2}(\overline{U}+\overline{U}^T)-1\!\!1$, we obtain

$$\operatorname{sym}(\overline{R}^T(\nabla m|\overline{R}_3) - 1\!\!1) \tag{4.19}$$

$$= \begin{pmatrix} m_1'(x)\cos\overline{\alpha}(x) - m_3'(x)\sin\overline{\alpha}(x) - 1 & 0 & \frac{1}{2}(m_1'(x)\sin\overline{\alpha}(x) + m_3'(x)\cos\overline{\alpha}(x)) \\ 0 & 0 & 0 \\ \frac{1}{2}(m_1'(x)\sin\overline{\alpha}(x) + m_3'(x)\cos\overline{\alpha}(x)) & 0 & 0 \end{pmatrix}.$$

Now we calculate the skew-symmetric part of a matrix by $\mathrm{skew}(\overline{U}) = \frac{\overline{U}-\overline{U}^T}{2}$. Hence,

$$\mathrm{skew}(\overline{R}^T(\nabla m|\overline{R}_3)) = \frac{1}{2}\begin{pmatrix} m_1'(x)\cdot\cos\overline{\alpha}(x) - m_3'\cdot\cos\overline{\alpha}(x) & 0 & 0 \\ 0 & 1 & 0 \\ m_1'(x)\cdot\sin\overline{\alpha}(x) + m_3'(x)\cdot\cos\overline{\alpha}(x) & 0 & 0 \end{pmatrix} \tag{4.20}$$

$$-\frac{1}{2}\begin{pmatrix} m_1'(x)\cos\overline{\alpha}(x) - m_3'(x)\sin\overline{\alpha}(x) & 0 & m_1'(x)\sin\overline{\alpha}(x) + m_3(x)\cos\overline{\alpha}(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & -m_1'(x)\sin\overline{\alpha}(x) - m_3'(x)\cos\overline{\alpha}(x) \\ 0 & 0 & 0 \\ m_1'(x)\sin\overline{\alpha}(x) + m_3'(x)\cos\overline{\alpha}(x) & 0 & 0 \end{pmatrix}.$$

Now we compute the trace of the matrix. For the sake of simplicity we introduce the second order approximation of the sin and cos functions, i.e.

$$\sin\gamma \approx \gamma - \frac{\gamma^3}{3!} = \gamma - \frac{\gamma^3}{6}, \quad \cos\gamma \approx 1 - \frac{\gamma^2}{2}, \tag{4.21}$$

This implies

$$\sin\gamma\cdot\cos\gamma \approx (\gamma - \frac{\gamma^3}{6})(1 - \frac{\gamma^2}{2}) = \gamma - \frac{\gamma^3}{2} - \frac{\gamma^3}{6} + \frac{\gamma^5}{12} \approx \gamma - \frac{2}{3}\gamma^3. \tag{4.22}$$

Hence, using these simplifications

$$\begin{aligned} \mathrm{tr}(\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3) - 1\!\!1))^2 &= (m_1'(x)\cos\overline{\alpha}(x) - m_3'(x)\sin\overline{\alpha}(x) - 1)^2 \\ &\approx (m_1'(x)(1 - \frac{\overline{\alpha}^2}{2}) - m_3'(x)(\overline{\alpha} - \frac{\overline{\alpha}^3}{6}) - 1)^2 \\ &= (m_1'(x))^2\cdot(1 + \frac{\overline{\alpha}^4}{4} - \overline{\alpha}^2) \\ &\quad + (m_3'(x))^2\cdot(\overline{\alpha}^2 + \frac{\overline{\alpha}^6}{36} - \frac{\overline{\alpha}^3}{3}) + 1 \\ &\quad - 2m_1'(x)m_3'(x)(1 - \frac{\overline{\alpha}^2}{2})(\overline{\alpha} - \frac{\overline{\alpha}^3}{6}) \\ &\quad - 2m_1'(x)(1 - \frac{\overline{\alpha}^2}{2}) + 2m_3'(x)(\overline{\alpha} - \frac{\overline{\alpha}^3}{6}) \\ &= \frac{\overline{\alpha}^4}{4}(m_1'(x))^2 + \frac{\overline{\alpha}^3}{3}(-((m_3'(x))^2 + 4m_1'(x)m_3'(x) \\ &\quad - m_3'(x)) + \overline{\alpha}^2(-(m_1'(x))^2 + (m_3'(x))^2 + m_1'(x)) \\ &\quad + 2\overline{\alpha}(m_3'(x) - m_1'(x)m_3'(x)) + (m_1'(x))^2 - 2m_1'(x) + 1\,, \end{aligned}$$

$$\begin{aligned} \|\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3) - 1\!\!1)\|^2 &= (m_1'(x)\cos\overline{\alpha}(x) - m_3'(x)\sin\overline{\alpha}(x) - 1)^2 \\ &\quad + \frac{1}{4}(m_1'(x)\sin\overline{\alpha}(x) + m_3'(x)\cos\overline{\alpha}(x))^2 \end{aligned} \tag{4.23}$$

$$
\begin{aligned}
&+\frac{1}{4}(m_1'(x)\sin\overline{\alpha}(x)+m_3'(x)\cos\overline{\alpha}(x))^2\\
&=(m_1'(x)\cos\overline{\alpha}(x))^2-m_1'(x)m_3'(x)\sin\overline{\alpha}\cos\overline{\alpha}\\
&\quad-2m_1'(x)\cos\overline{\alpha}+(m_3'(x)\sin\overline{\alpha}(x))^2+1\\
&\quad+\frac{1}{2}(m_1'(x)\sin\overline{\alpha})^2+\frac{1}{2}(m_3'(x)\cos\overline{\alpha})^2\\
&\approx(m_1'(x))^2(1-\frac{\overline{\alpha}^2}{2})^2-m_1'(x)m_3'(x)(\overline{\alpha}-\frac{2}{3}\overline{\alpha}^3)\\
&\quad-2m_1'(x)(1-\frac{\overline{\alpha}^2}{2})^2+(m_3'(x))^2(\overline{\alpha}-\frac{\overline{\alpha}^3}{6})^2+1\\
&\quad+\frac{1}{2}(m_1'(x))^2(\overline{\alpha}-\frac{\overline{\alpha}^3}{6})^2+\frac{1}{2}(m_3'(x)(1-\frac{\overline{\alpha}^2}{2})^2\\
&=(m_1'(x))^2(1+\frac{\overline{\alpha}^4}{4}-\overline{\alpha}^2)-m_1'(x)m_3'(x)\overline{\alpha}\\
&\quad+\frac{2}{3}\overline{\alpha}^3m_1'(x)m_3'(x)-2m_1'(x)(1+\frac{\overline{\alpha}^4}{4}-\overline{\alpha}^2)\\
&\quad+(m_3'(x))^2(\overline{\alpha}^2-\frac{\overline{\alpha}^4}{3})+1+\frac{1}{2}(m_1'(x))^2(\overline{\alpha}^2-\frac{\overline{\alpha}^4}{3})\\
&\quad+\frac{1}{2}m_3'(x)(1+\frac{\overline{\alpha}^4}{4}-\overline{\alpha}^2)\\
&=\overline{\alpha}^4(\frac{1}{4}(m_1'(x))^2-\frac{1}{2}m_1'(x)-\frac{1}{3}(m_3'(x))^2\\
&\quad-\frac{1}{6}(m_1'(x))^2+\frac{1}{8}m_3'(x))+\overline{\alpha}^3(\frac{2}{3}m_1'(x)m_3'(x))\\
&\quad+\overline{\alpha}^2(-m_1'(x)+\frac{1}{2}m_1'(x)+(m_3'(x))^2+\frac{1}{2}(m_1'(x))^2\\
&\quad-\frac{1}{2}m_3'(x))+\overline{\alpha}(-m_1'(x)m_3'(x))+(m_1'(x))^2+1\\
&\quad+\frac{1}{2}m_3'(x)-2m_1'(x)\,.
\end{aligned}
$$

In addition

$$
\begin{aligned}
\|\operatorname{skew}(\overline{R}^T(\nabla m|\overline{R}_3))\|^2&=\frac{1}{2}(m_1'(x)\sin\overline{\alpha}+m_3'(x)\cos\overline{\alpha})^2 \qquad (4.24)\\
&=\frac{1}{2}((m_1'(x))^2(\sin\overline{\alpha})^2+(m_3'(x))^2(\cos\overline{\alpha})^2\\
&\quad+2m_1'(x)m_3'(x)\sin\overline{\alpha}\cos\overline{\alpha})\\
&\approx\frac{1}{2}((m_1'(x))^2(\overline{\alpha}-\frac{\overline{\alpha}^3}{6})^2+(m_3'(x))^2(1-\frac{\overline{\alpha}^2}{2})^2\\
&\quad+2m_1'(x)m_3'(x)(\overline{\alpha}-\frac{2}{3}\overline{\alpha}^3))\\
&=\frac{1}{2}(\overline{\alpha}^2(m_1'(x))^2-\frac{\overline{\alpha}^4}{3}(m_1'(x))^2+(m_3'(x))^2+\frac{\overline{\alpha}^4}{4}(m_3'(x))^2\\
&\quad-\overline{\alpha}^2(m_3'(x))^2+2m_1'(x)m_3'(x)\overline{\alpha}-\frac{4}{3}\overline{\alpha}^3m_1'(x)m_3'(x))\,.
\end{aligned}
$$

