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tion of the section. Catskill is simply epochal but "Chemung" carries with it the conception of those physical and biological characteristics which mark the great closing period of the Devonian.

Chemung, therefore, and not Catskill is the epoch whose name should be applied to designate the whole group, while Catskill must be retained in its original signification only.

University of the City of New York.

Art. XLVIII.—*The Finite Elastic Stress-Strain Function;* by Geo. F. Becker.

Hooke's Law.—The law proposed by Hooke to account for the results of experiments on elastic bodies is equivalent to:—Strain is proportionate to the load, or the stress initially applied to an unstrained mass. The law which passes under Hooke's name is equivalent to:—Strain is proportional to the final stress required to hold a strained mass in equilibrium.* It is now universally acknowledged that either law is applicable only to strains so small that their squares are negligible. There are excellent reasons for this limitation. Each law implies that finite external forces may bring about infinite densities or infinite distortions, while all known facts point to the conclusion that infinite strains result only from the action of infinite forces. When the scope of the law is confined to minute strains, Hooke's own law and that known as his are easily shown to lead to identical results; and the meaning is then simply that the stress-strain curve is a continuous one cutting the axes of no stress and of no strain at an angle whose tangent is finite. Hooke's law in my opinion rests entirely upon experiment, nor does it seem to me conceivable that any process of pure reason "should reveal the character of the dependence of the geometrical changes produced in a body on the forces acting upon its elements."†

Purpose of this paper.—So far as I know no attempt has been made since the middle of the last century to determine the character of the stress-strain curve for the case of finite stress.‡ I have been unable to find even an analysis of a simple finite traction and it seems that the subject has fallen into neglect, for this analysis is not so devoid of interest as to be deliberately ignored, simple though it is.

* Compare Bull. Geol. Soc. Amer., vol. iv, 1893, p. 38.

† Saint-Venant in his edition of Clebsch. p. 39.

‡ J. Riccati, in 1747, a brief account of whose speculation is given in Todhunter's history of elasticity, proposed a substitute for Hooke's law.

In the first part of this paper finite stress and finite strain will be examined from a purely kinematical point of view; then the notion of an ideal isotropic solid will be introduced and the attempt will be made to show that there is but one function which will satisfy the kinematical conditions consistently with the definition. This definition will then be compared with the results of experiment and substantially justified.

In the second part of the paper the vibrations of sonorous bodies will be treated as finite and it will be shown that the hypothesis of perfect isochronism, or perfect constancy of pitch, leads to the same law as before, while Hooke's law would involve sensible changes of pitch during the subsidence of the amplitude of vibrations.

Analysis of shearing stress.—Let $\mathfrak{R}$, $\mathfrak{N}$ and $\mathfrak{T}$ be the resultant normal and tangential stresses at any point. Then if N_1, N_2 and N_3 are the so-called principal stresses and λ, μ, ν the direction cosines of a plane, there are two stress quadrics established by Cauchy which may be written

$$\mathfrak{R}^2 = N_1^2\lambda^2 + N_2^2\mu^2 + N_3^2\nu^2,$$
$$\mathfrak{N} = N_1\lambda^2 + N_2\mu^2 + N_3\nu^2.$$

Since also $\mathfrak{T}^2 = \mathfrak{R}^2 - \mathfrak{N}^2$,

$$\mathfrak{T}^2 = (N_1 - N_2)^2\lambda^2\mu^2 + (N_1 - N_3)^2\lambda^2\nu^2 + (N_2 - N_3)^2\mu^2\nu^2;$$

and these formulas include the case of finite stresses as well as of infinitesimal ones.

In the special case of a plane stress in the xy plane, $N_3 = 0$ and $\nu = 0$, and the formulas become

$$\mathfrak{R}^2 = N_1^2\lambda^2 + N_2^2\mu^2,$$
$$\mathfrak{N} = N_1\lambda^2 + N_2\mu^2,$$
$$\mathfrak{T}^2 = (N_1 - N_2)^2\lambda^2\mu^2.$$

In the particular case of a shear (or a *pure* shear) there are two sets of planes on which the stresses are purely tangential, for otherwise there could be no planes of zero distortion. On these planes $\mathfrak{N} = 0$, and if the corresponding value of λ/μ is α,

$$- N_1\alpha = N_2/\alpha.$$

If this particular quantity is called $Q/3$, one may write the equations of stress in a shear for any plane in the form

$$\mathfrak{R}^2 = \frac{Q^2}{9}\left(\mu^2\alpha^2 + \frac{\lambda^2}{\alpha^2}\right),$$
$$\mathfrak{N} = \frac{Q}{3}\left(\mu^2\alpha - \frac{\lambda^2}{\alpha}\right),$$
$$\mathfrak{T}^2 = \frac{Q^2}{9}\left(\alpha + \frac{1}{\alpha}\right)^2\lambda^2\mu^2.$$

For the axes of the shear the tangential stress must vanish, so that λ or μ must become zero, and therefore the axes of x and y are the shear axes. If $\mathfrak{N}_x$ and $\mathfrak{N}_y$ are the normal axial stresses, one then has

$$-\mathfrak{N}_x\alpha = \mathfrak{N}_y/\alpha = Q/3.$$

A physical interpretation must now be given to the quantity α. In a finite shearing strain of ratio α, it is easy to see that the normal to the planes of no distortion makes an angle with the contractile axis of shear the cotangent of which is α. If the tensile axis of the shear is the axis of y, and the contractile axis coincides with x, this cotangent is λ/μ. Hence in the preceding formulas α is simply the ratio of shear.

