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## ELASTICITY BEYOND THE ELASTIC LIMIT.\*

By M. REINER.

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1. Theories of elasticity have so far presupposed the existence of what Love (Art. 76) called a "state of ease" of "perfect elasticity" in which "a body can be strained without taking any set"; that state ranging between an "initial," "unstressed" and "unstrained" state (Art. 64) on one hand and the "elastic limit" on the other. Recent technological progress has gradually reduced, absolutely and, still more, relatively, the field in which this assumption holds good. Not only has increased accuracy of measurements of permanent sets lowered the elastic limit until in many cases as, for instance, annealed copper, it has nearly disappeared. More important, in materials which *do* show a definite elastic limit as, for instance, mild steel, deformations in most practical applications go *beyond* that limit. In addition, one has to consider elastic materials such as bitumen or cement-stone showing creep: their elastic potential gradually disappears through relaxation. Finally, there are such materials as rubber which can be caused to undergo very large deformations, a certain part of which will always be non-recoverable. It therefore becomes necessary to consider elasticity beyond the elastic limit. If we define elasticity with Love as "the property of recovery of an *original* size and shape," there would in all these cases be no question of elasticity because the *original* size and shape is not recovered. However, *some* of the deformation is *always* recovered: but *which* part of it is recoverable, becomes apparent only when all external forces, gravity included, *have been removed*. We may denote as the *ground-position* that position of the body which is then reached. To every deformation there corresponds a ground-position of its own, which generally will not be the *initial* position from which the deformation started. Let us denote by *deformation* a change of size and of shape in general, whether recoverable or not, and by *strain* that part of it which is recovered when all external forces have been removed. Generally, the strain will differ from the deformation not only in magnitude, but also in the orientation of the principal axes.

The ground-position is accordingly an unstrained and unstressed state, but it is not an *undeformed* state. A general theory of elasticity, then, has to relate the strain as now defined to (i) the stress produced by it and (ii) the

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external forces necessary to equilibrate the stresses in the body in accordance with d'Alembert's principle; while the classical theory of the "state of ease" refers to the special case when the strain is identical with the deformation. The considerations of the present paper are, however, also applicable in the latter case.

2. The classical theory was brought to completion by Murnaghan when considering finite strain. He derived the relation between the stress tensor  $T^{rs}$  and the strain tensor  $\epsilon_{rs}$  from a formula connecting the elastic potential  $\phi$  with the stress tensor.<sup>1</sup> That formula itself was derived by considering the virtual work of the stresses across a closed boundary of a portion of the material. This method is inapplicable in our case. If we write the fundamental law of thermodynamics for isothermal processes in the form of the Gibbs-Helmholtz equation

$$(2.1) \quad \delta w = \rho \delta \phi + \rho \delta \psi$$

where  $w$  is the strainwork per unit volume,  $\phi$  the intrinsic free energy-density and  $\psi$  the bound energy-density (compare Weissenberg, 1931), not only will  $\psi$  in our general case not vanish, but what is more remarkable, as Taylor and Quinney have shown in a metal which is subjected to cold working, part of the free energy is "latent" and not recoverable mechanically. We therefore must apply that other method used in the classical theory for the derivation of the stress-strain relation (e.g. by Stokes) which is a generalization of Hooke's law, writing

$$(2.2) \quad T_s^r = f(\epsilon_s^r)$$

and developing the function  $f$  by means of tensor analysis, as was done by Reiner in the analogous case of viscosity. The equation will then express a law of elasticity if  $\epsilon$  indicates the strain defined above as the recovered part of the deformation and if the relation connecting  $T_s^r$  and  $\epsilon_s^r$  is unequivocal. From the last condition there follows, that we can also write

$$(2.3) \quad \epsilon_s^r = \bar{f}(T_s^r).$$

In the experimental determination of the relation one would have in principle to proceed as follows: Subject a material to external forces and let it undergo a process of deformation of a certain type,<sup>2</sup> arrest the deformation and record

<sup>1</sup> We shall use in the present paper wherever possible Murnaghan's notation.

