

THE LAW OF ELASTICITY FOR ISOTROPIC AND QUASI-ISOTROPIC SUBSTANCES BY FINITE DEFORMATIONS

H. HENCKY, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS

Hooke's law, which is the foundation of the mathematical theory of elasticity, is unfit to describe satisfactorily the phenomena in elastic bodies even if we limit the scope of our research to ideal elastic deformations. We call a deformation ideally elastic, if the deformations disappear completely and if the energy stored up in the body is given back without loss, when the load is removed.

We exclude, therefore, in this research all relaxation and plasticity-phenomena, which are connected with such losses of energy.

The physical bases of our study are the experiments of P. W. Bridgman (at Harvard)¹ on the compressibility of matter, which are the first exact measurements of finite deformations. We will show that these experiments, if interpreted theoretically, are the foundation of a new rational theory of elasticity comprising in itself the old law of Hooke for small deformations.

With respect to hydrostatic compression our theory is in such agreement with the experiments of Bridgman, that the interpretation of the experiments in the light of this theory promises to be of importance for insight into the repulsive mechanism of the molecules.

We make the following assumptions: (1) The material is capable of being subdivided to any desired extent practically, but is nevertheless built up of small units, so that the elastic forces must be considered as the result of the mechanism of an elastic micromachinery. (2) In order to unveil this micromachinery we try to evade all arbitrary assumptions choosing our stress-strain relations as simple as possible and so as to include the common law of Hooke as a special case for small deformations. (3) The expression for the elastic energy must be independent of the way in which the body is loaded.

I. The Measurement of Strain

If we had never heard of the theory of elasticity and if all substances surrounding us had the elasticity of soft rubber so that we could obtain finite deformations with very small forces, we could define strain as either the ratio of the change of length to the original length or as the ratio of the change of length to the length after equilibrium is attained. Such an ambiguity warns us that we must revise our fundamental notions.

¹ Compare P. W. Bridgman, "Handbuch der Experimentalphysik," Vol. VIII, Part 2, pp. 247-395, where the original papers are cited. The author is indebted to Prof. Bridgman for valuable critical remarks and suggestions.

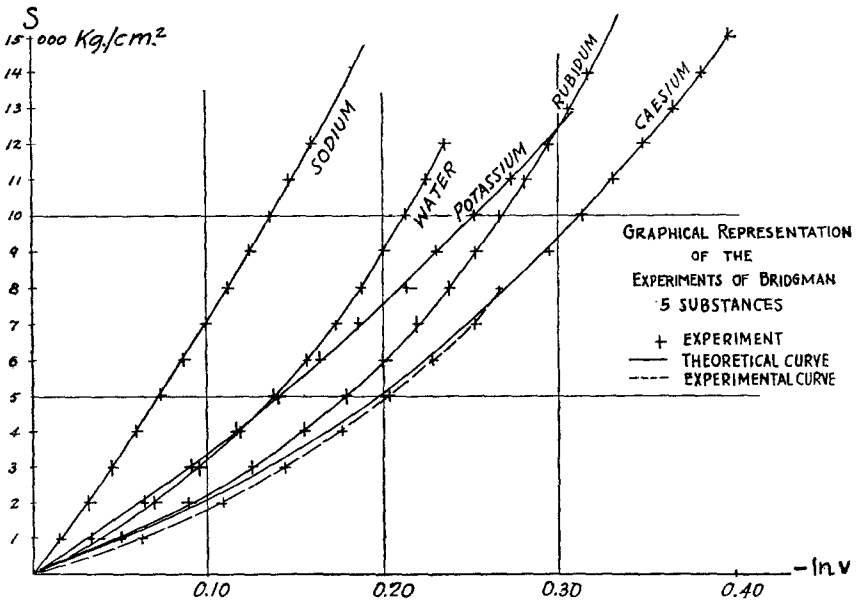
This is easy in the case in question, if we define the measure of an infinitesimal strain as the ratio of the increase in length to the length itself. If ϵ is the measure of finite strain we will have

$$d\epsilon = \frac{dx}{x} \quad (1)$$

Integrating and taking the length of the string at the beginning as $-a$ (at the end as $-b$) we get

$$\epsilon = \ln \left(\frac{b}{a} \right) \quad (1a)$$

Our measure of finite strain is therefore the natural logarithm of the affine relation.