In a shear of ratio α with a tensile axis in the direction of oy, minus $\mathfrak{N}_x\alpha$ is the negative stress acting in the direction of the x axis into the area α on which it acts. It is therefore the load or initial stress acting as a pressure in this direction. Similarly $\mathfrak{N}_y/\alpha$ is the total load or initial stress acting as a tension or positively in the direction oy. Hence a simple finite shearing strain must result from the action of two equal loads or initial stresses of opposite signs at right angles to one another the common value of the loads being in the terms employed $Q/3$.*

It is now easy to pass to a simple traction in the direction of oy since the principle of superposition is applicable to this case. Imagine two equal shears in planes at right angles to one another combined by their tensile axes in the direction oy, and let the component forces each have the value $Q/3$. To this system add a system of dilational forces acting positively and equally in all directions with an intensity $Q/3$. Then the sum of the forces acting in the direction of oy is Q and the sum of forces acting at right angles to oy is zero.

Inversely a simple finite load or initial stress of value Q is resoluble into two shears and a dilation, each axial component of each elementary initial stress being exactly one-third of the total load. Thus the partition of force in a finite traction is exactly the same as it is well known to be in an infinitesimal traction, provided that the stress is regarded as initial and not final.†

* This proposition I have also deduced directly from the conditions of equilibrium in Bull. Geol. Soc. Amer., vol. iv, 1893, page 36. It may not be amiss here to mention one or two properties of the stresses in a shear which are not essential to the demonstration in view. The equation of the shear ellipse may be written in polar coördinates $1/r^2 = \alpha^2\mu^2 + \alpha^{-2}\lambda^2$. Hence the resultant load on any plane whatever is $\mathfrak{R}r = \pm Q/3$. The final tangential stress is well known to be maximum for planes making angles of 45° with the axes; but it is easy to prove that the tangential load, $\mathfrak{T}r$, is maximum for the planes of no distortion. These are also the planes of maximum tangential strain. Rupture by shearing is determined by maximum tangential *load*, not stress.

† Thomson and Tait, Nat. Phil., section 682.

Application to system of forces.—Without any knowledge of the relations between stress and strain, the foregoing analysis can be applied to developing corresponding systems of stress and strain. Let a unit cube of an elastic substance presenting equal resistance in all directions be subjected to axial loads P, Q, R. Suppose these forces to produce respectively dilations of ratios h_1, h_2, h_3 and shears of ratios p, q, r. Then the following table shows the effects of each axial force on each axial dimension of the cube in any pure strain.

Active force	P			Q			R		
Axis of strain	x	y	z	x	y	z	x	y	z
Dilation	h_1	h_1	h_1	h_2	h_2	h_2	h_3	h_3	h_3
Shear	p	$1/p$	1	$1/q$	q	1	$1/r$	1	r
Shear	p	1	$1/p$	1	q	$1/q$	1	$1/r$	r

Grouping the forces and the strains by axes, it is easy to see that the components may be arranged as in the following table, which exhibits the compound strains in comparison with the compound loads which cause them, though without in any way indicating the functional relation between any force and the corresponding strain.

Pure Strains.

Axes	x	y	z
Dilation	$h_1h_2h_3$	$h_1h_2h_3$	$h_1h_2h_2$
Shear	$\frac{p^2}{qr}$	$\frac{qr}{p^2}$	1
Shear	1	$\frac{pq}{r^2}$	$\frac{r^2}{pq}$
Products	$\frac{h_1h_2h_3p^2}{qr}$	$\frac{h_1h_2h_3q^2}{pr}$	$\frac{h_1h_2h_3r^2}{pq}$

Loads or Initial Stresses.

Axes	x	y	z
Dilation	$\frac{P+Q+R}{3}$	$\frac{P+Q+R}{3}$	$\frac{P+Q+R}{3}$
Shear	$-\frac{Q+R-2P}{3}$	$\frac{Q+R-2P}{3}$	o
Shear	o	$\frac{P+Q-2R}{3}$	$-\frac{P+Q-2R}{3}$
Sums	P	Q	R

In many cases it is convenient to abbreviate the strain products. Thus if one writes $h_1h_2h_3=h$, $qr/p^2=\alpha$ and $pq/r^2=\beta$, the product are h/α, $h\alpha\beta$ and h/β.

Inferences from the table.—It is at once evident that the load sums correspond to the products of the strain ratios, and that zero force answers to unit strain ratios. There are also several reciprocal relations which are not unworthy of attention. If $R=0$ and $Q=-P$, the strain reduces to a pure shear. But the positive force, say Q, would by itself produce a dilation h_2, while the negative force, minus P, would produce cubical compression of ratio $h_1<1$. Now a shear is by definition undilatational and therefore, in this case, $h_1h_2=1$. Hence equal initial stresses of opposite signs produce dilatations of reciprocal ratios. The same two forces acting singly would each produce two shears while their combination produces but one. Q would contract lines parallel to oz in the ratio $1/q$ while minus P would elongate the same lines in the ratio $p/1$. Since the combination leaves these lines unaltered, $p/q=1$. Hence equal loads of opposite signs produce shears of reciprocal ratios. It is easy to show by similar reasoning that equal loads of opposite signs must produce pure distortions and extensions of reciprocal ratios.

Strain as a function of load.—One may at will regard strains as functions of load or of final stress; but there seem to be sufficient reasons for selecting load rather than final stress as the variable. To obtain equations giving results applicable to different substances, the equations must contain constants characteristic of the material as well as forces measured in an arbitrary unit. In other words the forces must be measured in terms of the resistance which any particular substance presents. Now these resistances should be determined for some strain common to all substances for forces of a given intensity. The only such strain is zero strain corresponding to zero force. Hence initial stresses or loads are more conveniently taken as independent variables.*

Argument based on small strains.

Physical hypothesis.—In the foregoing no relation has been assumed connecting stress and strain. The stresses and strains corresponding to one another have been enumerated, but the manner of correspondence has not been touched upon. One may now at least imagine a homogeneous elastic substance of such a character as to offer equal resistance to distortion in

* When the strains are infinitesimal, it is easy to see that load and final stress differ from one another by an infinitesimal fraction of either.

every direction and equal resistance to dilation in every direction. The two resistances may also be supposed independent of one another—for this is a more general case than that of dependence. The resistance finally may be supposed continuous and everywhere of the same order as the strains.