Now we can reassemble $W_{\mathrm{mp}}(\overline{U})$ for this particular situation. Physically it makes sense to set $\mu_c \equiv 0$. Hence,

$$
\begin{aligned}
W_{\mathrm{mp}}(\overline{R}^T(\nabla m|\overline{R}_3)) &= \mu\|\operatorname{sym}(\overline{R}^T(m\nabla|\overline{R}_3) - 1\!\!1)\|^2 + \mu_c\|\operatorname{skew}(\overline{R}^T(m\nabla|\overline{R}_3))\|^2 \\
&\quad + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}(\operatorname{sym}(\overline{R}^T(\nabla m|\overline{R}_3) - 1\!\!1))^2 \\
&\approx \overline{\alpha}^4\mu(\frac{1}{4}(m_1'(x))^2 - \frac{1}{2}m_1'(x) - \frac{1}{3}(m_3'(x))^2 - \frac{1}{6}(m_1'(x))^2 \\
&\quad + \frac{1}{8}m_3'(x)) + \overline{\alpha}^3\mu(\frac{2}{3}m_1'(x)m_3'(x)) + \overline{\alpha}^2\mu(-\frac{1}{2}m_1'(x) \\
&\quad + (m_3'(x))^2 + \frac{1}{2}(m_1'(x))^2 + \frac{1}{2}m_3'(x)) + \overline{\alpha}\mu(-m_1'(x)m_3'(x)) \\
&\quad + \mu((m_1'(x))^2 + 1 + \frac{1}{2}m_3'(x) - 2m_1'(x)) + \overline{\alpha}^4\frac{\mu\lambda}{4(2\mu+\lambda)}(m_1'(x))^2 \\
&\quad + \overline{\alpha}^3\frac{\mu\overline{\alpha}}{3(2\mu+\lambda)}(-(m_3'(x))^2 + 4m_1'(x)m_3'(x) - m_3'(x)) \\
&\quad + \overline{\alpha}^2\frac{\mu\lambda}{2\mu+\lambda}(-(m_1'(x))^2 + (m_3'(x))^2 + m_1'(x)) \\
&\quad + \overline{\alpha}\frac{2\mu\lambda}{2\mu+\lambda}(m_3'(x) - m_1'(x)m_3'(x)) \\
&\quad + \frac{\mu\lambda}{2\mu+\lambda}((m_1'(x))^2 - 2m_1'(x) + 1) \\
&= \overline{\alpha}^4(\frac{\mu}{4}(m_1'(x))^2 - \frac{\mu}{2}m_1'(x) - \frac{\mu}{3}(m_3'(x))^2 - \frac{\mu}{6}(m_1'(x))^2 \\
&\quad + \frac{\mu}{8}m_3'(x) + \frac{\mu\lambda}{4(2\mu+\lambda)}(m_1'(x))^2) + \overline{\alpha}^3(\frac{2\mu}{3}m_1'(x)m_3'(x) \\
&\quad + \frac{-\mu\lambda}{3(2\mu+\lambda)}(m_3'(x))^2 + \frac{4\mu\lambda}{3(2\mu+\lambda)}m_1'(x)m_3'(x) \\
&\quad - \frac{\mu\lambda}{3(2\mu+\lambda)}(m_3'(x))) + \overline{\alpha}^2(-\frac{\mu}{2}m_1'(x) + \mu(m_3'(x))^2 + \frac{\mu}{2}(m_1'(x))^2 \\
&\quad - \frac{\mu}{2}m_3'(x) - \frac{\mu\lambda}{2\mu+\lambda}(m_1'(x))^2 + \frac{\mu\lambda}{2\mu+\lambda}(m_3'(x))^2 \\
&\quad + \frac{\mu\lambda}{2\mu+\lambda}m_1'(x)) + \overline{\alpha}(-\mu m_1'(x)m_3'(x) + \frac{2\mu\lambda}{2\mu+\lambda}m_3'(x) \\
&\quad - \frac{2\mu\lambda}{2\mu+\lambda}m_1'(x)m_3'(x)) + \mu(m_1'(x))^2 + \mu + \frac{\mu}{2}m_3'(x) - 2\mu m_1'(x) \\
&\quad + \frac{\mu\lambda}{2\mu+\lambda}(m_1'(x))^2 - \frac{2\mu\lambda}{2\mu+\lambda}m_1'(x) + \frac{\mu\lambda}{2\mu+\lambda} \\
&= \overline{\alpha}^4((\frac{\mu}{12} + \frac{\mu\lambda}{4(2\mu+\lambda)})(m_1'(x))^2 - \frac{\mu}{2}m_1'(x) - \frac{\mu}{3}(m_3'(x))^2 \\
&\quad - \frac{\mu}{8}m_3'(x)) + \overline{\alpha}^3(m_1'(x)m_3'(x)(\frac{2\mu}{3} + \frac{4\mu\lambda}{3(2\mu+\lambda)}) - \frac{\mu\lambda}{3(2\mu+\lambda)}(m_3'(x))^2 \\
&\quad - m_3'(x)\frac{\mu\lambda}{3(2\mu+\lambda)}) + \overline{\alpha}^2(m_1'(x)\cdot(\frac{-\mu}{2} + \frac{\mu\lambda}{2\mu+\lambda}) + \frac{\mu\lambda}{2\mu+\lambda}
\end{aligned}
\tag{4.25}
$$

$$+ (m_3'(x))^2(\mu + \frac{\mu\lambda}{2\mu+\lambda}) + (m_1'(x))^2 \cdot (\frac{\mu}{2} - \frac{\mu\lambda}{2\mu+\lambda}) - \frac{\mu}{2} m_3'(x))$$
$$+ \overline{\alpha}(m_3'(x) m_1'(x)(-\mu - \frac{2\mu\lambda}{2\mu+\lambda}) + m_3'(x)\frac{2\mu\lambda}{2\mu+\lambda})$$
$$+ (m_1'(x))^2(\mu + \frac{\mu\lambda}{2\mu+\lambda}) + m_3'(x)\frac{\mu}{2} + m_1'(x)(-2\mu - \frac{2\mu\lambda}{2\mu+\lambda}) + \mu\,.$$

In order to build up $W_{\text{bend}}(\mathfrak{K}_{\mathfrak{b}})$ we need the following blocks ,

$$\operatorname{sym}(\overline{R}^T(\nabla \overline{R}_3|0)) = \begin{pmatrix} \overline{\alpha}'(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.26}$$

$$\|\operatorname{sym}(\overline{R}^T(\nabla \overline{R}_3|0))\|^2 = (\overline{\alpha}'(x))^2, \quad \operatorname{skew}(\overline{R}^T(\nabla \overline{R}_3|0)) = \frac{1}{2}\cdot 0 = 0,$$

$$\|\operatorname{skew}(\overline{R}^T(\nabla \overline{R}_3|0))\|^2 = 0, \quad \operatorname{tr}(\overline{R}^T(\nabla \overline{R}_3|0)) = \overline{\alpha}'(x),$$

$$\operatorname{tr}(\operatorname{sym}(\overline{R}^T(\nabla \overline{R}_3|0)))^2 = \operatorname{tr}\begin{pmatrix} (\overline{\alpha}'(x))^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (\overline{\alpha}'(x))^2.$$

Now for the bending term $W_{\text{bend}}(\mathfrak{K}_{\mathfrak{b}})$,

$$\begin{aligned} W_{\text{bend}}(\overline{R}^T(\nabla \overline{R}_3|0)) &= \mu\|\operatorname{sym}(\overline{R}^T(\nabla \overline{R}_3|0))\|^2 + \mu_c\|\operatorname{skew}(\operatorname{sym}(\overline{R}^T(\nabla \overline{R}_3|0)))\|^2 \\ &\quad + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}(\operatorname{sym}(\overline{R}^T(\nabla \overline{R}_3|0)))^2 \\ &= \mu(\overline{\alpha}'(x))^2 + \mu_c \cdot 0 + \frac{\mu\lambda}{2\mu+\lambda}(\overline{\alpha}'(x))^2 \\ &= \frac{2\mu(\mu+\lambda)}{2\mu+\lambda}\cdot(\overline{\alpha}'(x))^2\,. \end{aligned} \tag{4.27}$$

For $W_{\text{curv}}(\mathfrak{K}_{\mathfrak{s}})$ we need the following blocks,

$$\overline{R}e_1 = \begin{pmatrix} \cos\overline{\alpha}(x) \\ 0 \\ -\sin\overline{\alpha}(x) \end{pmatrix},$$

$$\overline{R}^T(\nabla \overline{R}_1|0) = \begin{pmatrix} \cos\overline{\alpha}(x) & 0 & -\sin\overline{\alpha}(x) \\ 0 & 1 & 0 \\ \sin\overline{\alpha}(x) & 0 & \cos\overline{\alpha}(x) \end{pmatrix} \cdot \begin{pmatrix} -\sin\overline{\alpha}(x)\overline{\alpha}'(x) & 0 & 0 \\ 0 & 0 & 0 \\ -\cos\overline{\alpha}(x)\overline{\alpha}'(x) & 0 & 0 \end{pmatrix} \tag{4.28}$$

$$= \begin{pmatrix} -\sin\overline{\alpha}(x)\cos\overline{\alpha}(x)\overline{\alpha}'(x) + \sin\overline{\alpha}(x)\cos\overline{\alpha}(x)\overline{\alpha}'(x) & 0 & 0 \\ 0 & 0 & 0 \\ -\sin^2\overline{\alpha}(x)\overline{\alpha}'(x) - \cos^2\overline{\alpha}(x)\overline{\alpha}'(x) & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\overline{\alpha}'(x) & 0 & 0 \end{pmatrix},$$

$$\overline{R}^T(\nabla \overline{R}_2|0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The third order tensor $k_s \in \mathfrak{T}(3)$ is given by

$$k_s = (\overline{R}^T(\nabla \overline{R}_1|0), \overline{R}^T(\nabla \overline{R}_2|0), \overline{R}^T(\nabla \overline{R}_3|0)) \tag{4.29}$$

$$= \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\overline{\alpha}'(x) & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \overline{\alpha}'(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right).$$

