An Incorrect Inequality in Micropolar Elasticity Theory

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The theory of micropolar elasticity presented by Eringen [1] is identical with the theory of Cosserat elasticity developed by Aero and Kuvshinskii [2, 3], Mindlin [4], Neuber [5], and, in the two dimensional case, by Schaefer [6]. The couple stress elasticity theory presented by Aero and Kuvshinskii [7], Mindlin [8], and Mindlin and Tiersten [9] is a special case of Cosserat or micropolar elasticity. In papers by Kaloni and Ariman [10] and Chauhan [11] the Cosserat or micropolar theory is called Eringen's theory and the couple stress theory is called Mindlin's theory; these authors do not realize that the couple stress theory is a special case of the Cosserat or micropolar theory and are thus led to make 'comparisons' of Eringen's theory with Mindlin's theory. Kaloni and Ariman [10] also incorrectly remark that Mindlin's theory contradicts thermodynamic restrictions. The source of the difficulties in the papers by Kaloni and Ariman [10] and Chauhan [11] is an erroneous thermodynamic inequality in the theory of micropolar elasticity as presented by Eringen [1].

Before discussing the incorrect inequality, it is necessary to review notations used by the various authors. The material coefficients μ and τ appear in the notation employed in [12] and [13] for isotropic Cosserat elasticity; μ is the classical Lamé shear modulus and τ is the modulus of local rotational stiffness. A table given as an appendix to [12] lists the equivalent notations in the papers of Aero and Kuvshinskii [2, 3], Mindlin [4], Neuber [5] and Eringen [1]. Mindlin [4] and Aero and Kuvshinskii [2, 3] also denote Lamé's modulus by μ while Neuber [5] uses G. Mindlin denotes the modulus of local rotational stiffness τ by β; Aero and Kuvshinskii used $-\gamma$ for τ and Neuber uses Ga for τ. Chauhan [11], Kaloni and Ariman [10], and Ariman [14] use the notation of Eringen [1]. If Eringen's use of the symbol μ is denoted by μ^*, then the classical Lamé shear modulus μ and the modulus of local rotational stiffness τ are denoted by Eringen as follows:

$$\mu = \mu^* + \frac{1}{2}\varkappa\,, \quad \tau = \frac{1}{2}\varkappa\,. \tag{1}$$

That the classical Lamé shear modulus is $\mu^* + (1/2)\varkappa$ in Eringen's [1] notation can be seen from a study of his equations (3.11) or (3.19). It can also be shown by using the table of equivalent notations presented in [12]. It is easy to show that thermodynamic restrictions require that

$$\mu \geqslant 0\,, \quad \tau \geqslant 0\,. \tag{2}$$

These results are obtained, for example, by Aero and Kuvshinskii [3] and are recorded in their equation (22a). Equation $(5.1)_2$ of Eringen [1] requires that

$$\mu^* \geqslant 0\,. \tag{3}$$

The inequality (3) is incorrect. A mathematical or physical argument for the existance of such a restriction has not been presented. The only argument to the validity of the inequality (3) presented in [1] is that it '... is well known from the classical theory of elasticity'. Eringen has apparently mistaken his quantity μ^* for the classical Lamé shear modulus μ. Recasting (3) in the notation of [12] and [13] by employing (1), one finds

$$\mu \geqslant \tau \, . \tag{4}$$

This is a second version of the incorrect inequality (3).

The incorrect inequality (3) or (4) obfuscates the fact that the couple stress theory is a special case of the Cosserat or micropolar theory. To understand how this occurs, note that Mindlin [4] and Neuber [5] show that the isotropic couple stress theory is properly obtained from the isotropic Cosserat theory as follows:

$$(\text{Couple Stress Theory}) = \lim_{\tau \to \infty} (\text{Cosserat Theory}) \, . \tag{5}$$

Kinematically, the passage to the limit indicated above corresponds to the complete constraint of local rotation because τ is the modulus of local rotational stiffness. The relationship (5) does not make sense when used in conjunction with the incorrect inequality (4) because (4) requires that μ become infinite if τ becomes infinite and μ is *not* infinite in either the Cosserat theory or the couple stress theory. Thus, accepting (4) Kaloni and Ariman [10] and Chauhan [11] failed to realize that the couple stress theory is a special case of the Cosserat or micropolar theory.

The remark that the couple stress theory contradicts thermodynamic restrictions is also based on the incorrect inequality (4). The correct inequalities (2) permit τ to be any nonnegative real number, but (4) requires that τ be less than or equal to μ. Thus, if μ is a fixed value, any $\tau > \mu$ contradicts (4) but not (2). This is the source of Kaloni and Ariman's [10] erroneous remark that Mindlin's theory contradicts thermodynamic restrictions. Finally, the graphs of the solutions given by Kaloni and Ariman [10], Chauhan [11] and Ariman [14] to problems in Cosserat elasticity theory are only for a portion of the possible range of parameter values because these authors accept the incorrect inequality (4) and do not permit τ to exceed μ in value; the correct inequality (2) does not restrict τ in this fashion and therefore permits a more extensive range of parameters.