In such an ideal isotropic substance it appears that the number of independent moduluses cannot exceed two; for a pure shear irrespective of its amount is the simplest conceivable distortion and no strain can be simpler than dilation, while to assume that either strain involved more than one modulus would be equivalent to supposing still simplier strains, each dependent upon one of the units of resistance. It is undoubtedly true that, unless the load-strain curve is a straight line, finite strains involve constants of which infinitesimal strains are independent; but these constants are mere coefficients and not moduluses: for the function being continuous must be developable by Taylor's Theorem, and the first term must contain the same variable as the succeeding terms, this variable being the force measured in terms of the moduluses. In this statement it must be understood that the moduluses are to be determined for vanishing strain.*

One can determine the general form of the variable in terms of the resistances or moduluses for the ideal isotropic solid defined above. The load effecting dilation in simple traction, as was shown above, is exactly one third of the total load, or say $Q/3$; and if a is the unit of resistance to linear dilation, $Q/3a$ is the quantity with which the linear dilation will vary. The components of the shearing stresses in the direction of the traction are each $Q/3$, and, if c is the unit of resistance to this

* One sometimes sees the incompleteness of Hooke's law referred to in terms such as "Young's modulus must in reality be variable." This is a perfectly legitimate statement provided that Young's modulus is defined in accordance with it; but the mode of statement does not seem to me an expedient one to indicate the failure of linearity. Let μ represent Young's modulus regarded as variable and F a force or a stress measured in arbitrary units. Then if y is the length of a unit cube when extended by a force, the law of extension may be written in the form $y = 1 + F/\mu$. Now let M be the value of Young's modulus for zero strain, and therefore an absolute constant. Then, assuming the continuity of the functions, one may write μ in terms of M thus,

$$\frac{1}{\mu} = \frac{1}{M}\varphi\left(\frac{F}{M}\right) = \frac{1}{M}\left(1 + \frac{AF}{M} + \frac{BF^2}{M^2} + \dots\right).$$

But this gives

$$y = 1 + \frac{F}{M} + \frac{AF^2}{M^2} + \frac{BF^3}{M^3} + \dots$$

so that $1/\mu$ merely stands for a development in terms of F/M. If therefore one defines Young's modulus as the tangent of the curve for vanishing strain, the fact of curvature is expressed by saying that powers of the force (in terms of Young's modulus) higher than the first enter into the complete expression for extension.

initial stress, the corresponding extension will vary with $2Q/3c$. In simple extension all faces of the unit cube remain parallel to their original positions, and the principle of superposition is applicable throughout the strain. Hence the total variable may be written $Q\left(\frac{1}{3a} + \frac{2}{3c}\right)$. The intensity of Q will not affect the values of the constants a and c which indeed should be determined for vanishing strain as has been pointed out.

The quantities a and c have been intentionally denoted by unusual letters. In English treatises it is usual to indicate the modulus of cubical dilation by k and the modulus of distortion by n. With this nomenclature $a=3k$ and $c=2n$. Using the abbreviation M for Young's modulus the variable then becomes

$$Q\left(\frac{1}{9k} + \frac{1}{3n}\right) = Q/M.$$

Since this is the form of the variable whether Q is finite or infinitesimal, the length of the strained cube according to the postulate of continuity must be developable in terms of Q/M and cannot consist, for example, solely of a series of terms in powers of $Q/9k$ plus a series of powers of $Q/3n$; in other words the general term of the development must be of the form $A_m(Q/M)^m$ and not $A_m(Q/9k)^m + B_m(Q/3n)^m$.

Form of the functions.—If α is the ratio of shear produced by the traction Q in the ideal isotropic solid under discussion, α must be some continuous function of $Q/3n$. So too if h is the ratio of linear dilation, h is some continuous function of $Q/9k$. The length of the strained mass is $\alpha^2 h$, and this must be a continuous function of Q/M. If then f, φ and ψ are three unknown continuous functions, one may certainly write

$$\alpha^2 = f\left(\frac{Q}{3n}\right); \qquad h = \varphi\left(\frac{Q}{9k}\right); \qquad \alpha^2 h = \psi\left(\frac{Q}{M}\right). \qquad (1)$$

It also follows from the definitions of α and h that

$$1 = f(0); \qquad 1 = \varphi(0); \qquad 1 = \psi(0). \qquad (2)$$

For the sake of brevity let $Q/3n = \nu$ and $Q/9k = \varkappa$. Then ν and $\varkappa$ may be considered algebraically as independent of one another even if an invariable relation existed between n and k; for since in simple traction, the faces of the isotropic cube maintain their initial direction, the principle of superposition is applicable; and to put $n=\infty$ or $k=\infty$ is merely equivalent to considering only *that part* of a strain due respectively to compressibility or to pure distortion.* Now the functions are related by the equation

$$f(\nu)\,\varphi(\kappa) = \psi(\nu + \kappa) \tag{3}$$

and if ν and κ are alternately equated to zero

$$f(\nu) = \psi(\nu) \text{ and } \varphi(\kappa) = \psi(\kappa).$$

Hence the three functions are identical in form† or (3) becomes

$$f(\nu) f(\kappa) = f(\kappa + \nu). \tag{4}$$

Developing the second member by Taylor's theorem and dividing by $f(\kappa)$ gives a value for $f(\nu)$, viz:

$$f(\nu) = 1 + \frac{d f(\kappa)}{f(\kappa) d\kappa} \cdot \nu + \ldots$$

Since the two variables are algebraically independent, this equation must answer to McLaurin's Theorem, which implies that the expression containing κ is constant, its value being say b. Then

$$\frac{d f(\kappa)}{f(\kappa)} = b d\kappa$$

Hence since $f(0) = 1$

$$f(\kappa) = \varepsilon^{b\kappa}$$

and since all three functions have the same form

$$f(\nu + \kappa) = \varepsilon^{b(\nu+\kappa)} = 1 + bQ/M + \ldots$$

* Compare Thomson and Tait Nat. Phil., section 179.