Hence,

$$\|k_s\|^q = |-\overline{\alpha}'|^q + |\overline{\alpha}'|^q,$$

$$\mathrm{sym}(k_s) := (\mathrm{sym}(k_s^1), \mathrm{sym}(k_s^2), \mathrm{sym}(k_s^3)), \tag{4.30}$$

$$\mathrm{sym}(k_s^1) = \frac{1}{2}\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\overline{\alpha}' & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\overline{\alpha}' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$$
$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & -\overline{\alpha}' \\ 0 & 0 & 0 \\ -\overline{\alpha}' & 0 & 0 \end{pmatrix},$$

$$\mathrm{sym}(k_s^2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{sym}(k_s^3) = \frac{1}{2}\begin{pmatrix} 2\overline{\alpha}' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{sym}(k_s) = \left(\begin{pmatrix} 0 & 0 & -\frac{1}{2}\overline{\alpha}' \\ 0 & 0 & 0 \\ -\frac{1}{2}\overline{\alpha}' & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \overline{\alpha}' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right),$$

$$\mathrm{skew}(k_s^1) = \frac{1}{2}\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\overline{\alpha}' & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -\overline{\alpha}' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 0 & 0 & \frac{1}{2}\overline{\alpha}' \\ 0 & 0 & 0 \\ -\frac{1}{2}\overline{\alpha}' & 0 & 0 \end{pmatrix},$$

$$\operatorname{skew}(k_s^2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \operatorname{skew}(k_s^3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\operatorname{skew}(k_s) = \left(\begin{pmatrix} 0 & 0 & \frac{1}{2}\overline{\alpha}' \\ 0 & 0 & 0 \\ -\frac{1}{2}\overline{\alpha}' & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right),$$

$$\begin{aligned} \|k_s\|^2 :&= \sum_{i=1}^{3} \|k_s^i\|^2, \\ \|\operatorname{sym}(k_s)\|^2 &= \frac{1}{4}((\overline{\alpha}')^2 + (\overline{\alpha}')^2) + 0 + \frac{1}{4} \cdot 4(\overline{\alpha}')^2 = \frac{3}{2}(\overline{\alpha}')^2, \\ \|\operatorname{skew}(k_s)\|^2 &= \frac{1}{4}(\overline{\alpha}')^2 + \frac{1}{4}(\overline{\alpha}')^2 = \frac{1}{2}(\overline{\alpha}')^2, \\ \operatorname{tr}(k_s) &= \operatorname{tr}(k_s^1) + \operatorname{tr}(k_s^2) + \operatorname{tr}(k_s^3) = 0 + 0 + \overline{\alpha}' = \overline{\alpha}'. \end{aligned} \tag{4.31}$$

In the following we substitute: $\overline{\alpha}_5 = \mu,\ ,\overline{\alpha}_6 = \mu_c,\ \overline{\alpha}_7 = \frac{\mu\lambda}{2\mu+\lambda},\ \overline{\alpha}_4 = 0,\ 1 + p = \hat{p}$, and we obtain for $W_{\text{curv}}(\mathfrak{K}_{\mathfrak{s}})$,

$$\begin{aligned} W_{\text{curv}}(\mathfrak{K}_{\mathfrak{s}}) &= \mu \cdot \frac{L_c^{1+p}}{12}(1 + \overline{\alpha}_4 L_c^q \|k_s\|^q) \cdot (\overline{\alpha}_5 \|\operatorname{sym}(k_s)\|^2 \\ &\quad + \overline{\alpha}_6 \|\operatorname{skew}(k_s)\|^2 + \overline{\alpha}_7 (\operatorname{tr}(k_s)^2))^{\frac{1+p}{2}} \\ &= \frac{\mu \cdot L_c^{\hat{p}}}{12} \cdot (1 + 0) \cdot (\mu \frac{3}{2}(\overline{\alpha}')^2 + \mu_c \frac{1}{2}(\overline{\alpha}')^2 + \frac{\mu\lambda}{2\mu+\lambda}(\overline{\alpha}')^2)^{\frac{\hat{p}}{2}} \\ &= \frac{L_c^{\hat{p}} \cdot \mu}{12} \cdot (|\overline{\alpha}'|^{\hat{p}} \cdot (\frac{3}{2}\mu + \frac{1}{2}\mu_c + \frac{\mu\lambda}{2\mu+\lambda}))^{\frac{\hat{p}}{2}}. \end{aligned} \tag{4.32}$$

With $W_{\text{mp}}(\overline{U}), W_{\text{curv}}(\mathfrak{K}_{\mathfrak{s}})$ and $W_{bend}(\mathfrak{K}_{\mathfrak{s}})$ now at hand, we can reassemble the integrand in the minimization problem (2.7), abbreviating $\lambda^* := \frac{\mu\lambda}{2\mu+\lambda}, \quad \hat{p} := 1 + p$,

$$\begin{aligned} I &= \int_\omega (h \cdot W_{\text{mp}}(\overline{U}) + h \cdot W_{\text{curv}}(\mathfrak{K}_{\mathfrak{s}}) + \frac{h^3}{12} \cdot W_{\text{bend}}(\mathfrak{K}_{\mathfrak{b}}))\, d\omega \\ &= \int_\omega h \left(\overline{\alpha}^2 (m_1'(\lambda^* - \frac{\mu}{2}) + (m_3')^2(\mu + \lambda^*) + (m_1')^2(\frac{\mu}{2} - \lambda^*) - \frac{\mu}{2} m_3' \right) \\ &\quad + h\overline{\alpha}\, (m_3' m_1'(-\mu + \lambda^*) + m_3' \lambda^*) \\ &\quad + h \left((m_1')^2 \cdot (\mu + \lambda^*) + m_3' \frac{\mu}{2} + m_1'(-2\mu + \lambda^*) + \mu + \lambda^* \right) \\ &\quad + h \cdot \left(L_c^{\hat{p}} \cdot \frac{\mu}{12} \cdot (\frac{3}{2}\mu + \frac{1}{2}\mu_c + \lambda^*)^{\frac{\hat{p}}{2}} \cdot |\overline{\alpha}'|^{\hat{p}} \right) \\ &\quad + \frac{h^3}{12} \cdot (\frac{2\mu^2}{2\mu+\lambda}) \cdot 2\lambda^* \cdot (\overline{\alpha}')^2\, d\omega, \end{aligned} \tag{4.33}$$

$$
\begin{aligned}
= \int_0^L \int_0^L & h\left(\overline{\alpha}^2(x)\cdot[m_1'(x)(-\frac{\mu}{2}+\lambda^*)+(m_3'(x))^2(\mu+\lambda^*)\right) \\
& + h\cdot\overline{\alpha}^2(x)\left((m_1'(x))^2(\frac{\mu}{2}-\lambda^*)-\frac{\mu}{2}(m_3'(x)]\right) \\
& + h\,(\overline{\alpha}(x)[m_3'(x)\cdot m_1'(x)\cdot(-\mu+\lambda^*)+\lambda^*\cdot m_3'(x)]) \\
& + h\left([m_1'(x))^2\cdot(\mu+\lambda^*)+\frac{\mu}{2}m_3'(x)]\right) \\
& + h\,(m_1'(x)(-2\mu+\lambda^*)+\mu+\lambda^*) \\
& + h\cdot(L_c^{\hat{p}}\cdot\frac{\mu}{12})\cdot(\frac{3}{2}\mu+\frac{1}{2}\mu_c+\lambda^*)^{\frac{\hat{p}}{2}}\cdot|\overline{\alpha}'(x)|^{\hat{p}} \\
& + \frac{h^3}{12}\cdot\frac{2\mu^2}{2\mu+\lambda}\cdot 2\lambda^*\cdot(\overline{\alpha}'(x))^2\,\mathrm{dx}\,\mathrm{dy}\,,
\end{aligned}
$$

where $\omega = [0, L] \times [0, L]$, for we may assume the paper to be a rectangle. Note that the integrand is independent of y such that the integration w.r.t. y can be performed immediately and we obtain

$$
\begin{aligned}
I = L\cdot\int_0^L & h\left(\overline{\alpha}^2(x)\cdot[m_1'(x)(-\frac{\mu}{2}+\lambda^*)+(m_3'(x))^2(\mu+\lambda^*)\right) \\
& + h\cdot\overline{\alpha}^2(x)\left((m_1'(x))^2(\frac{\mu}{2}-\lambda^*)-\frac{\mu}{2}(m_3'(x)]\right) \\
& + h\,(\overline{\alpha}(x)[m_3'(x)\cdot m_1'(x)\cdot(-\mu+\lambda^*)+\lambda^*\cdot m_3'(x)]) \\
& + h\left([m_1'(x))^2\cdot(\mu+\lambda^*)+\frac{\mu}{2}m_3'(x)]\right) \\
& + h\,(m_1'(x)(-2\mu+\lambda^*)+\mu+\lambda^*) \\
& + h\cdot(L_c^{\hat{p}}\cdot\frac{\mu}{12})\cdot(\frac{3}{2}\mu+\frac{1}{2}\mu_c+\lambda^*)^{\frac{\hat{p}}{2}}\cdot|\overline{\alpha}'(x)|^{\hat{p}} \\
& + \frac{h^3}{12}\cdot\frac{2\mu^2}{2\mu+\lambda}\cdot 2\lambda^*\cdot(\overline{\alpha}'(x))^2\,\mathrm{dx}\,.
\end{aligned}
\tag{4.34}
$$

5 Derivation of Euler-Lagrange equations

With respect to the result from the last sections we can now derive a system of differential equations by the methods of the calculus of variations. However, in this particular problem it will be easier to find out the Euler-Lagrange equations via direct derivation as used in the proof of the theorem of the Euler-Lagrange equation rather than apply the Euler-Lagrange equations formally.