<sup>2</sup> For "type" compare Love (Art. 73).

the magnitude of stress;<sup>3</sup> mark a sphere of unit radius in the material around some selected point; remove all external forces: this will induce relative displacements in the material changing the sphere into an ellipsoid called the reciprocal strain ellipsoid; wait until this movement dies out; measure the axes of the ellipsoid: they will provide a measure of strain; repeat this experiment reaching different magnitudes of the same type of deformation in a gradually increasing or decreasing order; record the strain, as determined, against the stress: provided the relaxation of the stress is negligible, the result is an empirical relation for (2.3) depending upon the type of deformation.<sup>4</sup> For instance, in the usual tensile test for metals in the work-hardening range, when the volume of an element of the material can be assumed as constant and the deformation has axial symmetry, only one axis of the ellipsoid need be measured *viz.*, either along or across the test piece and the empirical formula relates the axial traction  $p_{zz}$  to the axial strain  $\epsilon_{zz}$ .

3. In the first stage of our investigation we need not fix the *measure* of strain. Denoting the three axes of the above mentioned ellipsoid by  $l_i$  ( $i$  running over 1, 2, 3) or, what is sometimes more convenient, the axes of the strain ellipsoid by  $\lambda_i = 1/l_i$  "one may use (as Weissenberg (1946) pointed out) any function of the elongation ratios ( $\lambda_i$ ) in the direction of the main axes choosing the function to suit the particular field of investigation." One would, naturally, require that all these functions are reduced for infinitesimal strain to the Cauchy measure  $\lambda_i - 1$ . This is the case with the Kirchoff-measure which is based upon  $\frac{1}{2}[(\lambda_i)^2 - 1]$  and the Murnaghan-measure based upon  $\frac{1}{2}[1 - (l_i)^2]$ ; it is also so with the measure  $\ln(\lambda_i) = -\ln(l_i)$  originally proposed by Roentgen for rubber and, since systematically introduced by Hencky, now widely in use. We may also mention the measure  $(\lambda_i - l_i)$  proposed by Wall. All these measures comply also with a second requirement, *viz.* that the strain vanishes for  $\lambda_i = 1 = l_i$ . It is clear that a linear stress-strain relation in one measure will be non-linear in every other and the desire for linearity is often one of the motives behind the introduction of one or the other of the measures mentioned, our enumeration being far from complete.

4. Starting from (2.2) or (2.3), we follow the reasoning applied by Reiner, as has already been mentioned, in the analogous case of viscous

<sup>3</sup> It is necessary first to arrest the deformation as, generally, part of the stress will be due to viscous resistance, depending upon the velocity of deformation.

<sup>4</sup> "If . . . the stress-strain relations can be found experimentally, the strain-energy function can be calculated" (Sokolnikoff, p. 89).

resistance. We note that on the left side there stands a mixed tensor of rank two. Then in a development of either function  $f$  or  $\mathfrak{f}$  all terms on the right side must also be mixed tensors of rank two. The right side can therefore consist only of sums of mixed tensors of rank two multiplied by scalars and of inner products of such tensors which again are reduced to tensors of rank two. The general term of a development of the function  $f$  will therefore be of the form  $\epsilon_a^r \epsilon_\beta^a \epsilon_\gamma^\beta \cdots \epsilon_s^\lambda \cdot f(I)$ , where  $f(I)$  is a function of the three invariants: and we can therefore write

$$(4.1) \quad T_s^r = f_0 \delta_s^r + f_1 \epsilon_s^r + f_2 \epsilon_a^r \epsilon_s^a + f_3 \epsilon_a^r \epsilon_\beta^a \epsilon_s^\beta + \cdots.$$

This would mean an infinite number of such terms. However, in view of the Cayley-Hamilton equation of matrix theory, the following relation holds good<sup>5</sup>

$$(4.2) \quad \epsilon_a^r \epsilon_\beta^a \epsilon_s^\beta = \delta_s^r III - \epsilon_s^r II + \epsilon_a^r \epsilon_s^a I$$

where  $I$ ,  $II$  and  $III$  are the first, second and third invariants respectively.

Therefore

$$(4.3) \quad \begin{aligned} \epsilon_a^r \epsilon_\beta^a \epsilon_\gamma^\beta \epsilon_s^\gamma &= \delta_a^r \epsilon_s^a III - \epsilon_a^r \epsilon_s^a II + \epsilon_a^r \epsilon_\beta^a \epsilon_s^\beta I \\ &= \delta_s^r I \cdot III + \epsilon_s^r (III - I \cdot II) + \epsilon_a^r \epsilon_s^a (I^2 - II) \end{aligned}$$

and similarly with respect to higher terms. This enables us to write

$$(4.4) \quad T_s^r = F_0 \delta_s^r + F_1 \epsilon_s^r + F_2 \epsilon_a^r \epsilon_s^a$$