† This proposition is vital to the whole demonstration. Another way of expressing it is as follows:—If the functions are continuous,

$$a^2h = 1 + A\left(\frac{Q}{3n} + \frac{Q}{9k}\right) + B\left(\frac{Q}{3n} + \frac{Q}{9k}\right)^2 + \ldots$$

where A, B, etc; are constant coefficients. Then since n and k are algebraically independent, or since the principle of superposition is applicable, the development of a^2 is found by making $h=1$ and $k=\infty$. Thus

$$a^2 = 1 + \frac{AQ}{3n} + B\left(\frac{Q}{3n}\right)^2 + \ldots;$$

A, B, etc., retaining the same values as before. Consequently a^2 is the same function of $Q/3n$ that a^2h is of Q/M By equating a to unity and n to infinity, it appears that h also is of the same form as a^2h.

There is the closest connection between this method of dealing with the three functions and the principle, that, when an elastic mass is in equilibrium, any portion of it may be supposed to become infinitely rigid and incompressible without disturbing the equilibrium. For to suppose that in the development of a^2h, $k=\infty$ is equivalent to supposing a system of external forces equilibrating the forces $Q/9k$. This again is simply equivalent to assorting the applicability of the principle of superposition to the case of traction.

In pure elongation, unaccompanied by lateral contraction, it is easy to see that $h=a$ and that a varies as $Q/6n$. In this case also $6n=9k$ because Poisson's ratio is zero. Hence without resorting to the extreme cases of infinite n or k, it appears that h is the same function of $9k$ that a is of $6n$. This accords with the result reached in (5) without sufficing to prove that result.

Here b/M is the tangent of the load-strain curve for vanishing strain, and this by definition is $1/M$, so that $b=1$.

It appears then that the equations sought for the load-strain functions are

$$\alpha^2 = \varepsilon^{Q/3n}; \qquad h = \varepsilon^{Q/9k}; \qquad \alpha^2 h = \varepsilon^{Q/M}; \qquad (5)$$

a result which can also be reached from (4) without the aid of Taylor's theorem.

Tests of the equations.—These equations seem to satisfy all the kinematical conditions deduced on preceding pages. It is evident that opposite loads of equal intensity give shears, dilations and extensions of reciprocal ratios and that the products of the strain ratios vary with the sums of the loads. It is also evident that infinite forces and such only will give infinite strains. A very important point is that these equations represent a shear as held in equilibrium by the same force system whether this elementary strain is due to positve or negative forces. If any other quantity (not a mere power of Q or the sum of such powers), such as the final stress were substituted for the load Q, a pure shear would be represented as due to different force systems in positive and negative strains which would be a violation of the conditions of isotropy.* One might suppose more than two independent moduluses to enter into the denominator of the exponent; but this again would violate the condition of isotropy by implying different resistances in different directions. Any change in the numerical coefficients of the moduluses would imply a different partition of the load between dilation and distortion, which is inadmissible. It would be consistent with isotropy to suppose the exponent of the form $(Q/M)^{1+2c}$; but then, if c exceeds zero, the development of the function would contain no term in the first power of the variable and the postulate that strains and loads are to be of the same order would not be fulfilled. The reciprocal relations of load and strain would be satisfied and the loads would be of the same order as the strains, if one were to substitute a series of uneven powers of the variables for ν and $\varkappa$. Such series are for example the developments of tan ν

* Let a shearing strain be held in equilibrium by two loads, $Q/3$ and minus $Q/3$. If a second equal shear at right angles to the first is so combined with it that the tensile axes coincide, the entire tensile load is $2\,Q/3$. If on the other hand the two shears are combined by their contractile axes, the total pressure is $2\,Q/3$. In the first case the area of the deformed cube measured perpendicularly to the direction of the tension is $1/\alpha^2$, and if Q' is the final stress, $Q'/\alpha^2 = 2Q/3$ or $Q' = 2Q\alpha^2/3$. In the second case the area on which the pressure acts is α^2 and if the stress is Q'', $Q'' = -2Q/3\alpha^2$. Thus $Q' = -Q''\alpha^4$. Hence equal final stresses of opposite signs cannot produce shears of reciprocal ratios in an isotropic solid. The same conclusion is manifestly true of any quantity excepting Q or an uneven power of Q or the sum of such uneven powers.

and tan $\varkappa$. In a case of this kind, however, a^2h would not be a function of $Q/M = \nu + \varkappa$ excepting for infinitesimal strain; the exponent then taking the form of a series of terms $A_m(\nu^m + \varkappa^m)$ instead of $A_m(\nu + \varkappa)^m$. Finally it is conceivable that the expanded function should contain in the higher terms moduluses not appearing in the first variable term; but this would be inconsistent with continuity. In short I have been unable to devise any change in the functions which does not conflict with the postulate of isotropy as defined or with some kinematical condition.

Abbreviation of proof.—In the foregoing the attempt has been made to take a broad view of the subject in hand lest some important relation might escape attention Merely to reach the equations (5) only the following steps seem to be essential. Exactly one-third of the external initial stress in a simple traction is employed in dilation, and of the remainder one-half is employed in each of the two shears. An ideal isotropic homogeneous body is postulated as a material presenting equal resistance to strain in all directions, the two resistances to deformation and dilation being independent of one another; the strains moreover are to be of the same order as the loads, and continuous functions of them. In such a mass the simplest conceivable strains, shear and dilation, can each involve only a single unit of resistance or modulus. The principle of superposition is applicable to a simple traction applied axially to the unit cube however great the strain. It follows that the length of the strained unit cube is a function of Q/M.

Together these propositions and assumptions give (1) and without further assumptions the final equations sought (5) follow as a logical consequence.