Let us therefore define the constants :

$$
\begin{aligned}
K_{\text{curv}} &:= h\cdot(L_c^{\hat{p}}\cdot\frac{\mu}{12})\cdot(\frac{3}{2}\mu+\frac{1}{2}\mu_c+\lambda^*)^{\frac{\hat{p}}{2}} \quad , \\
K_{\text{bend}} &:= \frac{h^3}{12}\cdot\frac{2\mu^2}{2\mu+\lambda}\cdot 2\lambda^*\,.
\end{aligned}
\tag{5.35}
$$

$$t : [0, L] \subset \mathbb{R} \to \mathbb{R}^3 : x \mapsto t(x) = \begin{pmatrix} \overline{\alpha}(x) \\ m_1(x) \\ m_3(x) \end{pmatrix} .$$

Hence,

$$t'(x) = \begin{pmatrix} \overline{\alpha}'(x) \\ m_1'(x) \\ m_3'(x) \end{pmatrix} .$$

Therefore,

$$I = \int_0^L F(x, t(x), t'(x))\, \mathrm{dx}, \tag{5.36}$$

where F has the same analytical expression as in (4.33). Suppose that t is a minimizor of I. We can construct a competing function $\hat{t}$, with $u \in C^1([a, b], \mathbb{R}^3)$ and $\theta \geq 0$ (as before), such that $\hat{t}(x) = t(x) + \theta \cdot u(x)$, and $u(a) = u(b) = (0, 0, 0)^T$, with

$$\frac{d}{d\theta}_{|\theta=0} \int_0^L F(x,\ t(x) + \theta u(x),\ t'(x) + \theta u'(x))\, \mathrm{dx} = 0\,. \tag{5.37}$$

Let

$$V = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} := \begin{pmatrix} \overline{\alpha}(x) + \theta \cdot u_1(x) \\ m_1(x) + \theta \cdot u_2(x) \\ m_3(x) + \theta \cdot u_3(x) \end{pmatrix} ,$$

$$Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} := \begin{pmatrix} \overline{\alpha}'(x) + \theta \cdot u_1'(x) \\ m_1'(x) + \theta \cdot u_2'(x) \\ m_3'(x) + \theta \cdot u_3'(x) \end{pmatrix} .$$

Then the derivative is

$$\begin{aligned} \frac{dF}{d\theta} &= \frac{\partial F}{\partial v_1} \cdot \frac{dv_1}{d\theta} + \frac{\partial F}{\partial v_2} \cdot \frac{dv_2}{d\theta} + \frac{\partial F}{\partial v_3} \cdot \frac{dv_3}{d\theta} \\ &\quad + \frac{\partial F}{\partial z_1} \cdot \frac{dz_1}{d\theta} + \frac{\partial F}{\partial z_2} \cdot \frac{dz_2}{d\theta} + \frac{\partial F}{\partial z_3} \cdot \frac{dz_3}{d\theta} \\ &= \frac{\partial F}{\partial v_1} \cdot u_1 + \frac{\partial F}{\partial v_2} \cdot u_2 + \frac{\partial F}{\partial v_3} \cdot u_3 + \frac{\partial F}{\partial z_1} \cdot u_1' \\ &\quad + \frac{\partial F}{\partial z_2} u_2' + \frac{\partial F}{\partial z_3} \cdot u_3'\,, \end{aligned} \tag{5.38}$$

For $\theta = 0$, we get

$$\begin{aligned}
\frac{\partial F}{\partial v_1} = \frac{\partial F}{\partial \overline{\alpha}} &= 2h\left(m_1'(x)(-\frac{\mu}{2} + \lambda^*)\right)\overline{\alpha}(x) \\
&\quad + 2h\left(m_3'(x))^2(\mu + \lambda^*)\right)\overline{\alpha}(x) \\
&\quad + 2h\left(m_1'(x))^2 \cdot (\frac{\mu}{2} - \lambda^*) - \frac{\mu}{2} \cdot m_3'(x)\right)\overline{\alpha}(x) \\
&\quad + h\left(m_3'(x) \cdot m_1'(x) \cdot (-\mu + \lambda^*) + \lambda^* \cdot m_3'(x)\right)\,,
\end{aligned}$$

$$\frac{\partial F}{\partial v_2} = \frac{\partial F}{\partial m_1} = 0\,, \quad \frac{\partial F}{\partial v_3} = \frac{\partial F}{\partial m_3} = 0\,,$$

$$\frac{\partial F}{\partial z_1} = \frac{\partial F}{\partial \overline{\alpha}'} = \hat{p} \cdot K_{\text{curv}} \cdot |\overline{\alpha}'(x)|^{\hat{p}-1} \cdot \frac{\overline{\alpha}'(x)}{|\overline{\alpha}'(x)|} + 2K_{\text{bend}} \cdot \overline{\alpha}'(x)\,, \tag{5.39}$$

$$\begin{aligned}
\frac{\partial F}{\partial z_2} = \frac{\partial F}{\partial m_1'} &= h \cdot \overline{\alpha}^2(x)(-\frac{u}{2} + \lambda^*) \\
&\quad + 2h \cdot \overline{\alpha}^2(x)(\frac{\mu}{2} - \lambda^*)m_1'(x) + h \cdot \overline{\alpha}(x)m_3'(x)(-\mu + \lambda^*) \\
&\quad + 2h \cdot (\mu + \lambda^*)m_1'(x) + h \cdot (-2\mu + \lambda^*) \\
&= h\left((\mu\overline{\alpha}^2(x) - 2\lambda^*\overline{\alpha}^2(x) + 2\mu + 2\lambda^*)m_1'(x)\right) \\
&\quad + h\left(\lambda^*\overline{\alpha}(x) - \mu\overline{\alpha}(x))m_3'(x)\right) \\
&\quad - h \cdot \left(\frac{\mu}{2}\overline{\alpha}^2(x) - \lambda^*\overline{\alpha}^2(x) + 2\mu + \lambda^*\right)\,, \\
\frac{\partial F}{\partial z_3} = \frac{\partial F}{\partial m_3'} &= h\left(2(\mu + \lambda^*) \cdot \overline{\alpha}^2(x) \cdot m_3(x) - \frac{\mu}{2} \cdot \overline{\alpha}^2(x)\right) \\
&\quad + h\left((-\mu + \lambda^*) \cdot \overline{\alpha}(x) \cdot m_1'(x) + \lambda^* \cdot \overline{\alpha}(x) + \frac{\mu}{2}\right)\,.
\end{aligned}$$

By the stationarity condition (1.33), we obtain

$$\int_a^b \frac{\partial F}{\partial \overline{\alpha}} u_1 + \frac{\partial F}{\partial \overline{\alpha}'} u_1' + \frac{\partial F}{\partial m_1'} u_2' + \frac{\partial F}{\partial m_3'} u_3' \,\mathrm{dx} = 0\,. \tag{5.40}$$

Hence,

$$\begin{aligned}
&\Rightarrow \int_a^b \frac{\partial F}{\partial \overline{\alpha}} u_1(x) + \frac{\partial F}{\partial m_1} u_2(x) + \frac{\partial F}{\partial m_3} u_3(x)\,\mathrm{dx} \\
&- \int_a^b \frac{d}{dx}\frac{\partial F}{\partial \overline{\alpha}'} u_1(x) + \frac{d}{\mathrm{dx}}\frac{\partial F}{\partial m_1'} u_2(x) + \frac{d}{\mathrm{dx}}\frac{\partial F}{\partial m_3'} u_3(x)\,\mathrm{dx} = 0\,, \qquad (5.41) \\
&\Rightarrow \int_a^b (\frac{\partial F}{\partial \overline{\alpha}} - \frac{d}{\mathrm{dx}}\frac{\partial F}{\partial \overline{\alpha}'})u_1(x) + (-\frac{d}{\mathrm{dx}}\frac{\partial F}{\partial m_1'})u_2(x) + (-\frac{d}{\mathrm{dx}}\frac{\partial F}{\partial m_3'})u_3(x)\,\mathrm{dx} = 0\,,
\end{aligned}$$

$$\Rightarrow \int_a^b \langle \begin{pmatrix} \frac{\partial F}{\partial \overline{\alpha}} - \frac{d}{\mathrm{dx}}\frac{\partial F}{\partial \overline{\alpha}'} \\ -\frac{d}{\mathrm{dx}}\frac{\partial F}{\partial m_1'} \\ -\frac{d}{\mathrm{dx}}\frac{\partial F}{\partial m_3'} \end{pmatrix} , \begin{pmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \end{pmatrix} \rangle \,\mathrm{dx} = 0 \,. \tag{5.42}$$

Since u is arbitrary in $C^1([a,b],\mathbb{R}^3)$, by the fundamental lemma of the calculus of variations (Lemma 3.2), we must have for each component

$$\frac{d}{\mathrm{dx}}\frac{\partial F}{\partial \overline{\alpha}'} = \frac{\partial F}{\partial \overline{\alpha}} \,, \qquad \frac{d}{\mathrm{dx}}\frac{\partial F}{\partial m_3'} = 0 \,, \qquad \frac{d}{\mathrm{dx}}\frac{\partial F}{\partial m_1'} = 0 \,. \tag{5.43}$$

Hence, we finally get a second order differential system of 3 equations for the three unknown functions α, m_1, m_3 given by

$$\frac{\partial F}{\partial \overline{\alpha}} = \hat{p} \cdot K_{\mathrm{curv}} \left((\hat{p}-2) \left|\overline{\alpha}'(x)\right|^{\hat{p}-4} \cdot (\overline{\alpha}'(x))^2 + \left|\overline{\alpha}'(x)\right|^{\hat{p}-2} \cdot \overline{\alpha}''(x) \right) + 2K_{\mathrm{bend}}\overline{\alpha}''(x) \,,$$

$$\begin{aligned} 0 &= 2h(-\frac{\mu}{2} + \lambda^*) \cdot \overline{\alpha}(x) \cdot \overline{\alpha}'(x) + 4h(\frac{\mu}{2} - \lambda^*) \cdot \overline{\alpha}(x) \cdot \overline{\alpha}'(x) \cdot m_1'(x) \\ &+ 2h(\frac{\mu}{2} - \lambda^*) \cdot \overline{\alpha}^2(x) \cdot m_1''(x) + h(-\mu + \lambda^*) \cdot \overline{\alpha}'(x) \cdot m_3'(x) \\ &+ h(-\mu + \lambda^*) \cdot \overline{\alpha}(x) \cdot m_3''(x) + 2h(\mu + \lambda^*) \cdot m_1''(x) \,, \end{aligned} \tag{5.44}$$

$$\begin{aligned} 0 &= 4h(\mu + \lambda^*) \cdot \overline{\alpha}(x) \cdot \overline{\alpha}'(x) \cdot m_3'(x) + 4h(\mu + \lambda^*) \cdot \overline{\alpha}^2(x) \cdot m_3''(x) \\ &- h \cdot \mu \cdot \overline{\alpha}(x) \cdot \overline{\alpha}'(x) + h(-\mu + \lambda^3) \cdot \overline{\alpha}'(x) \cdot m_1'(x) \\ &+ h(-\mu + \lambda^3) \cdot \overline{\alpha}(x) \cdot m_1''(x) + \lambda^* \overline{\alpha}'(x) \,. \end{aligned}$$

6 Further simplified problem

As we can see from the result of the preceding section, the system of equations (1.40) is still very difficult to solve, even in view of the preceding simplification. In order to demonstrate "some visible result", we make a further simplification on I by eliminating $W_{\mathrm{mp}}(\overline{U})$. This is justified for pure bending situations (no membrane action, viz. $W_{\mathrm{mp}} := 0$).