and analogously

$$(4.5) \quad \epsilon_s^r = \mathfrak{F}_0 \delta_s^r + \mathfrak{F}_1 T_s^r + \mathfrak{F}_2 T_a^r T_s^a$$

where the  $F$  are functions of the three invariants  $I_\epsilon$ ,  $II_\epsilon$  and  $III_\epsilon$  of the strain-, and the  $\mathfrak{F}$  functions of the three invariants  $I_T$ ,  $II_T$  and  $III_T$  of the stress-tensor. Prager has recently derived equations built up in a manner similar to (4.4) and (4.5), but subject to specializations due to certain simplifying assumptions. Our equations are general and express nothing more than that both stress and strain are tensors of rank two, the principal axes of which coincide; and that the functions  $F$  and  $\mathfrak{F}$  are scalars. We may call a material in such a state *isotropic*.

However, we also require that in the ground-position, when the stress is removed, the strain should also vanish, and *vice versa*. Therefore

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<sup>5</sup> These developments are entirely analogous to those of Reiner for the viscous liquids, but it was thought desirable to make the present paper self-contained.

$$(4.6) \quad F_0 = F_{01}I_\epsilon + F_{02}II_\epsilon + F_{03}III_\epsilon$$

$$(4.7) \quad \mathcal{F}_0 = \mathcal{F}_{01}I_T + \mathcal{F}_{02}II_T + \mathcal{F}_{03}III_T$$

where the new  $F$  and  $\mathcal{F}$  are again, in general, functions of all three invariants  $I$ ,  $II$  and  $III$ .

The functions  $F$  are *moduli* of elasticity, the functions  $\mathcal{F}$  *coefficients* of elasticity; the latter, generally, *not* the reciprocals of the former. There are therefore, generally, five of each kind, each one possessing  $\infty^3$  values in accordance with the values which the invariants may have in every particular case. In the expressions for  $F$  and  $\mathcal{F}$  as functions of the invariants, there will appear a number of parameters, which are the elastic "constants" of the material. The  $F$  and  $\mathcal{F}$  may, of course, themselves be constants; in special cases some of them may vanish, in other cases they may not be independent; and this would reduce their number from five to less.

The  $F$  and  $\mathcal{F}$  can be given physical interpretations only when a definite measure of strain is assumed and we shall examine what consequences the adoption of any such measure may have.

5. Before dealing with the problem in a general way, it will be useful to examine the special case of simple shear dealt with by Love in Art. 37. This is given kinematically by the equations

$$(5.1) \quad x_1 = x + sy; \quad y_1 = y; \quad z_1 = z.$$

Putting

$$(5.2) \quad s = 2 \tan \alpha,$$

Love calculates

$$(5.3) \quad \lambda_1 = \frac{1 - \sin \alpha}{\cos \alpha}; \quad \lambda_2 = \frac{1 + \sin \alpha}{\cos \alpha}; \quad \lambda_3 = 1$$

and he proves that the directions of the principal axes of strain are the bisectors of the angle  $(\pi/2) + \alpha$  with the  $x$ -axis, and the angle through which the principal axes are turned is the angle  $\alpha$ . The stress caused by the strain will have the principal components  $T_1, T_2, T_3$  which from (4.4) and (4.6) are

$$(5.4) \quad T_i = F_{01}I + F_{02}II + F_{03}III + F_{1\epsilon_i} + F_{2\epsilon_i}^2.$$

The components of stress with respect to the system  $x, y, z$  will be from Love's equations, Art. 49:

$$\begin{aligned}
 (5.5) \quad T_{xx} &= \frac{1}{2}(T_1 + T_2) - \frac{1}{2}(T_1 - T_2) \sin \alpha \\
 T_{yy} &= \frac{1}{2}(T_1 + T_2) + \frac{1}{2}(T_1 - T_2) \sin \alpha \\
 T_{zz} &= T_3 \\
 T_{xy} &= -\frac{1}{2}(T_1 - T_2) \cos \alpha; \quad T_{yz} = T_{zx} = 0.
 \end{aligned}$$