Taking now an element of the body and straining it in three perpendicular directions, x_1, x_2, x_3 , we get the new length $\overline{dx}_1, \overline{dx}_2, \overline{dx}_3$ from the old ones, dx_1, dx_2, dx_3 . The corresponding strains are

$$\epsilon_i = \ln \left(\frac{\overline{dx}_i}{dx_i} \right) \quad (i = 1, 2, 3) \quad (2a)$$

As we will presently see, it is very convenient to take the average of these principal strains

$$\epsilon = \frac{1}{3} \cdot (\epsilon_1 + \epsilon_2 + \epsilon_3) \quad (2b)$$

we find

$$3\epsilon = \ln \left(\frac{\overline{dx}_1}{dx_1} \cdot \frac{\overline{dx}_2}{dx_2} \cdot \frac{\overline{dx}_3}{dx_3} \right) = \ln \left(\frac{d\overline{V}}{dV} \right) \quad (2c)$$

ϵ arises from that part of the extension or compression which is equal in all directions. The same sort of deformation is caused by a lowering or raising of the temperature. If we subtract ϵ from the principal strains, we get a deformation without a change in the volume.

This division of the deformation into 2 parts, one part $(\epsilon_i - \epsilon)$ ($i = 1, 2, 3$) leaving the volume constant and another part ϵ equal in all directions is of the outmost importance, if we wish to go from an elastic state of matter to the liquid state simply by changing the coefficients of elasticity. Moreover this division suggests a very simple expression for the elastic energy which we must consider next.

II. The Elastic Energy Stored Up in the Body

The normal unit-stresses S_i ($i = 1, 2, 3$) acting on the surfaces of the deformed element can be also averaged and divided into two systems, one being the average of these normal stresses:

$$S = \frac{1}{3} \cdot (S_1 + S_2 + S_3)$$

and the remaining stress system $S_i - S$ ($i = 1, 2, 3$).

We assume that the stress system $S_i - S$ does not cause any change in the volume. The system S is the only one possible in an ideal liquid. The work done in the body is connected with the mass and must therefore be related to the volume in the beginning. Assuming the energy of dilatation and compression as independent from the energy of the proper changes of form caused by the stresses $S_i - S$ we have with the modulus of rigidity G and the compression-modulus K :

$$2.W = 2.G. \{ (\epsilon_1 - \epsilon)^2 + (\epsilon_2 - \epsilon)^2 + (\epsilon_3 - \epsilon)^2 \} + 2.9.K. \int_{\epsilon=0}^{\epsilon=\epsilon} f(\epsilon).d\epsilon \quad (4)$$

The expression for the energy of compression is perfectly arbitrary. Without special assumptions about the mechanism of the repulsive forces we cannot deduce theoretically the form of function $f(\epsilon)$. We may choose this function as simple as the experiments suggest.

The first part of the energy, however, cannot be chosen otherwise or the theory would lose every practical interest by becoming too complicated.

Differentiating Equation (4) we can compute now the change in the elastic energy caused by a virtual change in $\delta\epsilon_i$ ($i = 1, 2, 3$)

$$\delta W = 2G. \{ (\epsilon_1 - \epsilon) \delta (\epsilon_1 - \epsilon) + (\epsilon_2 - \epsilon) \delta (\epsilon_2 - \epsilon) + (\epsilon_3 - \epsilon) \delta (\epsilon_3 - \epsilon) \} + 9K.f(\epsilon).\delta\epsilon$$