After this simplification, we only maintain W_{curv} and W_{bend}. Hence, the problem turns into

$$\begin{aligned} I &= \int_\omega (h \cdot W_{\mathrm{curv}}(\mathfrak{K}_{\mathfrak{s}}) + \frac{h^3}{12} \cdot W_{\mathrm{bend}}(\mathfrak{K}_{\mathfrak{b}}))\, d\omega \,, \quad \omega \subset \mathbb{R}^2 \\ &= \int_\omega h \left(L_c^{\hat{p}} \frac{\mu}{12} (\frac{3}{2}\mu + \frac{1}{2}\mu_c + \lambda^*)^{\frac{\hat{p}}{2}} \left|\overline{\alpha}'\right|^{\hat{p}} \right) + \frac{h^3}{12} (\frac{2\mu^2}{2\mu + \lambda}) 2\lambda^* (\overline{\alpha}')^2 \, d\omega \,. \end{aligned}$$

Hence ,

$$I = \int\limits_{\omega} K_{\text{curv}}\, |\overline{\alpha}'(x)|^{\hat{p}} + K_{\text{bend}} (\overline{\alpha}'(x))^2 \,\mathrm{d(x,y)} \tag{6.45}$$

$$= \int\limits_0^L \int\limits_0^L K_{\text{curv}}\, |\overline{\alpha}'(x)|^{\hat{p}} + K_{\text{bend}} (\overline{\alpha}'(x))^2 \,\mathrm{dx}\,\mathrm{dy}\,.$$

Since, $\overline{\alpha}$ depends only on x, the minimization of I is equivalent to the minimization of the inner integral, which, for the sake of simplicity, will still be denoted by I.

$$I = \int\limits_0^L K_{\text{curv}}\, |\overline{\alpha}'(x)|^{\hat{p}} + K_{\text{bend}} (\overline{\alpha}'(x))^2 \,\mathrm{dx}; \quad \overline{\alpha}(0) = 0, \quad \text{and} \quad \overline{\alpha}(L) = \beta\,. \tag{6.46}$$

Let $u \in C^1([0,L],\mathbb{R}), u(0) = 0, u(L) = 0$ and $\theta \geq 0$. Construct a competing function $\hat{\overline{\alpha}}(x) := \overline{\alpha}(x) + \theta u(x)$. Thus we must have

$$\frac{d}{d\theta}_{|\theta=0} \int\limits_0^L K_{\text{curv}} \left|(\overline{\alpha}(x) + \theta u(x))'\right|^{\hat{p}} + K_{\text{bend}} \left((\overline{\alpha}(x) + \theta u(x))'\right)^2 \,\mathrm{dx} = 0\,. \tag{6.47}$$

Therefore

$$\int\limits_0^L K_{\text{curv}} \cdot \frac{d}{d\theta} |(\overline{\alpha}(x) + \theta u(x))'|^{\hat{p}} + K_{\text{bend}} \cdot \frac{d}{d\theta} ((\overline{\alpha}(x) + \theta u(x))')^2 \,\mathrm{dx}|_{\theta=0} \; = 0\,, \tag{6.48}$$

$$\frac{d}{d\theta} |(\overline{\alpha}(x) + \theta u(x))'|^{\hat{p}} = \hat{p} \cdot |\overline{\alpha}'(x) + \theta u'(x)|^{\hat{p}-1} \cdot \frac{\overline{\alpha}(x)' + \theta u'(x)}{|\overline{\alpha}'(x) + \theta u'(x)|} \cdot u'(x)\,,$$

$$\frac{d}{d\theta} [(\overline{\alpha}(x) + \theta u(x))']^2 = 2 \cdot [\overline{\alpha}'(x) + \theta u'(x)] \cdot u'(x)\,.$$

Hence, by evaluate the equation (6.47) at $\theta = 0$

$$\int\limits_0^L K_{\text{curv}} \cdot \hat{p} \cdot |\overline{\alpha}'(x)|^{\hat{p}-1} \cdot \frac{\overline{\alpha}'(x)}{|\overline{\alpha}'(x)|} \cdot u'(x) + 2K_{\text{bend}} \cdot \overline{\alpha}'(x) u'(x) \,\mathrm{dx} = 0\,, \tag{6.49}$$

$$\int\limits_0^L [K_{\text{curv}} \cdot \hat{p}\, |\overline{\alpha}'(x)|^{\hat{p}-1} \cdot \frac{\overline{\alpha}'(x)}{|\overline{\alpha}'(x)|} + 2K_{\text{bend}} \overline{\alpha}'(x)] \cdot u'(x) \,\mathrm{dx} = 0\,,$$

$$\Rightarrow - \int\limits_0^L \frac{d}{\mathrm{dx}} [K_{\text{curv}} \cdot \hat{p} \cdot |\overline{\alpha}'(x)|^{\hat{p}-1} \cdot \frac{\overline{\alpha}'(x)}{|\overline{\alpha}'(x)|} + 2K_{\text{bend}} \cdot \overline{\alpha}'(x)] \cdot u(x) \,\mathrm{dx} = 0\,,$$

Since u is arbitrary in $C^1([0,L],\mathbb{R})$, it follows by the fundamental lemma (3.2) of the calculus of variations

$$\frac{d}{\mathrm{dx}}\left(K_{\mathrm{curv}}\cdot\hat{p}\cdot|\overline{\alpha}'(x)|^{\hat{p}-1}\cdot\frac{\overline{\alpha}'(x)}{|\overline{\alpha}'(x)|}+2K_{\mathrm{bend}}\cdot\overline{\alpha}'(x)\right)=0,\qquad \overline{\alpha}(0)=0,\ \overline{\alpha}(L)=\beta. \tag{6.50}$$

Therefore

$$K_{\mathrm{curv}}\cdot\hat{p}\left((\hat{p}-2)\,|\overline{\alpha}'(x)|^{\hat{p}-4}\,(\overline{\alpha}'(x))^2\overline{\alpha}''(x)+|\overline{\alpha}'(x)|^{\hat{p}-2}\,\overline{\alpha}''(x)\right)+2K_{\mathrm{bend}}\overline{\alpha}''(x)=0\,. \tag{6.51}$$

Finally the equation to be solved is

$$\begin{aligned}&\left(K_{\mathrm{curv}}\cdot\hat{p}\left((\hat{p}-2)\,|\overline{\alpha}'(x)|^{\hat{p}-4}\,(\overline{\alpha}'(x))^2+|\overline{\alpha}'(x)|^{\hat{p}-2}\right)+2K_{\mathrm{bend}}\right)\overline{\alpha}''(x)=0\,, \\ &\qquad\qquad \overline{\alpha}(0)=0,\quad \overline{\alpha}(L)=\beta\,.\end{aligned} \tag{6.52}$$

The equation(6.51) is the differential equation we derived with the elimination of $W_{\mathrm{mp}}(\overline{U})$.

7 Analytical solution

As we can see from equation (6.51), the items in the bracket are always positive, for, in particular, the constants in the bracket are built up with positive factors. Hence, we can only have $\overline{\alpha}''(x)=0$. It follows $\overline{\alpha}'(x)$ is a constant, say C_3 . Hence, $\overline{\alpha}(x)=C_3x$. With the boundary condition, the solution is

$$\overline{\alpha}(x)=\frac{\beta}{L}x\,,$$

which shows that the microrotation angle $\overline{\alpha}$ varies linearly over the length in this simplified setting.

8 Existence of minimizers in the general case

In this section we will identify some conditions on the Lagrangian L which ensure that the function $I[\cdot]$ does indeed have a minimizer, at least within an appropriate Sobolev space.

8.1 Coercivity, lower semicontinuity

Let us start with some largely heuristic insights as to when the functional

$$I[u]:=\int_\Omega L(Du(x),u(x),x)\,\mathrm{dx}\,, \tag{8.53}$$

defined for appropriate functions $u : \Omega \mapsto \mathbb{R}$, satisfying

$$u = g \text{ on } \partial\Omega\,, \tag{8.54}$$

should have a minimizer.

a. Coercivity.

We first of all note that even a smooth function f mapping $\mathbb{R}$ to $\mathbb{R}$ and bounded below need not attain its infimum. Consider, for instance, some hypothesis controlling $I[u]$ for "large" arguments u. Certainly the most effective way to ensure this would be to hypothesize that $I[u]$ "grows rapidly as $|u| \to \infty$".
More specifically, let us assume that

$$1 < q < \infty\,, \tag{8.55}$$

is fixed. We will then suppose

$$\begin{cases} \text{there exist constants } C_1 > 0,\ C_2 > 0 \quad \text{such that} \\ L(p, z, x) \geq C_1 |p|^q - C_2 \\ \text{for all } p \in \mathbb{R}^n,\ z \in \mathbb{R},\ x \in \Omega\,. \end{cases} \tag{8.56}$$

Therefore

$$I[u] \geq C_1 ||Du||^q_{L^q(\Omega)} - r\,, \tag{8.57}$$

for $r := C_2|\Omega|$. Thus $I[u] \to \infty$ as $||Du||_{L^q} \to \infty$. It is customary to call (8.57) a **coercivity condition** on $I[\cdot]$.