Introducing the expressions for the principal stresses from (5.4) into (5.5) gives

$$\begin{aligned}
 (5.6) \quad T_{xx} &= F_{01}I + F_{02}II + F_{03}III + \frac{1}{2}\{F_1[(\epsilon_1 + \epsilon_2) - (\epsilon_1 - \epsilon_2) \sin \alpha] \\
 &\quad + F_2[(\epsilon_1^2 + \epsilon_2^2) - (\epsilon_1^2 - \epsilon_2^2) \sin \alpha]\} \\
 T_{yy} &= F_{01}I + F_{02}II + F_{03}III + \frac{1}{2}\{F_1[(\epsilon_1 + \epsilon_2) + (\epsilon_1 - \epsilon_2) \sin \alpha] \\
 &\quad + F_2[(\epsilon_1^2 + \epsilon_2^2) + (\epsilon_1^2 - \epsilon_2^2) \sin \alpha]\} \\
 T_{zz} &= F_{01}I + F_{02}II + F_{03}III \\
 T_{xy} &= -\frac{1}{2}[F_1(\epsilon_1 - \epsilon_2) + F_2(\epsilon_1^2 - \epsilon_2^2)] \cos \alpha.
 \end{aligned}$$

We now assume definite measures of strain. If  $l_0$  is a length extended in simple elongation by  $\Delta l$  to  $l$ , the measure of the extension may relate  $\Delta l$  to either  $l_0$  or  $l$ , or it may relate an element of elongation  $dl$  to the instantaneous length  $l$ . These three possibilities correspond to the Kirchhoff-measure.

$$(5.7) \quad \epsilon_i^K = \frac{1}{2}(\lambda_i^2 - 1)$$

the Murnaghan-measure

$$(5.8) \quad \epsilon_i^M = \frac{1}{2}(1 - l_i^2) = \frac{1}{2}(1 - 1/\lambda_i^2)$$

and to the logarithmic or Hencky-measure

$$(5.9) \quad \epsilon_i^H = \ln \lambda_i = -\ln l_i.$$

Introducing the expressions  $\lambda_i$  from (5.3), we find the principal strain-components, the strain-invariants and the stress-components in the  $x$ ,  $y$  and  $z$  directions as entered in the following Table:

$\epsilon_i$	$\frac{1}{2}(\lambda_i^2 - 1)$	$\ln \lambda_i$	$\frac{1}{2}(1 - 1/\lambda_i^2)$
$\epsilon_1$	$-\tan \alpha \frac{1 - \sin \alpha}{\cos \alpha}$	$\ln \frac{1 - \sin \alpha}{\cos \alpha}$	$-\tan \alpha \frac{\cos \alpha}{1 - \sin \alpha}$
$\epsilon_2$	$\tan \alpha \frac{\cos \alpha}{1 - \sin \alpha}$	$\ln \frac{\cos \alpha}{1 - \sin \alpha}$	$\tan \alpha \frac{1 - \sin \alpha}{\cos \alpha}$
$\epsilon_3$	0	0	0
$I_\epsilon$	$2(\tan \alpha)^2$	0	$-2(\tan \alpha)^2$
$II_\epsilon$	$-(\tan \alpha)^2$	$-(\ln \frac{\cos \alpha}{1 - \sin \alpha})^2$	$-(\tan \alpha)^2$
$III_\epsilon$	0	0	0
$T_{xx}$	$(\tan \alpha)^2 [2F_{01} - F_{02} + 2F_1 + F_2(1 + 4 \tan^2 \alpha)]$	$\ln \frac{\cos \alpha}{1 - \sin \alpha} [(F_2 - F_{02}) \ln \frac{\cos \alpha}{1 - \sin \alpha} + F_1 \sin \alpha]$	$-(\tan \alpha)^2 (2F_{01} + F_{02} - F_2)$
$T_{yy}$	$(\tan \alpha)^2 (2F_{01} - F_{02} + F_2)$	$\ln \frac{\cos \alpha}{1 - \sin \alpha} [(F_2 - F_{02}) \ln \frac{\cos \alpha}{1 - \sin \alpha} - F_1 \sin \alpha]$	$-(\tan \alpha)^2 [2F_{01} + F_{02} + 2F_1 - F_2(1 + 4 \tan^2 \alpha)]$
$T_{zz}$	$(\tan \alpha)^2 (2F_{01} - F_{02})$	$-(\ln \frac{\cos \alpha}{1 - \sin \alpha})^2 F_{02}$	$-(\tan \alpha)^2 (2F_{01} + F_{02})$
$T_{xy}$	$\tan \alpha (F_1 + 2F_2 \tan^2 \alpha)$	$\ln \frac{\cos \alpha}{1 - \sin \alpha} F_1$	$\tan \alpha (F_1 - 2F_2 \tan^2 \alpha)$