Data from experiment.—No molecular theory of matter is essential to the mechanical definition of an isotropic substance. An isotropic homogeneous body is one a sphere of which behaves to external forces of given intensity and direction in the same way however the sphere may be turned about its center. There may be no real absolutely isotropic substance, and if there were such a material we could not ascertain the fact, because observations are always to some extent erroneous. It is substantially certain, however, that there are bodies which approach complete symmetry so closely that the divergence is insensible or uncertain. Experience therefore justifies the assumption of an isotropic substance as an approximation closely representing real matter.

All the more recent careful experiments, such as those of Amagat and of Voigt, indicate that Cauchy's hypothesis, leading for isotropic subtances to the relation $3k = 5n$, is very far from being fulfilled by all substances of sensibly symmetrical properties. This is substantially a demonstration that the molecular constitution of matter is very complex,* but provided that the mass considered is very large relatively to the distances between molecules this complexity does not interfere with the hypothesis that pure shear and simple dilation can each be characterized by one constant only.

The continuity of the load-strain function both for loads of the same sign and from positive to negative loads is regarded as established by experiment for many substances; and equally well established is the conclusion that for small loads, load and strain are of the same order.† In other words Hooke's law is applicable to minute strains. Perfect elastic recovery is probably never realized, but it is generally granted that some substances approach this ideal under certain conditions so closely as to warrant speculation on the subject.

These results appear to justify the assumptions made in the paragraph headed "physical hypothesis" as representing the most important features of numerous real substances. On the other hand viscosity, plasticity and ductility have been entirely ignored; so that the results are applicable only to a part of the phenomena of real matter.

Stress-strain function.—It is perfectly easy to pass from the load-strain function to the stress-strain function for the ideal solid under discussion. The area of the extended cube is its volume divided by its length or h^2/a^2h. Hence if Q' is the stress, or force per unit area, $Q'h^2/a^2 = Q$. Therefore the stress-strain function is

$$(a^2h)^{a^2/h^2} = \varepsilon^{Q'/M}$$

an equation which though explicit in respect to stress and very compact is not very manageable. If one writes $a^2h = y$ and $h/a = x$, the first member of this equation becomes y^{1/x^2}. Here x and y are the coördinates of the corner of the strained cube.

Verbal statement of law.—If one writes $a^2h-1=f$, the last of equations (5) gives

$$df = (1+f)\,dQ/M$$

or the increment of strain is proportional to the increment of load and to the length of the strained mass. This is of course the "compound interest law" while Hooke's law answers to simple interest.

* Compare Lord Kelvin's construction of the system of eight molecules in a substance not fulfilling Poisson's hypothesis in his Lectures on Molecular Dynamics.

† Compare B. de Saint-Venant in his edition of Navier's Leçons, 1864, p. 14, and Lord Kelvin, Encyc. Brit., 9th ed., Art. Elasticity, Section 37.

Curves of absolute movement.—Let σ be Poisson's ratio

or
$$\sigma=\frac{3k-2n}{2(3k+2n)}.$$

Let $x_\circ y_\circ$ be the original positions of a particle in an unstrained bar, and let xy be their positions after the bar has been extended by a load Q. Then $x=x_\circ h/a$ and $y=y_\circ a^2 h$. It also follows from (5) that $a^{6n}=h^{9k}$, whence it may easily be shown that the path of the particle is represented by the extraordinarily simple equation*

$$xy^{\sigma}=x_\circ y_\circ^{\sigma} \qquad (6)$$

If one defines Poisson's ratio as the ratio of lateral contraction to axial elongation, its expression is by definition

$$\sigma=-\frac{dx}{x}\Big/\frac{dy}{y}=-\frac{ydx}{xdy};$$

and this, when integrated on the hypothesis that σ is a constant, gives (6). Thus for this ideal solid, the ratio of lateral contraction to linear elongation is independent of the previous strain.

The equation (6) gives results which are undeniably correct in three special cases. For an incompressible solid $\sigma=1/2$, and (6) becomes x^2y=constant, or the volume remains unchanged. For a compressible solid of infinite rigidity $\sigma=-1$ and (6) becomes x/y=constant so that only radial motion is possible. For linear elongation unaccompanied by lateral extension $\sigma=0$, and (6) gives x=constant.†

* On Cauchy's hypothesis $\sigma=1/4$, which, introduced into this equation, implies that the volume of the strained cube is the square root of its length.

† It seems possible to arrive at the conclusion that σ is constant by discussion of these three cases Let e and $-f$ be small axial increments of strain due to a small increment of traction applied to a mass already strained to any extent. Let it also be supposed that the moduluses are in general functions of the coördinates, so that n and k are only limiting values for no strain. Then, by the ordinary analysis of a small strain (Thomson and Tait, section 682), one may at least write for an isotropic solid

$$e=P\left(\frac{1}{3n[1+f_1(x)]}+\frac{1}{9k[1+f_2(x)]}\right),$$
$$-f=P\left(\frac{1}{6n[1+f_3(x)]}-\frac{1}{9k[1+f_4(x)]}\right),$$

where $f(x)$ is supposed to disappear with the strain. These values represent each element of the axial extension and each element of the lateral contraction as wholly independent. The value of σ is $-f/e$. Now for an incompressible substance, as mentioned in the text, $\sigma=1/2$ and the formula gives

$$\sigma=\frac{1}{2}\cdot\frac{1+f_1(x)}{1+f_3(x)}, \text{ so that } f_1(x)=f_3(x).$$

Again for $n=\infty$ only dilation is possible, or $\sigma=-1$, while the formula gives

Argument from finite vibrations.

Sonorous vibrations finite.—In the foregoing pages the attempt has been made to show, that a certain definition of an isotropic solid in combination with purely kinematical propositions leads to a definite functional expression for the load-strain curve. The definition of an isotropic solid is that usual except among elasticians who adhere to the rariconstant hypothesis, and it seems to be justified by experiments on extremely small strains. But the adoption of this definition for bodies under finite strain is, in a sense, exterpolation. It is therefore very desirable to consider the phenomena of such strains as cannot properly be considered infinitessimal.