This work can be also calculated as done by the stresses, remembering that the stresses are to be measured with respect to unit areas in the strained state. In that case we get

$$\delta W.d x_1 d x_2 d x_3 = S_1.\overline{d x_2}.\overline{d x_3}.\delta\overline{d x_1} + S_2.\overline{d x_3}.\overline{d x_1}.\delta\overline{d x_2} + S_3.\overline{d x_1}.\overline{d x_2}.\delta\overline{d x_3}$$

Putting $v = \frac{d\bar{V}}{dV}$ (the relative volume) we can write for the last expression of the work done:

$$\delta W = v \cdot \{ (S_1 - S) \delta (\epsilon_1 - \epsilon) + (S_2 - S) \delta (\epsilon_2 - \epsilon) + (S_3 - S) \delta (\epsilon_3 - \epsilon) \} + v \cdot 3S \cdot \delta \epsilon$$

Putting the two expressions of δW equal and considering that the four variations $\delta(\epsilon_i - \epsilon)$ ($i = 1, 2, 3$) and $\delta \epsilon$ are absolutely independent and arbitrary, we get the generalized law of elasticity in the following form:

$$S_1 - S = \frac{2.G}{v} \cdot (\epsilon_1 - \epsilon)$$

$$S_2 - S = \frac{2.G}{v} \cdot (\epsilon_2 - \epsilon) \quad (5a)$$

$$S_3 - S = \frac{2.G}{v} \cdot (\epsilon_3 - \epsilon)$$

and

$$S = \frac{3.K}{v} \cdot f(\epsilon) \quad (5b)$$

Mathematically the appearance of the relative volume, v , means that the so-called stress-tensor is a tensor-density with respect to the transformations of the general affine group.

By multiplying the stress-tensor S_i with the transformation-determinant v , we get a real tensor $v \cdot S_i = S_i'$, which can be equalized to the components of the pure affine deformation ϵ_i in the manner of our law (Equations (5a) and (5b)).

For this reduced tensor S_i' the law of superposition holds, if we choose $f(\epsilon) = \epsilon$ in Equation (5b). By "superposition-law" we mean that the relation between the changes in both stress and strain as they are connected by Equations (5a) and (5b) is absolutely independent of the stresses and strains already present. The extension and compression law

$$S = \frac{K}{v} \cdot \ln v, \quad (6a)$$

which gives for $v = 0$ $S = -\infty$, and is the simplest law possible for finite deformations. Here we have an unlimited compressibility, because the volume can be reduced to an arbitrary small amount, if we increase the stress.

The experiments of Bridgman show that only a small number of solids obey the law (6a). Most of the experiments suggest another law, *viz.*, that there is a limiting relative volume φ which cannot be reached by a finite pressure.

Comparison with the experiments shows that the formula

$$S = K \cdot \frac{1 - \varphi}{v - \varphi} \cdot \frac{\ln v}{v} \quad (6b)$$

reproduces most of the experimental material, with such a precision that it cannot be considered as an accidental coincidence but must have a physical meaning.

III. The Experiments of Bridgman Concerning the Compressibility of Solids, Liquids, and Gases under High Pressures

The measurements of Bridgman cover a range of stress from 0 to 15,000 kg/cm². Most of the results are given by a parabolic interpolation-formula, from which our constants K and φ can be computed easily.

Taking two corresponding stresses $S'\varphi''$ and strains v', v'' , respectively, we can eliminate K and get with (6b)

$$\frac{S'.v'}{S''.v''} \cdot \frac{\log v''}{\log v'} = a = \frac{v'' - \varphi}{v' - \varphi}$$

and therefore

$$\varphi = \frac{v'' - a.v'}{1 - a} \quad (6c)$$

For most of the materials examined by Bridgman φ turns out to be really a constant.

Near atmospheric pressures the formula does not give as good results as at higher pressures. Metallic caesium is remarkable in that it is very irregular from 0 to 7500 kg/cm² but follows the theoretical curve very closely for higher pressures. It seems that the individual structure of the molecule has much more influence at lower pressures. In the following table we have taken those substances which Bridgman has marked out as very irregular, as particularly fit for a crucial test of the proposed mathematical description. It seems that even gases acquire a constant modulus of compression at pressures of 10,000–15,000 kg/cm² and behave then elastically exactly as solids.