In infinitesimal strain we may neglect  $(\tan \alpha)^2$  and introducing  $\tan \alpha = \alpha = s/2$  all three measures give the same  $\epsilon_2 = -\epsilon_1 = s/2$ ,  $T_{xy} = F_1 s/2$ , while normal tractions  $T_{xx}$ ,  $T_{yy}$  and  $T_{zz}$  vanish. In this case simple shear is accompanied by a shearing stress only. In finite strain such a simple relation is not possible, whatever the values of the elastic moduli. We may in the Kirchhoff measure make  $F_{02} = 2F_{01}$  and  $T_{zz}$  vanishes. We may in addition make  $F_2 = 0$  and  $T_{yy}$  will vanish. But there *must* remain a tension in the direction  $x$  which is  $T_{xx} = 2F_1(\tan \alpha)^2$  and we cannot put  $F_1 = 0$  because then  $T_{xy}$  also would vanish. Alternately, we may put  $F_2 = -2F_1/[1 + 4(\tan \alpha)^2]$  which would make  $T_{xx}$  vanish, but leave a pressure  $T_{yy} = -2F_1(\tan \alpha)^2/[1 + 4(\tan \alpha)^2]$ . Conditions are similar in other measures. In an isotropic material finite simple shear is accompanied by either a tension in the direction of the displacement or compression in the direction of its gradient or both. Weissenberg (1947) has demonstrated the existence of such stresses in elastic liquids in a series of striking experiments.

6. The present theory is distinguished from the usual theory of elasticity of finite strains mainly by the appearance of the modulus  $F_2$ . In the usual theory, Equation (5.4) would be

$$(6.1) \quad T_i = F_0 + F_1 \epsilon_i$$

which constitutes three equations with two unknowns, *viz.*, the moduli  $F_0$  and  $F_1$ . In order that these equations should be consistent, certain relations between  $T_i$  and  $\epsilon_i$  must be satisfied. The matrix of the coefficients is of rank two. The augmented matrix

$$(6.2) \quad K \equiv \left\| \begin{array}{ccc} 1 & \epsilon_1 & T_1 \\ 1 & \epsilon_2 & T_2 \\ 1 & \epsilon_3 & T_3 \end{array} \right\|$$

must therefore also be of the rank two. This requires the determinant

$$(6.3) \quad \left| \begin{array}{ccc} 1 & \epsilon_1 & T_1 \\ 1 & \epsilon_2 & T_2 \\ 1 & \epsilon_3 & T_3 \end{array} \right| \equiv 0$$

or

$$(6.4) \quad \frac{T_1 - T_2}{\epsilon_1 - \epsilon_2} = \frac{T_2 - T_3}{\epsilon_2 - \epsilon_3} = \frac{T_3 - T_1}{\epsilon_3 - \epsilon_1} = F(I_\epsilon, II_\epsilon, III_\epsilon, I_T, II_T, III_T).$$

Equation (6.4) has been proposed by Weissenberg (1947) as a law of elasticity. As has, however, been shown here, it is not general enough and is not

independent of the *measure* of strain. For instance, should experiments show that simple shear is accompanied by a tension in the direction of displacement, the Murnaghan measure could not be used. On the other hand, should experiments show that it is accompanied by compression in the direction of the gradient of the displacement, the Kirchhoff measure could not be used. In the form of Equation (5.4) the law of elasticity is independent of the measure and does not prejudice the outcome of experiment.

7. Considering that, by including the modulus  $F_2$  (or the coefficient  $\mathcal{F}_2$ ), we are independent of the measure of strain, we may for our further investigation assume any measure. We shall select the Hencky-measure for two reasons:

(i) Because of  $\epsilon_i = \ln \lambda_i$ ,  $\dot{\epsilon}_i = \dot{\lambda}_i / \lambda_i$ . Denoting by  $e_i$  the principal "velocity-extension" of hydrodynamics, we accordingly get  $e_i = \dot{\epsilon}_i$ , provided the principal axes do not rotate. Therefore in *pure* strain, in the Hencky-measure, to use Murnaghan's words "the variation of the strain tensor (is equal) to the space derivative of the virtual displacement vector." This is of advantage, especially if we consider that it may be possible in many cases to arrange "the removal of the external forces" (compare 2 and 3 above) in such a way that the axes do not rotate and the strain is accordingly pure.