It is usual to treat the strains of tuning forks and other sonorous bodies as so small that their squares may be neglected, and the constancy of pitch of a tuning fork executing vibrations of this amplitude has been employed by Sir George Stokes to extend the scope of Hooke's law to moving systems. It does not appear legitimate, however, to regard strongly excited sonorous bodies as only infinitesimally strained. Tuning forks sounding loud notes perform vibrations the amplitudes of which are sensible fractions of their length. Now it is certain that no elastician would undertake to give results for the strength of a bridge similarly strained, or in other words he would deny that such flexures were so small as to justify neglect of their squares.*

Sonorous vibrations isochronous.—The vibrations of sonorous bodies seem to be perfectly isochronous, irrespective of the amplitude of vibration. Were this not the case, a tuning-fork strongly excited would of course sound a different note from that which it would give when feebly excited. Neither

$$\sigma = -1 \cdot \frac{1+f_2(x)}{1+f_4(x)}, \text{ so that } f_2(x)=f_4(x).$$

For pure elongation the lateral contraction is by definition zero, or $\sigma=0$, and the formula is

$$\sigma = \frac{f_3(x)-f_4(x)}{2[1+f_2(x)]+1+f_1(4)}, \text{ whence } f_4(x)=f_3(x).$$

Hence all four functions of x are identical and σ reduces to its well known constant-form.—With σ as a constant, equation (6) follows from the definition of σ; and substituting $a^2h=y/y_0$ and $h/a=x/x_0$ gives $a^{6n}=h^{9k}$. If $W=6n\ln a$ one may then write

$$a=\varepsilon^{W/6n}; \quad h=\varepsilon^{W/9k}; \quad a^2h=\varepsilon^{W/M}=1+W/M+ \ldots .$$

Here experiment shows that W may be regarded either as load or stress; and reasoning indicates that it must be considered as load if M is determined for vanishing strain

* It is scarcely necessary to point out that many of the uses to which springs are put, in watches for example, afford excellent evidence of the continuity of the load-strain function for finite distortions.

musicians nor physicists have detected any such variation of pitch which, if sensible, would render music impossible. The fact that the most delicate and accurate microchronometrical instruments yet devised divide time by vibrations of forks, is an additional evidence that these are isochronous. Lord Kelvin has even suggested the vibrations of a spring in a vacuum as a standard of time almost certainly superior to the rotation of the earth, which is supposed to lose a few seconds in the course of a century.*

It is therefore a reasonable hypothesis in the light of experiment that the load-strain function is such as to permit of isochronous vibrations; but to justify this conclusion from an experimental point of view, it must also be shown that Hooke's law is incompatible with sensibly isochronous vibration. I shall therefore attempt to ascertain what load-strain function fulfills the condition of perfect isochronism (barring changes of temperature) and then to make a quantitative comparison between the results of the law deduced and those derived from Hooke's law.

Application of moment of momenta.—If the cube circumscribed about the sphere of unit radius is stretched by opposing initial stresses and then set free, it will vibrate; and the plane through the center of inertia perpendicular to the direction of the stress will remain fixed. Each half of the mass will execute longitudinal vibrations like those of a rod of unit length fixed at one end, and it is known that the cross section of such a rod does not affect the period of vibration, because each fiber parallel to the direction of the external force will act like an independent rod. Hence attention may be confined to the unit cube whose edges coincide with the positive axes of coördinates, the origin of which is at the center of inertia of the entire mass.

The principle of the moment of momenta is applicable to one portion of the strain which this unit cube undergoes during vibration. The moment of a force in the xy plane relatively to the axis of oz, being its intensity into its distance from this axis, is the moment of the tangential component of the force and is independent of the radial force component. Now dilation is due to radial forces and neither pure dilation nor any strain involving dilation can be determined by discussion of the moments of external forces. Hence the principle of the moment of momenta applies only to the distortion of the unit cube. This law as applied to the xy plane consequently governs only the single shear in that plane.

The principle of the moment of momenta for the xy plane may be represented by the formula

* Nat. Phil., sections 406 and 830.

$$\frac{d}{dt}\Sigma m\left(x_1\frac{dy_1}{dt} - y_1\frac{dx_1}{dt}\right) = \Sigma\,(x_1 Y - y_1 X), \qquad (7)$$

where the second member expresses the moments of the external forces, which are as usual measured per unit area, and $x_1 y_1$ are the coördinates of any point the mass of which is m.

Reduction of equation (7).—Let x and y represent the position of the corner of the strained cube; then the abscissa of the center of inertia of the surface on which the stress Y acts is $x/2$, and since Y is uniform, $\Sigma x_1 Y = xY/2$. Similarly $\Sigma y_1 X = yX/2$. Now xY and yX may also be regarded as the loads or initial stresses acting on the two surfaces of the mass parallel respectively to ox and oy, and in a shear these two loads are equal and opposite. Hence the second member of (7) reduces to xY. It has been shown above that, if Q is an initial tractive load, $Q/3$ is the common value of the two equal and opposite loads producing one shear. But to obtain comparable results for shear dilation and extension, $Q/3$ must be measured in appropriate units of resistance. Since M is the unit of resistance appropriate to extension, the separate parts of the force must be multiplied by M and divided by resistances characteristic of the elementary strains. Now

$$\frac{M}{2n}\cdot\frac{Q}{3} + \frac{M}{2n}\cdot\frac{Q}{3} + \frac{M}{3k}\cdot\frac{Q}{3} = Q,$$

and it is evident that $2n/M$ is the unit in which $Q/3$ should be measured for the single shear.* Thus the second member of (7) becomes $MQ/6n$.