Turning once more to our basic task of finding minimizers for the functional $I[\cdot]$, we observe from inequality (8.57) that it seems reasonable to define $I[u]$ not only for smooth functions u, but also for functions u in the Sobolev space $W^{1,q}(\Omega)$ that satisfy the boundary condition (8.54) in the trace sense. After all, the wider the class of functions u for which $I[u]$ is defined, the more candidates we will have for a minimizer.
We will henceforth write

$$\mathcal{A} := \{\mathbf{u} \in W^{1,q}(\Omega)\ |u = g \text{ on } \partial\Omega \text{ in the sense of trace }\}\,, \tag{8.58}$$

to denote this class of *admissible* functions ω. Note in view of (8.56) that $I[u]$ is defined for each $u \in \mathcal{A}$.

b. Lower semicontinuity.

Next, let us observe that although a continuous function $f : \mathbb{R} \mapsto \mathbb{R}$ satisfying a coercivity condition does indeed attain its infimum, our integral functional $I[\cdot]$ in general will not. To understand the problem, set

$$m := \inf_{u \in \mathcal{A}} I[u]\,, \tag{8.59}$$

and choose functions $u_k \in \mathcal{A} (k = 1, ...)$ such that

$$I[u_k] \to m \quad \text{as } k \to \infty\,. \tag{8.60}$$

By (8.56) it is clear that the infimum exists. We call $\{u_k\}_{k=1}^{\infty}$ a **minimizing sequence.**

We would now like to show that some subsequence of $\{u_k\}_{k=1}^{\infty}$ converges to an actual minimizer. For this, however, we need some kind of compactness, and this is definitely a problem since the space $W^{1,q}(\Omega)$ is infinite dimensional. Indeed, if we utilize the coercivity inequality, it turns out that we can only conclude that the minimizing sequence lies in a bounded subset of $W^{1,q}(\Omega)$. But this does *not* imply that there exists any subsequence which converges strongly in $W^{1,q}(\Omega)$.

We therefore turn our attention to the **weak topology**. Since we are assuming $1 < q < \infty$, so that $L^q(\Omega)$ is reflexive, we conclude that there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$ and a function $u \in W^{1,q}(\Omega)$ such that

$$\begin{cases} u_{k_j} \rightharpoonup u \quad \text{weakly in } L^q(\Omega) \\ Du_{k_j} \rightharpoonup Du \quad \text{weakly in } L^q(\Omega;\mathbb{R}^n). \end{cases} \tag{8.61}$$

We will hereafter abbreviate (8.61)by saying

$$u_{k_j} \rightharpoonup u \quad \text{weakly in } W^{1,q}(\Omega). \tag{8.62}$$

Furthermore, it will be true that $u = g$ on $\partial\Omega$ in the sense of trace, and so for the weak limit $u \in \mathcal{A}$.

Consequently by shifting to the weak topology we have recovered enough compactness from the coercivity inequality (8.57) to deduce (8.62) for an appropriate subsequence. But now another difficulty arises, for in essentially all cases of interest, the **funtional $I[\cdot]$ is not continuous with respect to weak convergence**. In other words, we *can not* deduce from (8.60) and (8.62) that

$$I[u] = \lim_{j\to\infty} I[u_{k_j}], \tag{8.63}$$

and thus we cannot directly deduce that u is a minimizer. The problem is that $Du_{k_j} \rightharpoonup Du$ does not imply $Du_{k_j} \to Du$ a.e. It is quite possible for instance that the gradients Du_{k_j}, although bounded in L^q, are oscillating more and more rapidly as $k_j \to \infty$.

What saves us is the final, key observation that we do not really need the full strength of (8.63). It would suffice instead to know only

$$I[u] \leq \liminf_{j\to\infty} I[u_{k_j}]. \tag{8.64}$$

Then from (8.60) we could deduce $I[u] \leq m$. But owing to (8.59), $m \leq I[u]$. Consequently u is indeed a minimizer.

Lemma 8.1

We say that a function $I[\cdot]$ is (sequentially) weakly lower semicontinuous on $W^{1,q}(\Omega)$, provided

$$I[u] \leq \liminf_{k\to\infty} I[u_k]\,,$$

whenever

$$u_k \rightharpoonup u \quad \textit{weakly in } W^{1,q}(\Omega).$$

Our goal therefore is now to identify reasonable conditions on the nonlinear term L that ensure that $I[\cdot]$ is weakly lower semicontinuous.

8.2 Convexity

Now we analyze the following second variation inequality,

$$\sum_{i,j=1}^{n} L_{p_i p_j}(Du(x), u(x), x)\xi_i \xi_j \geq 0 \quad (\xi \in \mathbb{R}^n, x \in \Omega)\,,$$

holding as a necessary condition, whenever u is a smooth minimizer. This inequality strongly suggests that it might be reasonable to assume that L is convex in its first argument, i.e. for the gradient Du. Indeed we have the result.

Lemma 8.2 (Weak lower semicontinuity)
Assume that L is bounded below, and in addition the mapping $p \mapsto L(p, z, x)$ is convex for each $z \in \mathbb{R}, x \in \Omega$. Then $I[\cdot]$ is weakly lower semicontinuous on $W^{1,q}(\Omega)$.

Proof. 1. Choose any sequence $\{u_k\}_{k=1}^{\infty}$ with

$$u_k \rightharpoonup u \text{ weakly in } W^{1,q}(\Omega), \tag{8.65}$$

and set $l := \liminf_{k\to\infty} I[u_k]$. We must show that

$$I[u] \leq l. \tag{8.66}$$

2. Note first from (8.65) that

$$\sup_k ||u_k||_{W^{1,q}(\Omega)} < \infty. \tag{8.67}$$

Upon passing to a subsequence if necessary, we may as well also suppose

$$l = \lim_{k\to\infty} I[u_k]. \tag{8.68}$$

Furthermore we use the compactness theorem that $u_k \to u$ strongly in $L^q(\Omega)$; and thus, passing if necessary to yet another subsequence, we have

$$u_k \to u \text{ a.e. in } \Omega. \tag{8.69}$$

3. Fix $\varepsilon > 0$. Then (8.69) and Egroff's Theorem assert

$$u_k \to u \text{ uniformly on } E_\varepsilon, \tag{8.70}$$

where E_ε is a measurable set with

$$|\Omega - E_\varepsilon| \leq \varepsilon. \tag{8.71}$$

Now write

$$F_\varepsilon := \{x \in \Omega, |u(x)| + |Du(x)| \leq \frac{1}{\varepsilon}\}, \tag{8.72}$$

then

$$|\Omega - F_\varepsilon| \to 0 \quad \text{as } \varepsilon \to 0. \tag{8.73}$$

We finally set

$$G_\varepsilon := E_\varepsilon \cap F_\varepsilon, \tag{8.74}$$

and observe from $(8.71), (8.73)$ that $|\Omega - G_\varepsilon| \to 0$ as $\varepsilon \to 0$.
4. Now let us observe since L is bounded below, we may as well assume

$$L \geq 0, \tag{8.75}$$

(for otherwise we could apply the following arguments to $\overline{L} = L + \beta \geq 0$ for some appropriate constant β). Consequently

$$\begin{aligned} I[u_k] = \int_\Omega L(Du_k, u_k,)\,\mathrm{dx} &\geq \int_{G_\varepsilon} L(Du_k, u_k, x)\,\mathrm{dx} \\ &\geq \int_{G_\varepsilon} L(Du, u_k, x)\,\mathrm{dx} + \int_{G_\varepsilon} D_pL(Du, u, x) \cdot (Du_k - Du)\,\mathrm{dx}. \end{aligned} \tag{8.76}$$

the last inequality following from the convexity of L in its first argument. Now in view of (8.70), (8.72) and (8.74):

$$\lim_{k\to\infty} \int_{G_\varepsilon} L(Du, u_k, x)\,\mathrm{dx} = \int_{G_\varepsilon} L(Du, u, x)\,\mathrm{dx}. \tag{8.77}$$

In addition, since $D_pL(Du, u_k, x) \to D_pL(Du, u, x)$ uniformly on G_ε and $Du_k \rightharpoonup Du$ weakly in $L_q(U; \mathbb{R}^n)$, we have

$$\lim_{k\to\infty} \int_{G_\varepsilon} D_pL(Du, u_k, x) \cdot (Du_k - Du)\,\mathrm{dx} = 0. \tag{8.78}$$

Owing now to (8.77) we deduce from (8.78) that

$$l = \lim_{k\to\infty} I[u_k] \geq \int_{G_\varepsilon} L(Du, u, x)\,\mathrm{dx}. \tag{8.79}$$

This inequality holds for each $\varepsilon > 0$. We now let ε tend to zero, and recall (8.75) and the Monotone Convergence Theorem to conclude that

$$l \geq \int_\Omega L(Du, u, x)\,\mathrm{dx} = I[u],$$

as required.
Remark. It is important to understand how the foregoing proof deals with the weak convergence $Du_k \rightharpoonup Du$. The key is the convexity inequality (8.76), on

the right hand side of which Du_k appears linearly. Weak convergence is, by its very definition, compatible with linear expressions, and so the limit (8.78) holds. Remember that it is not in general true that $Du_k \to Du$ a.e., even if we pass to a subsequence.
The convergence of u_k to u in L^q is much stronger, and so we do not need any convexity assumption concerning the second variance $z \mapsto L(p, z, x)$.

We can at last establish that $I[\cdot]$ has a minimizer among the functions in $\mathcal{A}$.