(ii) Secondly, from

$$(7.1) \quad V/V_0 = \lambda_1 \cdot \lambda_2 \cdot \lambda_3,$$

there follows

$$(7.2) \quad \epsilon_v = \ln(V/V_0) = \ln \lambda_1 + \ln \lambda_2 + \ln \lambda_3 = \epsilon_1 + \epsilon_2 + \epsilon_3 = I_\epsilon.$$

Therefore, in the Hencky measure, and only in that measure, the cubical dilation is equal to the first invariant of the strain tensor.<sup>6</sup> Accordingly, only in this measure has the resolution of the tensor in an isotropic and a deviatoric component physical significance.

8. By carrying out the resolutions

$$(8.1) \quad T_s^r = T \delta_s^r + T'^s_r; \quad \epsilon_s^r = \epsilon \delta_s^r + \epsilon'^s_r$$

where

$$(8.2) \quad T_a^a = 3T, \quad \epsilon_a^a = 3; \quad T'^a_a = \epsilon'^a_a = 0$$

we get from (4.4) and (4.6)

<sup>6</sup> This is, of course, also the case in infinitesimal strain

$$(8.3) \quad \begin{aligned} T &= F_{01} + F_{02} + II'_\epsilon + F_{03}III'_\epsilon \\ T'_{s^r} &= F_1\epsilon'_{s^r} + F_2(\epsilon'_\alpha{}^r\epsilon'_{s^a} + 2II'_\epsilon/3 \cdot \delta_s{}^r) \end{aligned}$$

and from (4.5) and (4.7)

$$(8.4) \quad \begin{aligned} \epsilon &= \mathcal{F}_{01}T + \mathcal{F}_{02}II'_T + \mathcal{F}_{03}III'_T \\ \epsilon'_{s^r} &= \mathcal{F}_1T'_{s^r} + \mathcal{F}_2(T'_\alpha{}^rT'_{s^a} + 2II'_T/3 \cdot \delta_s{}^r) \end{aligned}$$

where the accents indicate the deviator and the  $F$  and  $\mathcal{F}$  are now functions of the invariants of the deviator, different from the functions  $F$  and  $\mathcal{F}$  appearing in (4.4) to (4.7).

If we introduce  $\epsilon'_{s^r}$  from (8.4) into (8.3), considering that  $\delta_s{}^r$ ,  $T'_{s^r}$  and  $T'_\alpha{}^rT'_{s^a}$  stand for the zero, first and second powers in the stress components, we find

$$(8.5) \quad \begin{aligned} F_1 &= \left| \frac{\mathcal{F}_1^2 + II'_T/3 \cdot \mathcal{F}_2^2}{\mathcal{F}_1[\mathcal{F}_2III'_T - 2II'_T/3 \cdot \mathcal{F}_1]\mathcal{F}_2} \right| \\ F_2 &= \frac{\mathcal{F}_2}{\text{as above}} \cdot \end{aligned}$$

The moduli of elasticity  $F$  are therefore generally not the reciprocals of the coefficients of elasticity  $\mathcal{F}$ .

We now carry out in imagination a series of experiments such as mentioned at the end of Section 2.

(i) Firstly, we apply a uniform hydrostatic pressure; here the stress tensor is a scalar tensor

$$(8.6) \quad T_s{}^r = -p\delta_s{}^r$$

where  $p$  is what is commonly called "pressure" and the stress invariants are

$$(8.7) \quad T = -p, \quad II'_T = III'_T = 0$$

$T'_{s^r}$  and  $T'_\alpha{}^rT'_{s^a}$ , therefore, vanish and the second of (8.4) gives  $\epsilon'_{s^r} = 0$ , while the first yields

$$(8.8) \quad \epsilon = -p\mathcal{F}_{01}(T, 0, 0).$$

This defines a *coefficient of volume elasticity*

$$(8.9) \quad k' = -3\epsilon/p = 3\mathcal{F}_{01}.$$

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<sup>7</sup> For the derivation compare Reiner.

Considering that  $II'_\epsilon$  and  $III'_\epsilon$  vanish, the first of (8.3) gives

$$(8.10) \quad T = -p = \epsilon \mathcal{F}_{01}.$$

This defines the *modulus of compression*

$$(8.11) \quad k = -p/3\epsilon = F_{01}/3$$

and  $k' = 1/k$ .