This, then, is the value which the moment of the external forces assumes when these hold the strained unit cube in equilibrium. This unit cube forms an eighth part of the cube circumscribed about the sphere of unit radius. When the entire mass is considered, the sum of all the moments of the external forces is zero; since they are equal and opposite by pairs. If the entire mass thus strained is suddenly released and allowed to perform free vibrations, the sum of all the moments of momenta will of course remain zero. On the other hand the quantity $MQ/6n$ will remain constant. For this load determines the limiting value of the strain during vibration and is independent of the particular phase of vibration, or of the time counted from the instant of release. It may be considered as the moment of the forces which the other parts of the entire material system exert upon the unit cube.

* In this paper changes of temperature are expressly neglected. The changes of temperature produced by varying stress in a body performing vibrations of small amplitude can be allowed for by employing "kinetic" moduluses, which are a little greater than the ordinary "static" moduluses. Thomson and Tait, Nat. Phil., section 687.

Turning now to the first member of (7), values of x_1 and y_1 appropriate to the case in hand must be substituted. Each point of the unit cube during shear moves on an equilateral hyperbola, so that if x_o, y_o are the original coördinates of a point, $x_1y_1 = x_oy_o$. For the corner of the cube, whose coördinates are x and y, the path is $xy = 1$. Now $x_1/x_o = x$ and $y_1/y_o = y$ so that

$$x_1dy_1 - y_1dx_1 = x_oy_o(xdy - ydx)$$

If ψ is the area which the radius vector of the point x, y describes during strain, it is well known that $2d\psi = xdy - ydx$ and, since in this case $xy = 1$, it is easy to see that

$$2d\psi = 2d\ lny$$

Since the quantities x and y refer to a single point, the sign of summation does not affect them, and the first member of (7) may be written

$$\frac{d^2lny}{dt^2}\Sigma 2mx_oy_o$$

Here one may write for m, $\rho dx_odу_o$, where ρ is the constant density of the body; and since the substance is uniform, summation may be performed by double integration between the limits unity and zero. This reduces the sum to $\rho/2$.

Value of a. Equation (7) thus becomes

$$\frac{d^2lny}{dt^2} = \frac{2}{\rho}\cdot\frac{MQ}{6n}$$

the second member being constant. Counting time from the instant of release, or from the greatest strain, and integrating y between the limits $y = a$ and $y = 1$ gives

$$lna = \frac{MQ}{6n}\cdot\frac{t^2}{\rho}$$

It is now time to introduce the hypothesis that the vibrations are isochronous. It is a well known result of theory and experiment that a rod of unit length with one end fixed, executing its gravest longitudinal vibrations, performs one complete vibration of small amplitude in a time expressed by $4\sqrt{\rho/M}$. In the equation stated above t expresses the time of one-quarter of a complete vibration or the interval between the periods at which $y = 1$ and $y = a$. Hence for a small vibration, t as here defined is $\sqrt{\rho/M}$. If the vibrations are to be isochronous irrespective of amplitude, this must also be the value of t in a finite vibration. Hence at once

$$a = \varepsilon^{Q/6n} = \varepsilon^{\psi},$$

the same result reached in (5).

This result may also be expressed geometrically. The quantity $Q/6n$ is simply the area swept by the radius vector of the point $x_0 = 1$, $y_0 = 1$. This area is also the integral of ydx from $x = 1/a$ to $x = 1$, or the integral of xdy from $y = a$ to $y = 1$. Thus ψ represents any one of three distinct areas. In terms of hyperbolic functions, $a = \text{Sin}\ \psi + \text{Cos}\ \psi$ and the amount of shear is $2\text{Sin}\ \psi$.

It appears then that isochronous vibrations imply that in pure shear the area swept by the radius vector of the corner of the cube, or lna, is simply proportional to the load. The law proposed by Hooke implies that the length $a - 1$ is proportional to the same load. The law commonly accepted as Hooke's makes $a - 1$ proportional to the final stress, or $(a-1)/a$ proportional to the load.

Value of h.—Knowing the value of a, the value of h can be found without resort to the extreme case $n = \infty$. In the case of pure elongation, unattended by lateral contraction, $h = a$ and $9k = 6n$. If a_1 and h_1 are the ratios for this case,

$$a_1 = \varepsilon^{Q/9k};\ h_1 = \varepsilon^{Q/9k};\ a_1^2 h_1 = \varepsilon^{Q/3k}$$

If three such elongations in the direction of the three axes are superimposed, the volume becomes

$$(a_1^2 h_1)^3 = \varepsilon^{Q/k},$$

and this represents a case of pure dilation without distortion. Here however $a_1 = h_1$ and therefore the case of no distortion, irrespective of the value of n, is given by

$$h^9 = \varepsilon^{Q/k}$$

The values of a and h derived from the hypothesis of isochronous vibrations when combined evidently give the same value of a^2h which was obtained from kinematical considerations and the definition of isotropy in equation (5).

Law of elastic force.—Let s be the distance of a particle on the upper surface of a vibrating cube from its original position or

$$s = a^2h - 1 = \varepsilon^{Q/M} - 1.$$

Then the elastic force per unit volume is minus Q, or

$$\rho\frac{d^2s}{dt^2} = -Q = -Mln(s+1) = -Ms + \frac{Ms^2}{2} - \dots$$

When the excursions of the particle from the position of no strain are very small, this becomes

$$\rho\frac{d^2s}{dt^2} = -Ms$$

a familiar equation leading to simple harmonic motion.

Limitation of harmonic vibrations.—While the theory of harmonic vibrations is applicable to very small vibrations on any theory in which the load strain curve is represented as continuous and as making an angle with the axes whose tangent is finite, it appears to be inapplicable in all cases where the excursions are sufficient to display the curvature of the locus. If the attraction toward the position of no strain in the direction of *oy* is proportional to $y-1$, then in an isotropic mass there will also be an attraction in the direction of *ox* which will be proportional to $1-x$. The path of the particle at the corner of a vibrating cube will therefore be the resultant of two harmonic motions whose phases necessarily differ by exactly by one-half of the period of vibration, however great and however different the amplitudes may be. This resultant is well known to be a straight line. Hence the theory precludes all displacements excepting those which are so small that the path of the corner of the cube may properly be regarded as rectilinear. It seems needless to insist that such cannot be the case for finite strains in general.