The fact that the seemingly very complicated results of Bridgman can be represented by the simple Formula (6b) using a constant modulus of compression K is of considerable importance for the theory of the repulsive forces in the molecules.

The assumption of a limiting volume as a characteristic feature of our mathematical description and the confirmation of this assumption within certain limits by the experiments of Bridgman must also have a mechanical meaning.

For the time being, however, as it seems impossible to understand the

mechanism leading to Formula (6*b*) so we must consider this formula as empirical.

Substance	Temperature, centigrade	Limiting relat. volume	Modulus K in kg/cm^2 $\times 10^4$	Extreme deviations in φ Range of stress, 3000-15,000 atm.	in K
Solids:					
Iron	30	0.8	170
Sodium	30	Nearly inf.	6.46	6.43-6.48
Potassium	45	Nearly inf.	3.09	3.03-3.14
Caesium	50	0	1.706	Irregular from 0-7000 atm.	1.700-1.710
Rubidium	50	0.44	1.66	0.40-0.45	1.65-1.68
Liquids:*					
Mercury	20	0.75	24.9	0.68-0.82	24.6-25.3
Water	40	0.50	2.34	0.49-0.52	2.33-2.36
CS ₂	20	0.56	1.29	0.55-0.57	1.27-1.31
Methylalkohol	20	0.56	1.17	0.55-0.57	1.15-1.18
Aethylalkohol	20	0.56	1.17	0.52-0.59	1.15-1.18
Aether	20	0.52	0.87	0.48-0.57	0.83-0.90
Gases:					
Hydrogen	65	Nearly 0	0.82	Only by 13,000-15,000 atm.	
Helium	65	Nearly 0	0.47		

* These liquids behave more or less irregularly at low pressures.

Summary

Hooke's law concerning the relation between stress and strain is not valid for finite deformations. A law of elasticity is developed which contains in itself the law of Hooke as a special case and is theoretically incontestable. Specializing this law in a manner suggested by the experiments of Bridgman concerning the compressibility of matter it is possible to reproduce analytically most of the results of Bridgman. A table giving the two coefficients of compressibility—consisting of the modulus of compression K and a hypothetical limiting relative volume φ —and a diagram show the coincidence of formula and experiment.

Discussion

DR. KRAEMER: This is a little off the subject of rheology, perhaps, but I listened with considerable interest to Dr. Hencky's allusion to allotropy in liquids at high pressure. Not long ago an announcement was made that the dielectric constants of the ether change at low temperatures before freezing occurs, and the suggestion was made in that case that allotropy may occur in liquids. I was wondering if Dr. Hencky has any idea as to what constitutes allotropy in terms of molecular structure in the liquid.

DR. HENCKY: I have no ideas about that.

DR. KARRER: The question of what length to choose has interested us in the rubber industry, because the formulas, as Dr. Hencky has pointed out, are all right as long as the changes in length are not comparable with the original dimensions. Therefore we have for some years been using the logarithm for the actual length compared with the mean

length. I was interested in the plasticity and softness of rubber. In both cases we have deformations which are great compared with the original dimensions, and in that case we considered the logarithm as Dr. Hencky has done.

DR. NÁDAI: I wanted to ask Dr. Hencky several questions regarding the general suggestion to treat large deflections, and also some questions regarding the definition of the elastic constants under these conditions. Instead of using the ordinary definition you introduce a new definition here for strain and I feel that it is a very clever suggestion to treat large deformations in rubber in this way. But may I ask, is there a possibility of developing a general mathematical theory similar to the theory for small deformations? Obviously one has to combine the law of elasticity with the equilibrium equations and these have to be written down for the changed shape of the body, not for the original one.