(ii) In the second experiment we apply a tangential stress

$$(8.12) \quad T'_s r = \begin{vmatrix} 0 & T_{xy} & 0 \\ T_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = T'_s r$$

so that

$$(8.13) \quad T = 0; \quad II'_T = -T_{xy}^2; \quad III'_T = 0$$

and

$$(8.14) \quad T'_\alpha r T'_s \alpha = T_{xy}^2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

This makes (8.4)

$$(8.15) \quad \epsilon = -T_{xy}^2 \mathcal{F}_{02}(0, II'_T, 0)$$

$$\epsilon'_s r = \mathcal{F}_1(0, II'_T, 0) T_{xy} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} + \mathcal{F}_2(0, II'_T, 0) \frac{T_{xy}^2}{3} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{vmatrix}$$

and defines three coefficients of elasticity, *viz.*

$$(8.16) \quad \begin{aligned} \delta' &= -\epsilon/II'_T = -\mathcal{F}_{02} \\ \mu' &= 2\mathcal{F}_1 \\ \alpha' &= -2\mathcal{F}_{2/3}. \end{aligned}$$

The coefficient  $\mathcal{F}_1$  connects shearing stress with shearing strain and is accordingly a generalized *coefficient of shear elasticity* or of *rigidity*. The isotropic component of the strain,  $\epsilon$ , is a measure of the cubical dilation. If  $\delta'$  does not vanish, a simple shearing stress will produce an increase (or decrease for negative  $\delta'$ ) of the volume measured by  $\delta' T_{xy}^2$ .<sup>8</sup>

<sup>8</sup> It is remarkable that Sir William Thomson (Lord Kelvin) should have foreseen in 1875 the possible existence of such a phenomenon on purely theoretical grounds, *vide* the following quotation: "It is possible that a shearing stress may produce in a truly isotropic solid condensation or dilatation in proportion to the square of its value; and it is possible that such effect may be sensible in india-rubber or cork, or other bodies susceptible of great deformations or compressions with persistent elasticity." Footnote p. 34, *Math. & Phys. Papers*, Vol. III, London, 1890. Weissenberg has observed negative elastic dilatancy in porous rubber (not yet published).

Accordingly,  $\delta'$  may be termed the *coefficient of (elastic) dilatancy* (compare Reiner). Should  $\delta'$  vanish, but not  $\alpha'$ , then a simple shearing stress will produce (in the case of a positive  $\alpha'$ ) an extension normal to the plane of shear (in our case the  $z$ -direction) which is equal to  $\alpha' T_{xy}^2$ , together with two lateral contractions equal to  $\alpha'/2 \cdot T_{xy}^2$ , so that the volume is not changed. If  $\delta'$  should not vanish, there will be superposed a change of volume. We may call  $\alpha'$  the *coefficient of cross-elasticity*.

(iii) If we force upon the material a tangential strain, we shall similarly find three moduli of elasticity

$$(8.17) \quad \begin{aligned} \delta &= -F_{02}/4 \\ \mu &= F_{1/2} \\ \alpha &= -F_2/6 \end{aligned}$$

of which  $\mu$  is a generalized *shear modulus* or *modulus of rigidity*.  $\delta$ , the *modulus of dilatancy*, will measure a hydrostatic tension necessary to maintain simple shear; and  $\alpha$ , a *modulus of cross-elasticity*, measures a stress produced by simple shear, in the direction normal to its plane.

(iv) Simple pull, in infinitesimal elasticity employed to determine Young's modulus and Poisson's ratio, gives us

$$(8.18) \quad T_s r = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T_{zz} \end{vmatrix}$$

so that

$$(8.19) \quad T = T_{zz}/3; T'_s r = \frac{T_{zz}}{3} \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{vmatrix}; T'_\alpha r T'_s \alpha = \frac{T_{zz}^2}{9} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{vmatrix}$$

and

$$(8.20) \quad II'_T = -T_{zz}^2/3; \quad III'_T = 2T_{zz}^3/27.$$

This makes

$$(8.21) \quad \begin{aligned} \epsilon_s r &= T_{zz}/3 (k'/3 + T_{zz}\delta' + (2T_{zz}^2/9)\mathfrak{F}_{03}) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &+ (T_{zz}/6)(\mu' - \alpha'T_{zz}) \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{vmatrix} \end{aligned}$$

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<sup>o</sup> Note that a simple shear is measured traditionally by twice the tangential component of the strain-tensor.

and defines a generalized Youngs' modulus

$$(8.22) \quad E^{-1} = \epsilon_{zz}/T_{zz} = -\frac{1}{3}[k'/3 + \mu' + T_{zz}(\delta' - \alpha') + (2T_{zz}^2/9)\mathcal{F}_{03}]$$

and a generalized Poisson-ratio

$$(8.23) \quad \sigma = -\epsilon_{xx}/\epsilon_{zz} = -\frac{k'/3 - \mu'/2 + T_{zz}(\delta' + \alpha'/2) + (2T_{zz}^2/9)\mathcal{F}_{03}}{k'/3 + \mu' + T_{zz}(\delta' - \alpha') + (2T_{zz}^2/9)\mathcal{F}_{03}}.$$