There is at least one elastic solid substance, vulcanized india rubber, which can be stretched to several times its normal length without taking a sensible permanent set. Now if the ideal elastic solid stretched to double its original length (or more) were allowed to vibrate, the hypothesis of simple harmonic vibration implies that this length would be reduced to zero (or less) in the opposite phase of the vibration, a manifest absurdity.

Variation of pitch by Hooke's law.—It remains to be shown that if the commonly accepted law were applicable to finite strain, sonorous vibrations would be accompanied by changes of pitch which could scarcely have escaped detection by musicians and physicists. Experiments have shown that the elongation of steel piano wire may be pushed to 0·0115 before the limit of elasticity is reached.* Since virtuosos not infrequently break strings in playing the piano, it is not unreasonable to assume that a one per cent elongation is not seldom attained. In simple longitudinal vibration the frequency of vibration is expressed by 1/4 of $\sqrt{M/\rho}$, and if according to Hooke's law, $s = Q/M$, where Q is the load, the number of vibrations, v, may be written

$$v = \frac{1}{4}\sqrt{\frac{Q}{s\rho}}\ .$$

If, on the other hand, according to the theory of this paper, $ln(1+s) = Q/M$ the number of vibrations, u, may be written

* From experiments on English steel piano wire by Mr. D. McFarlane.

$$u = \frac{1}{4}\sqrt{\frac{Q}{\rho\, ln(1+s)}} \text{ so that } \frac{v}{u} = \sqrt{\frac{ln(1+s)}{s}}.$$

If $s = 0{\cdot}01$, this expression gives $v/u = 400/401$.

It would appear then that on the hypothesis of Hooke, a note due to longitudinal vibrations of about the pitch G_3 would give a lower note when sounding fortissimo than when sounding pianissimo, and that the difference would be one vibration per second, or one in four hundred. But according to Weber's experiments experienced violin players distinguish musical intervals in melodic progressions no greater than 1000/1001, while simultaneous tones can be still more sharply discriminated.* The value of s corresponding to $v/u = 1000/1001$ is only 0·004, and consequently strains reaching only about one-third of the elastic limit of piano wire should give sensible variations of tone during the subsidence of vibrations if Hooke's law were correct.

Longitudinal vibrations are not so frequently employed to produce notes as transverse vibrations. The quantity M/ρ enters also into the expression for the frequency of transverse vibrations though in a more complex manner. In the case of rods not stretched by external tension, the ratio v/u would take the same form as in the last paragraph. One theory of the tuning-fork represents it as a bar vibrating with two nodes, and therefore as comparable to a rod resting on two supports.

A pair of chronometrical tuning-forks could be adjusted to determine much smaller differences in the rate of vibration than 1000/1001; for the relative rate of the forks having been determined on a chronographic cylinder for a certain small amplitude, one fork could be more strongly excited than the other and a fresh comparison made. The only influences tending to detract from the delicacy of this method of determining whether change of amplitude alters pitch, would seem to be the difficulty of sustaining a constant amplitude and the difference of temperature in the two forks arising from the dissipative action of viscosity.

Conclusion.—The hypothesis that an elastic isotropic solid of constant temperature is such as to give absolutely isochronous longitudinal vibrations leads to the conclusion $ln\,(a^2h) = Q/M$ without any apparent alternative. Comparison with the results of Hooke's law shows that, if this law were applicable to finite vibrations, easily sensible changes of pitch would occur during the subsidence of vibrations in strongly excited sonorous bodies.—The logarithmic law is the same deduced in the earlier part of the paper from the ordinary definition of the ideal elastic isotropic solid, based upon experiments on

* Helmholtz, Tonempfindungen, page 491.

very small strains, in combination with purely kinematical considerations.—There can be no doubt that the law here proposed would simplify a great number of problems in the dynamics of the ether and of sound, as well as questions arising in engineering and in geology, because of the simple and plastic nature of the logarithmic function. In the present state of knowledge, the premises of the argument can scarcely be denied; whether the deductions have been logically made must be decided finally by better judges than myself.

Washington, D. C., July, 1893.

ART. XLIX. — *On Powellite from a new Locality;* by GEORGE A. KOENIG and LUCIUS L. HUBBARD, Michigan Mining School, Houghton.

THE material for this investigation was found in the 21st level of No. 8 shaft South Hecla Copper Mine, Houghton Co., Michigan, in the fall of 1892. It came into the possession of the authors through Mr. John T. Reader of the Tamarack mine.

Three small pieces were obtained, which had evidently resulted from the breaking up of a larger piece—altogether about 45 grams. In its present condition it appears homogeneous, that is to say, there is no other mineral associated with it except a trace of native copper. The finder states that the three pieces were lying in a cavity. The specimen, when examined with the lens, exhibits an irregular polysynthetic structure. Some of the aggregates terminate in a more or less distinct pyramid. Three of the small groups were detached for measurement. The signals were not very distinct, and the results, therefore, are only approximate.

Basal edge 49° 15′; pole edge 79° 59′.

The powder yields microscopic, positive and uniaxial images and the *tetragonal* symmetry may be assumed as fairly established. Mellville* gives for powellite:

$$111 \wedge 1\bar{1}1 = 49° 12; \; 111 \wedge 101 = 80° 1'.$$

Two cleavages were observed, one distinct and apparently parallel to 111, the other imperfect, parallel to 001. Hardness = 4·5. Spec. Gr. = 4·349. The color is pale bluish green; in patches it is deep olive- or asparagus-green. The luster is vitreous and unctuous. A thin section of the mineral under the microscope showed high double refraction (colors of the

* This Journal, No. 242, p. 138, Febr., 1891.