Lemma 8.3 (Existence of minimizer)
Assume that L satisfies the coercivity inequality (8.56) *and is convex in the variable p. Suppose also the admissible set $\mathcal{A}$ is nonempty. Then there exists at least one function $u_0 \in \mathcal{A}$ solving*

$$I[u_0] = \min_{u \in \mathcal{A}} I[u].$$

Proof. 1. Set $m := \inf_{u \in \mathcal{A}} I[u]$. If $m = +\infty$ we are done, and so we henceforth assume m is finite. Select a minimizing sequence $\{u_k\}_{k=1}^{\infty}$. Then by construction

$$I[u_k] \to m. \tag{8.80}$$

2. We may as well take $C_2 = 0$ in inequality (8.56), since we could otherwise just as well consider $\overline{L} := L + C_2$. Thus $L \geq C_1 |p|^q$, and so

$$I[u] \geq C_1 \int_{\Omega} |Du_0|^q \, \mathrm{dx}\,. \tag{8.81}$$

Since m is finite, we conclude from (8.80) and (8.81) that

$$\sup_k ||Du_k||_{L^q(\Omega)} \leq K < \infty. \tag{8.82}$$

3. Now for a given function $u_0 \in \mathcal{A}$. Since u_k and u_0 both equal g on $\partial\Omega$ in the trace sense, we have $u_k - u_0 \in W_0^{1,q}(\Omega)$. Therefore by the triangular inequality and Poincare's inequality implies

$$\begin{aligned} ||u_k||_{L^q(\Omega)} &\leq ||u_k - u_0||_{L^q(\Omega)} + ||u_0||_{L^q(\Omega)} \\ &\leq C||D(u_k - u_0)||_{L^q(\Omega)} + C \leq C, \end{aligned} \tag{8.83}$$

by (8.82). Hence $\sup_k ||u_k||_{L^q(\Omega)} < \infty$. This estimate and (8.82) imply $\{u_k\}_{k=1}^{\infty}$ is bounded in $W^{1,q}(\Omega)$.
4. Consequently there exist a subsequence $\{u_k\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$ and a function $u_0 \in W^{1,q}(\Omega)$ such that

$$u_{k_j} \rightharpoonup u_0 \text{ weakly in } W^{1,q}(\Omega).$$

We assert next that $u_0 \in \mathcal{A}$. To see this, note that for $u \in \mathcal{A}$ as above, $u_k - u \in W_0^{1,q}(\Omega)$ is a closed, linear subspace of $W_0^{1,q}(\Omega)$, and so, by Mazur's Theorem, is weakly closed. Hence $u_0 - u \in W_0^{1,q}(\Omega)$. Consequently the trace of u on $\partial\Omega$ is g.

In view of Theorem 1 then, $I[u_0] \le \liminf_{j\to\infty} I[u_{k_j}] = m$. But since $u \in \mathcal{A}$, it follows that

$$I[u_0] = m = \min_{u\in\mathcal{A}} I[u].$$

We turn next to the problem of uniqueness. In general there can be many minimizers, and so to ensure uniqueness we need the further assumptions. Suppose for instance

$$L = L(p,x) \quad \text{does not depend on } z, \tag{8.84}$$

and

$$\begin{cases} \text{there exists } \theta > 0 \text{ such that} \\ \sum_{i,j=1}^{n} L_{p_i p_j}(p,x)\xi_i\xi_j \ge \theta|\xi|^2 \ (p,\xi \in \mathbb{R}^n; x \in \Omega). \end{cases} \tag{8.85}$$

Condition (8.85) means that the mapping $p \mapsto L(p,x)$ is uniformly convex for each x in p.

Lemma 8.4 (Uniqueness of minimizer)
Suppose (8.84),(8.85) *hold. Then a minimizer* $u \in A$ *of* $I[\cdot]$ *is unique.*

Proof. 1. Assume $u, \overline{u} \in \mathcal{A}$ are both minimizer of $I[\cdot]$ over $\mathcal{A}$. Then $v := \frac{u+\overline{u}}{2} \in \mathcal{A}$. We claim

$$I[v] \le \frac{I[u] + I[\overline{u}]}{2}, \tag{8.86}$$

with a strict inequality, unless $u = \overline{u}$ a.e.
2. To see this, note from the uniform convexity assumption that we have

$$L(p,x) \ge L(q,x) + D_pL(q,x)\cdot(p-q) + \frac{\theta}{2}|p-q|^2 \quad (x\in\Omega, p,q\in\mathbb{R}^n). \tag{8.87}$$

Set $q = \frac{Du+D\overline{u}}{2}, p = Du$, and integrate over U:

$$I[v] + \int_\Omega D_pL(\frac{Du+D\overline{u}}{2},x)\cdot(\frac{Du-D\overline{u}}{2})\,\mathrm{dx} + \frac{\theta}{8}\int_\Omega |Du-D\overline{u}|^2\,\mathrm{dx} \le I[u]. \tag{8.88}$$

Similarly, set $q = \frac{Du-D\overline{u}}{2}, p = D\overline{u}$ in (8.87) and integrate:

$$I[v] + \int_\Omega D_pL(\frac{Du+D\overline{u}}{2},x)\cdot(\frac{Du-D\overline{u}}{2})\,\mathrm{dx} + \frac{\theta}{8}\int_\Omega |Du-D\overline{u}|^2\,\mathrm{dx} \le I[\overline{u}]. \tag{8.89}$$

Add and divide by 2, to deduce

$$I[v] + \frac{\theta}{8}\int_\Omega |Du-D\overline{u}|^2\,\mathrm{dx} \le \frac{I[u]+I[\overline{u}]}{2}. \tag{8.90}$$

This proves(8.86).
3. As $I[u] = I[\overline{u}] = min_{u_0\in A} I[u_0] \le I[v]$, we deduce $Du = D\overline{u}$ a.e. in Ω. Since $u = \overline{u} = g$ on $\partial\Omega$ in the sense of trace , it follows that $u = \overline{u}$ a.e. ■

Finally, we proof that the minimizer to the problem 2.7 in our simplified setting exists.

9 Application to the simplified Cosserat model

Let us recall,

$$I(m,\overline{R}) = \int_\omega h\cdot W_{\mathrm{mp}}(\overline{U}) + h\cdot W_{\mathrm{curv}}(\mathfrak{K}_{\mathfrak{s}}) + \frac{h^3}{12}\cdot W_{\mathrm{bend}}(\mathfrak{K}_{\mathfrak{b}})\ \mathrm{dx\,dy}, \qquad (9.91)$$

$$\text{with } m:\mathbb{R}^2\mapsto\mathbb{R}^3, \overline{R}:\mathbb{R}^2\mapsto \mathrm{SO}(3,\mathbb{R}).$$

Here, the following abbreviations are recalled,

$$W_{\mathrm{mp}}(\overline{U}) = \mu\|\,\mathrm{sym}(\overline{U}-1\!\!1)\|^2 + \mu_c\|\,\mathrm{skew}(\overline{U})\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\cdot\mathrm{tr}(\mathrm{sym}(\overline{U}-1\!\!1))^2\ ,$$

$$W_{\mathrm{curv}}(\mathfrak{K}_{\mathfrak{s}}) = \mu\frac{L_c^{1+p}}{12}(1+\alpha_4 L_c^q\|\mathfrak{K}_{\mathfrak{s}}\|^q)(\alpha_5\|\,\mathrm{sym}\,\mathfrak{K}_{\mathfrak{s}}\|^2 + \alpha_6\|\,\mathrm{skew}(\mathfrak{K}_{\mathfrak{s}})\|^2 \qquad (9.92)$$

$$+\,\alpha_7\,\mathrm{tr}(\mathfrak{K}_{\mathfrak{s}})^2)^{\frac{1+p}{2}}\ ,$$

$$\mathfrak{K}_{\mathfrak{s}} = \left(\overline{R}^T(\nabla(\overline{R}e_1)|0), \overline{R}^T(\nabla(\overline{R}e_2)|0), \overline{R}^T(\nabla(\overline{R}e_3)0)\right),$$

(reduced third order curvature tensor),

$$W_{\mathrm{bend}}(\mathfrak{K}_{\mathfrak{b}}) = \mu\|\,\mathrm{sym}(\mathfrak{K}_{\mathfrak{b}})\|^2 + \mu_c\|\,\mathrm{skew}(\mathfrak{K}_{\mathfrak{b}})\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\cdot\mathrm{tr}(\mathrm{sym}(\mathfrak{K}_{\mathfrak{b}}))^2\ ,$$

$$\mathfrak{K}_{\mathfrak{b}} = \overline{R}^T(\nabla\overline{R}_3|0) = {\mathfrak{K}_3}^3, \quad \text{(second order , non-symmetric bending tensor).}$$

We refer to (4.24), the simplified form of m and $\overline{R}$ under the condition $m:\mathbb{R}^2\mapsto\mathbb{R}^3, \overline{R}:\mathbb{R}^2\mapsto\mathrm{SO}(3,\mathbb{R})$. We write therefore,

$$\begin{aligned}
W_a &= \|\,\mathrm{sym}(\overline{R}^T(\nabla|\overline{R}_3) - 1\!\!1)\|^2 \qquad (9.93)\\
&= (m_1'(x))^2(\cos\overline{\alpha}(x))^2 - m_1'(x)m_3'(x)\sin\overline{\alpha}(x)\cos\overline{\alpha}(x) - 2m_1'(x)\cos\overline{\alpha}(x)\\
&\quad + (m_3'(x))^2(\sin\overline{\alpha})^2 + 1 + \frac{1}{2}(m_1'(x))^2(\sin\overline{\alpha})^2 + \frac{1}{2}(m_3'(x))^2(\cos\overline{\alpha})^2,\\
W_b &= \|\,\mathrm{skew}(\overline{R}^T(\nabla m|\overline{R}_3))\|^2 = \frac{1}{2}(m_1'(x)\sin\overline{\alpha} + m_3'(x)\cos\overline{\alpha})^2 \ge 0,\\
W_c &= \mathrm{tr}(\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3) - 1\!\!1))^2 = (m_1'(x)\cos\overline{\alpha}(x) - m_3'(x)\sin\overline{\alpha}(x) - 1)^2 \ge 0.
\end{aligned}$$

First of all, we observe that $I(m,\overline{R})$ can be written as $I^*(m_1,m_3,\alpha)$, which is satisfying the coercivity condition (8.57). This is now shown. By **Young's inequality**, $a\cdot b\le\frac{1}{2}(a^2+b^2)$ we have

$$m_1'(x)m_3'(x)\sin\overline{\alpha}(x)\cos\overline{\alpha}(x) \le \frac{1}{2}(m_1'(x))^2(\cos\overline{\alpha})^2 + \frac{1}{2}(m_3'(x))^2(\sin\overline{\alpha})^2. \quad (9.94)$$