Either  $E$  or  $\sigma$  can be used to determine a further coefficient of elasticity

$$(8.24) \quad \beta' = 2\mathcal{F}_{03}/9.$$

Summarizing, we can now write (8.4) as follows:

$$(8.26) \quad \begin{aligned} \epsilon_v &= -k'p - 3\delta'II'T + (2\gamma/2)\beta'III'T \\ \epsilon'_{s'r} &= (\mu'/2)T'_{s'r} - 3(\alpha'/2)(T'_{a'r}T'_{s'a} + 2II'T/3 \cdot \delta_s r) \end{aligned}$$

and (8.3) as follows:

$$(8.27) \quad \begin{aligned} p &= -k\epsilon_v + 4\delta II'\epsilon - (9\beta/2)III'\epsilon \\ T'_{s'r} &= 2\mu\epsilon'_{s'r} - 6\alpha(\epsilon'_{a'r}\epsilon'_{s'a} + 2II'\epsilon/3 \cdot \delta_s r) \end{aligned}$$

where  $p$  is the hydrostatic pressure and  $\epsilon_v$  the cubical dilatation,  $\epsilon'_{s'r}$  the deviator of strain and  $T'_{s'r}$  the deviator of stress,  $k, \delta, \beta, \mu, \alpha$  moduli of elasticity and  $k', \delta', \beta', \mu', \alpha'$  coefficients of elasticity. These are generally functions of all three invariants of stress and strain respectively, but may also degenerate to constants. A hydrostatic tension will cause a cubical dilatation and *vice versa*; but a cubical dilatation may also be caused in the absence of a hydrostatic tension by either simple shearing stress or traction. Likewise, a hydrostatic pressure may be required to maintain simple shear or a volume-constant simple extension. Finally, a simple shearing stress may not only produce a corresponding shearing strain, but also "sideways" a volume-constant extension. Likewise simple shear may require for its maintenance not only a corresponding shearing stress but also "sideways" a traction. The general elastic body has accordingly three additional properties absent in classical elasticity, namely dilatancy of two kinds, (shear- and tractional dilatancy) and cross-elasticity. It is not so much the property of dilatancy predicted by Kelvin as early as 1875 and observed as a permanent set by Reynolds as early as 1885, which is challenging, but the cross-elasticity, which is connected with the functions  $\mathcal{F}_2$  and  $F_2$  respectively. We, therefore, consider this property again from a different aspect.

9. Let  $n$  be the normal to an element of interface in the interior or of surface on the boundary of the body under consideration. Let the traction

$T_n$  be resolved into three orthogonal components  $T_{nq}$  where  $q$  runs through  $n$ ,  $t$  and  $c$ ;  $t$  being the direction parallel to the face and  $c$  the direction cross-wise to  $n$  and  $t$ , so that

$$(9.1) \quad T_{nc} = 0.$$

Let  $\epsilon_n$  be resolved in the same directions. We find, then, from the second of (8.4)

$$(9.2) \quad \epsilon'_{nc} = \mathcal{F}_2 T'_{na} T'_{ac}$$

the term following within brackets disappearing because  $r \neq s$  ( $n \neq c$ ). Now

$$(9.3) \quad T'_{na} T'_{ac} = T'_{nn} T'_{nc} + T'_{nt} T'_{tc} + T'_{nc} T'_{cc}.$$

As (9.2) is not affected by an isotropic stress component,  $T'_{nc}$  vanishes also and this reduces (9.3) to

$$(9.4) \quad T'_{na} T'_{ac} = T'_{nt} T'_{tc}.$$

Now on the right side of (9.4)  $T'_{nt}$  does not vanish, by definition; and if one imagines in the standard cube which defines  $T_{xx}$  etc.,  $x, y, z$  replaced by  $n, t, c$ , it is clear that  $T'_{tc}$  will, in general, not vanish. Therefore  $\epsilon'_{nc}$  is finite. This brings out very strikingly a consequence of the existence of  $\mathcal{F}_2$  and supports the designation "cross-elasticity." We have, however, shown that  $\mathcal{F}_2$  (or  $F_2$ ) can generally not be omitted without prejudicing experimental results.

TEL AVIV, PALESTINE.

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