Additional Remarks

After this note was submitted, the writer was kindly informed by a referee and by Professor T. Ariman that the correct form of the inequality was obtained and presented by Eringen [15]. This is true; however, the incorrect conclusions of Kaloni and Ariman [10] arising from the use of the incorrect inequality are repeated in [15]. In particular, Eringen [15] repeats the 'comparisons' of micropolar theory (i.e., Cosserat theory) with the couple stress theory and the incorrect assertion that the couple stress theory is physically unreasonable because '$\varkappa$ cannot be as great as twice the shear modulus'. In the present notation the preceding assertion is equivalent to asserting that τ cannot exceed μ. It was obtained by Kaloni and Ariman [10] using the incorrect inequality (4).

Eringen corrected this inequality in [15], but failed to correct this assertion of physical unreasonableness which is a consequence of the use of the incorrect inequality.

The misleading Figures 2, 3, 4 and 5 of [10] are repeated as Figures 29, 30, 31 and 32 of [15]. It seems appropriate to comment more fully on these figures, which were only alluded to in the text above, because they illustrate the misleading theme that is repeated in [10], [11] and [15]. These figures were intended to compare the solution for the stress concentration factor in the problem of the cylindrical cavity in a field of uniaxial stress obtained from the Cosserat theory (Eringen's theory or the micropolar theory) with that obtained from the couple stress theory (Mindlin's theory). The solution to the problem of the cylindrical cavity in a field of uniaxial stress in the context of the Cosserat theory was given by Neuber [16]. The maximum normal stress $\tau_{\theta\theta}$ occurs on the circumference of the cavity at points whose tangents are parallel to the axis of applied stress. Denoting the uniform applied stress far away from the cavity by p, the stress concentration factor for the cylindrical cavity is given by

$$K_C = \frac{\tau_{\theta\theta}}{p} = \frac{3+F}{1+F} \tag{6}$$

where

$$F = \frac{8\,(1-\nu)\,N^2}{4+(L\,N)^2+2\,L\,N\,[K_0(N\,L)/K_1(N\,L)]}\,, \tag{7}$$

$$N = \sqrt{\frac{\tau}{\mu+\tau}}\,, \quad 0 \leqslant N \leqslant 1 \tag{8}$$

and where K_0, K_1 are modified Bessel functions of the second kind, ν is Poisson's ratio, and L is a dimensionless ratio of the cavity radius to a material parameter of dimension length. This notation is explained in more detail in [12] and [13]. The solution of this problem in the context of the couple stress theory was obtained by Mindlin [8] and Mindlin and Tiersten [9]. The couple stress theory solution is given by (6), (7) and (8) when $N = 1$. When $N = 0$ the solution given above coincides with the classical elasticity solution (i.e., $K_C = 3$). From the nature of the functional depend-

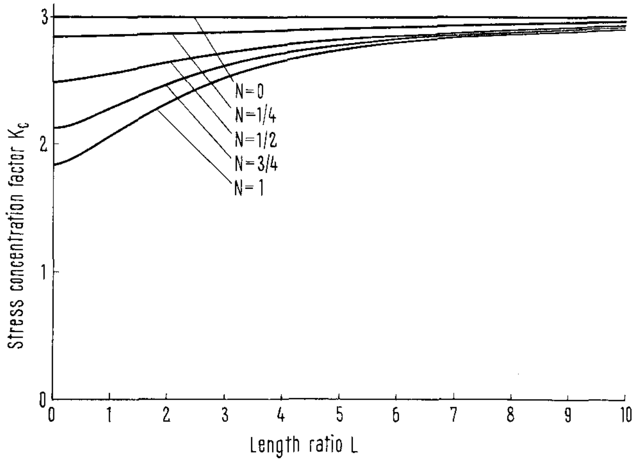

Figure 1
The stress concentration factor K_C as a function of L for $\nu = 0.3$ and various values of N.

ence of K_C upon N it can be seen that $N = 1$ and $N = 0$ represent the two extremal values for K_C. In Figure 1 the stress concentration factor K_C is plotted as a function of the length ratio L for $\nu = 0.3$ and various values of N. In comparing Figure 1 with Figures 2, 3, 4 and 5 of Kaloni and Ariman [10] note that the curve labeled '$N = 1$' here is labeled 'Mindlin's theory' there and that the curves labeled 'Eringen's theory' there are curves that lie between $N = 0$ and $N = 1/4$ here. Figures 2, 3, 4 and 5 of Kaloni and Ariman are misleading in that they imply that the Cosserat theory and the couple stress theory give different results, whereas, in fact, the couple stress theory is a special case of the Cosserat theory. Kaloni and Ariman [10] were themselves misled by their acceptance of the incorrect inequality (4) which restricted the range of their parameters unnecessarily. It is easy to see that the incorrect inequality (4) requires that N appearing in the solution above not exceed 1/4 while the correct inequality requires only that it not exceed 1.

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Zusammenfassung

Es wird festgestellt, dass in verschiedenen Veröffentlichungen über mikropolare Elastizität eine inkorrekte Ungleichung verwendet worden ist und dass dies zu unrichtigen Ergebnissen geführt hat.

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