It is not clear to me what will happen if we do this. But perhaps the result will be very complicated. The second question which interests me is this: I understand that the elastic constants are independent of hydrostatic pressure. Now here arises a question in connection with certain phenomena as to the propagation of waves in the interior of our earth. We know that we have a large pressure in the interior of the earth, and we know that we can measure an elastic constant by sound vibrations. In the interior of the earth we have a pressure of several hundred thousands or millions of atmospheres changing the relative volume as much as 50%. Therefore we know the density and can measure the velocity of earthquake vibrations coming from the interior of the earth. As the period of the vibrations is equal to the square root of the elastic constant over the density we can get the rigidity factor from this formula and I am inclined to think that this rigidity factor will be other than the constant used in the law of finite elastic strain.

We must therefore distinguish between two kinds of elastic constants, those which are true to the actual state and a hypothetical kind of elastic constants appearing in the formula of the elastic energy. Perhaps Dr. Hencky can inform us as to what he thinks about this.

DR. HENCKY: To Dr. Karrer I can answer that the logarithmic measure of deformation was already used by the technologist, Ludwig.* I think everybody comes to the same idea automatically. And now the question of Dr. Nádai about the full equations and the elastic coefficients in general.

The connection of finite deformations with the equations of equilibrium has been worked out by Duhem, and E. and F. Cosserat and is reproduced in an improved and elegant manner in the "Elementare Mechanik" by Hamel.**

Taking an already stressed body and studying the behavior of the equations of equilibrium against any small change in the stresses I myself have tried to prepare a way to the solutions of problems analogous to those suggested by Dr. Nádai, but I did not think it useful to embody these researches in my present paper which has only to do with the physical foundation of elasticity. Further work in this direction will show that the difficulties with the constants are not contradictions in the theory. However, much depends on simple solutions of the equations and just the problem of oscillation suggested by Dr. Nádai is very attractive and important, so that I will store it in my memory. (Laughter.)

MR. PEEK: I think the theory should be completed concerning the strains in different directions so that problems of shear can be treated also. Can such a simple general treatment be given? My second question relates to the connection between stress and strain. Is the relationship chosen by you the most general for the case of an

* Hutte, *Des Ingenieurs Taschenbuch*, 25, Auflage 1925. Pp. 324-331, by Dr. Siebel.

** G. Hamel, *Elementare Mechanik* (1922), Teubner, Leipzig.

isotropic body? It seems to me that there is no strict generality, however advisable the formulation may be practically. Am I correct about that?

DR. HENCKY: I can answer Mr. Peek immediately on the first question. The assumed law is absolutely complete. With the law of elasticity once written down in the form connecting the principal strains with the equally directed principal stresses every question concerning the behavior in a certain direction is answered by geometry.

Concerning the second question I should say that the assumed independence of the changes of volume and the changes of form is arbitrary. Putting the question to nature in this manner it is possible that the experiment will contradict the assumption as indeed it does for very large deformations, but not knowing for the time being the precise connection between change of form and change of volume, I am tactful and will not anticipate. (Laughter.)

MR. PEEK: Well, then, if you make that separation, it simply means that there will be two relations between stress and strain but not necessarily linear.

DR. HENCKY: The equation connecting the changes of form with the stresses must be linear for the sake of simplicity; the equation for the changes in volume is empirical. The elastic energy should be deduced from the electric potential of the atoms but as we are far from the possibility of doing this, I have simply assumed a convenient form. (Laughter.)

MR. MOONEY: I should like to ask Dr. Hencky about the dimensions of this quantity φ in his equations. It seems to have different dimensions in denominator and numerator. Is there no mistake there?

DR. HENCKY: No, it is right.

MR. MOONEY: What dimensions has φ ?

DR. HENCKY: φ has no dimensions at all. It is a limiting relation between the volume at the end and at the beginning of the deformation. For water it is 0.5; for mercury 0.75.

MR. MOONEY: I think it is all right. Dr. Nádai just points out to me that φ is a relative volume.

DR. HENCKY: Yes, that is what I said. (Laughter.)