Taking into account our previous simplification w.r.t. m and $\overline{R}$ (4.14), we arrive at

$$\begin{aligned}
W_a &\ge (m_1'(x))^2(\cos\overline{\alpha}(x))^2 - \frac{1}{2}(m_1')^2(\cos\overline{\alpha}(x))^2 - \frac{1}{2}(m_3'(x))^2(\sin\overline{\alpha}(x))^2\\
&\quad + (m_3'(x))^2(\sin\overline{\alpha}(x))^2 + \frac{1}{2}(m_1'(x))^2(\sin\overline{\alpha}(x))^2 + \frac{1}{2}(m_3'(x))^2(\cos\overline{\alpha}(x))^2\\
&\quad - 2m_1'(x)\cos\overline{\alpha}(x) + 1\\
&\ge \frac{1}{2}(m_1'(x))^2 + \frac{1}{2}(m_3'(x))^2 - 2m_1'(x)\cos\overline{\alpha}(x). \qquad (9.95)
\end{aligned}$$

Since we know W_b and W_c are both positive, we have

$$W_{\text{mp}} \geq \mu W_a + \mu_c W_b + \frac{\mu\lambda}{2\mu+\lambda} W_c \geq \mu W_a. \tag{9.96}$$

Now set $m'(x) = \begin{pmatrix} m_1'(x) \\ m_3'(x) \end{pmatrix}$, then using that $|\cos\overline{\alpha}| \leq 1$, shows

$$W_{\text{mp}} \geq \mu \cdot \frac{1}{2}\|m'(x)\|^2 - 2|m_1'(x)| \quad \text{hence,}$$
$$\int_a^b W_{\text{mp}}\,\mathrm{dx} \geq \frac{\mu}{2}\int_a^b \|m'(x)\|^2\,\mathrm{dx} - 2\mu\int_a^b \|m'(x)\|\,\mathrm{dx}. \tag{9.97}$$

Now use **Hölders-inequality**,

$$\int_a^b 1 \cdot \|m'(x)\|\,\mathrm{dx} \leq \left(\int_a^b 1^2\,\mathrm{dx}\right)^{\frac{1}{2}} \cdot \left(\int_a^b \|m'(x)\|^2\,\mathrm{dx}\right)^{\frac{1}{2}}$$
$$= (b-a)^{\frac{1}{2}}\|m'(x)\|_{L^2(a,b)}. \tag{9.98}$$

Then, we obtain the following inequality,

$$\int_a^b W_{\text{mp}}\,\mathrm{dx} \geq \frac{\mu}{2}\|m'(x)\|^2_{L^2(a,b)} - 2\mu(b-a)^{\frac{1}{2}}\|m'(x)\|_{L^2(a,b)}, \tag{9.99}$$

Let $\xi = \int_a^b \|m'(x)\|\,\mathrm{dx} = \|m'(x)\|_{L^2(a,b)}$, then from the inequality (9.99) we get

$$\int_a^b W_{\text{mp}}\,\mathrm{dx} \geq \frac{\mu}{2}\xi^2 - 2\mu\sqrt{b-a}\cdot\xi. \tag{9.100}$$

By **Young's inequality** again, we know $2\mu\sqrt{b-a}\xi \leq \varepsilon\xi^2 + \left(\frac{\mu\sqrt{b-a}}{\sqrt{\varepsilon}}\right)^2$ with some yet unspecified coefficient $\varepsilon > 0$, then

$$I[m'(x),\alpha] \geq \int_a^b W_{\text{mp}}\,\mathrm{dx}$$
$$\geq \frac{\mu}{2}\xi^2 - \varepsilon\xi^2 - \left(\frac{\mu\sqrt{b-a}}{\sqrt{\varepsilon}}\right)^2$$
$$= (\frac{\mu}{2} - \varepsilon)\xi^2 - \left(\frac{\mu\sqrt{b-a}}{\sqrt{\varepsilon}}\right)^2. \tag{9.101}$$

By specifying $\varepsilon > 0$ such that $\frac{\mu}{2} - \varepsilon > 0$, we obtain the estimate in the form (8.57), therefore

$$I[m'(x),\alpha] \geq (\frac{\mu}{2} - \varepsilon)\xi^2 - \left(\frac{\mu\sqrt{b-a}}{\sqrt{\varepsilon}}\right)^2 \tag{9.102}$$
$$= (\frac{\mu}{2} - \varepsilon)\|m'(x)\|^2_{L^2(a,b)} - \left(\frac{\mu\sqrt{b-a}}{\sqrt{\varepsilon}}\right)^2, \quad \text{with } \frac{\mu}{2} > \varepsilon > 0.$$

Furthermore, from 6.45 the simplification of small system of differential equations

$$
\begin{aligned}
I[m'(x),\alpha] &= \int_a^b W_{\text{curv}}(\mathfrak{K}_{\mathfrak{s}}) + W_{\text{bend}}(\mathfrak{K}_{\mathfrak{b}})\,\mathrm{dx}, \quad \text{with} \quad \overline{\alpha}(a)=0,\\
&= \int_a^b K_{\text{curv}}|\overline{\alpha}'(x)|^{\hat{p}} + K_{\text{bend}}(\overline{\alpha}'(x))^2\,\mathrm{dx},\\
&= K_{\text{curv}}\cdot\int_a^b ||\overline{\alpha}'(x)||^{\hat{p}}\,\mathrm{dx} + K_{\text{bend}}\cdot\int_a^b ||\overline{\alpha}'(x)||^2\,\mathrm{dx}\\
&\geq K_{\text{curv}}\cdot||\overline{\alpha}'(x)||^{\hat{p}}_{L^2_{(a,b)}},
\end{aligned}
\tag{9.103}
$$

together with **Poincare's inequality** shows coercivity of I in $W^{1,2}$ w.r.t. m **and** coercivity of I in $W^{1,\hat{p}}$ w.r.t. α.

Now we prove that the map $m(x)\mapsto \mathcal{W}(m'(x)(\alpha))$, with $m'(x)=(m_1'(x),m_3'(x))^T$, and $\mathcal{W}(m'(x),\alpha)=W_a(m'(x),\alpha)+W_b(m'(x),\alpha)+W_c(m'(x),\alpha)$ is convex w.r.t. m'.
To this end let $x=m_1', y=m_3'$, we proof that $W_a(x,y,\alpha)$ is convex in the variable (x,y) first. Since the linear term $-2x\cos\overline{\alpha}+1$ is already convex, it is sufficient to consider

$$
W_a(x,y)=x^2(\cos\overline{\alpha})^2-x\sin\overline{\alpha}y\,\cos\overline{\alpha}+y^2(\sin\overline{\alpha})^2+\frac{1}{2}x^2(\sin\overline{\alpha})^2+\frac{1}{2}y^2(\cos\overline{\alpha})^2.
$$

The second derivative of $W_a(x,y)$, i.e. the Hesse-matrix is given by

$$
\nabla^2 W_a(x,y)=\begin{pmatrix} 2(\cos\overline{\alpha})^2+(\sin\overline{\alpha})^2 & -\sin\overline{\alpha}\cos\alpha\\ -\sin\overline{\alpha}\cos\overline{\alpha} & 2(\sin\overline{\alpha})^2+(\cos\overline{\alpha})^2\end{pmatrix}. \tag{9.104}
$$

Now consider the relevant quadratic form with argument $h=(h_1,h_2)$, then we have,

$$
\begin{aligned}
&(h_1,h_2)\cdot\nabla^2 W_a'(h_1,h_2)\cdot(h_1,h_2)^T \qquad (9.105)\\
&\quad= h_1^2(2(\cos\overline{\alpha})^2+(\sin\overline{\alpha})^2)-2h_1h_2\sin\overline{\alpha}\cos\overline{\alpha}+h_2^2(2(\sin\overline{\alpha})^2+(\cos\overline{\alpha})^2)\\
&\quad= h_1^2+(h_1\cos\overline{\alpha}+h_2\cos\overline{\alpha})^2+h_2^2\geq ||h||^2,
\end{aligned}
$$

which shows uniform convexity of $W_a(x,y)$. Similarly, $W_b(x,y)$ and $W_c(x,y)$ are both convex.

Now consider minimizing sequences $(m_{1,k},m_{3,k},\alpha_k)$ with $I(m_{1,k},m_{3,k},\alpha_k)$ bounded. The admissible set is defined as

$$
\begin{aligned}
\mathcal{A}=&\{m_1,m_3\in W^{1,2}(a,b),\alpha\in W^{1,\hat{p}}(a,b)\,|\,m_1(a)=g_1(a),\\
&m_3(a)=g_3(a),\alpha(a)=\alpha_d(a),m_1(b)=g_1(b),m_3(b)=g_3(b),\\
&\alpha(b)=\alpha_d(b)\quad\text{in the sense of trace}\}.
\end{aligned}
\tag{9.106}
$$

By coercivity we can concentrate on weakly convergent subsequences,

$$
\begin{aligned}
m_{1,k} &\rightharpoonup m_1 \quad \text{in} \quad W^{1,2}(a,b),\\
m_{3,k} &\rightharpoonup m_3 \quad \text{in} \quad W^{1,2}(a,b),\\
\alpha_k' &\rightharpoonup \alpha \quad\ \ \text{in} \quad W^{1,\hat{p}}(a,b).
\end{aligned}
\tag{9.107}
$$

Moreover, we can arrange for a subsequence of α_k which also converges strongly in $L^2(a, b)$ by computation.
Since α_k converges strongly, the convexity of $\mathcal{W}(m', \alpha)$ w.r.t. m' is enough to ensure weak lower semi-continuity. The bending and curvature terms are also convex in α', hence altogether we have,

$$I(m', \alpha) \leq \liminf I(m'_k, \alpha_k). \tag{9.108}$$

Since (m_k, α_k) is a minimizing sequence, i.e.

$$\lim I(m'_k, \alpha_k) \to I(\overline{m}', \overline{\alpha}') \quad \overline{m}', \overline{\alpha}' \in \mathcal{A}, \tag{9.109}$$

we conclude that the weak limit (m', α) is indeed a minimizer, i.e.

$$I(m', \alpha) = \inf I(\overline{m}', \overline{\alpha}).$$

This finishes the argument. ■