FINITE STRAIN J_2 DEFORMATION THEORY

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(Received November 7, 1980)

ABSTRACT

A finite strain version of the J_2 deformation theory of plasticity is given. The material model is an isotropic, nonlinearly elastic solid. The range of states is investigated for which the equations governing incremental responses are elliptic.

1. Introduction

A particular finite-elasticity constitutive law which can be considered as a prototype model for certain limited classes of time-independent deformations of plastic solids is the subject of this paper. This law was first formulated by Hutchinson and Neale [1] in connection with an investigation of localized necking failures in thin sheet metals. It has subsequently been employed in a number of other studies of finite-strain and bifurcation phenomena in plastic solids (e.g. [2–4]).

In small-strain plasticity the most commonly employed constitutive laws are the "J_2 flow theory" and "J_2 deformation theory" relations. In both of these constitutive theories the plastic strain increments satisfy incompressibility, and they are connected to a general multiaxial stress state through J_2 — the second invariant of the deviatoric stress tensor. J_2 flow theory involves relations between stress increments and strain increments, which lead to path-dependence of total stress and total strain for arbitrary stress histories. In contrast, J_2 deformation theory is a small-strain nonlinear elasticity constitutive law. Although deformation theory is clearly inadequate for characterizing the most general path-dependent features of plastic behavior,

Proceedings of the IUTAM Symposium on Finite Elasticity, Lehigh University, August 10–15, 1980. Invited paper presented by J. W. Hutchinson.

Carlson, D.E. and Shield, R.T. (eds.)
Proceedings of the IUTAM Symposium on Finite Elasticity
ISBN 90 247 2629 8

there are nonetheless some restricted classes of plastic behavior for which its use can be rigorously justified. For example, small-strain J_2 deformation theory is simply the integrated result of the corresponding J_2 flow theory if the loading history is "proportional", i.e., if all stress components are increased monotonically in fixed proportion to one another. An example involving non-proportional loading increments arises in classical bifurcation analyses, where the use of deformation theory can be justified by showing it to be equivalent to a flow theory which permits the development of yield surface vertices [5]. The wide-spread use of the small strain deformation theory suggests a corresponding role for a finite strain version.

2. Finite strain J_2 deformation theory

The finite strain J_2 deformation theory developed in [1] is a nonlinear elastic law, where the solid is assumed to be isotropic and incompressible. Its development makes extensive use of Hill's theory [6] for finitely deformed isotropic elastic solids.

For an isotropic nonlinear elastic solid, the principal directions of Cauchy stress $\boldsymbol{\sigma}$ must coincide with the axes of the Eulerian strain ellipsoid. Also, to fully specify the state of strain in a material element we need only know the three principal stretches λ_i relative to some reference configuration and the principal directions of strain. Thus, the constitutive law is completely determined once the relations between the principal components of Cauchy stress σ_i and principal stretches λ_i are known. The incremental form of the constitutive law can be obtained using Hill's "principal-axes techniques" [6].

The strain measure adopted is the logarithmic strain tensor $\boldsymbol{\varepsilon}$ which, by definition, is coaxial with the Lagrangian strain ellipsoid and has principal values

$$\varepsilon_i = \ln \lambda_i. \tag{2.1}$$

The logarithmic strain rates $\dot{\varepsilon}_1 = \dot{\lambda}_1/\lambda_1$, etc. are then the Eulerian strain-rate components $\dot{\varepsilon}_{ij}$ on the axes of the Eulerian strain ellipsoid. For incompressible deformations, the constraint $\lambda_1\lambda_2\lambda_3 = 1$ with the choice (2.1) implies the simple condition $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ as well as $\dot{\varepsilon}_{kk} = 0$. Inherent advantages of the logarithmic strain measure in setting up constitutive inequalities for both elastic and elastic-plastic solids have been discussed by Hill [6, 7]. This strain measure ("natural strain") has conventionally been used over the years by metallurgists to report (true) stress-strain data for metals.

By analogy with small-strain J_2 deformation theory we introduce the following stress invariant

$$\sigma_e = (3J_2)^{1/2} = (3s_is_i/2)^{1/2}, \tag{2.2}$$

where $s_i = \sigma_i - (\sigma_1 + \sigma_2 + \sigma_3)/3$ are the principal components of the Cauchy stress deviator. We commonly refer to σ_e as the "effective stress". An "effective strain" ε_e is defined as follows

$$\varepsilon_e = (2\varepsilon_i\varepsilon_i/3)^{1/2}. \tag{2.3}$$

For simple tension in the 1-direction σ_e and ε_e correspond to the axial stress σ_1 and strain ε_1, respectively. The strain energy and complementary strain energy functions, $W(\varepsilon_e)$ and $W^c(\sigma_e)$, are assumed to be functions of only ε_e and σ_e, respectively. The constitutive law has the following form

$$\begin{aligned} \varepsilon_i &= \frac{\partial W^c}{\partial \sigma_i} = \frac{3}{2}\frac{\varepsilon_e}{\sigma_e} s_i = \frac{3}{2}\frac{1}{E_s} s_i, \\ \sigma_i &= \frac{\partial W}{\partial \varepsilon_i} - p = \frac{2}{3} E_s \varepsilon_i - p, \end{aligned} \tag{2.4}$$

where $E_s = \sigma_e/\varepsilon_e$ denotes the secant modulus, obtainable from the uniaxial tension curve, and p is an arbitrary hydrostatic pressure. Note that $\dot{W} = \sigma_e\dot{\varepsilon}_e = s_i\dot{\varepsilon}_i = \sigma_i\dot{\varepsilon}_i$.

For certain applications (e.g., bifurcation analyses) it is convenient to express the constitutive law in incremental or rate form. Using the definitions for effective stress σ_e, effective strain ε_e and secant modulus E_s, we obtain the following from (2.4)

$$\begin{aligned} \dot{s}_i &= \dot{\sigma}_i + \dot{p} \\ &= \frac{2}{3} E_s\dot{\varepsilon}_i - s_i(E_s - E_t)\frac{s_k\dot{\varepsilon}_k}{\sigma_e^2}, \end{aligned} \tag{2.5}$$

where $E_t = d\sigma_e/d\varepsilon_e$ is the tangent modulus. For the shear components of stress-rate and Eulerian strain-rate $\dot{\varepsilon}_{ij}$ on the principal (Eulerian ellipsoid) axes, Hill's method of principal axes [6] gives

$$\overset{*}{\sigma}_{12} = (\sigma_1 - \sigma_2)\coth(\varepsilon_1 - \varepsilon_2)\dot{\varepsilon}_{12}, \qquad \text{etc.} \tag{2.6}$$

where the asterisk denotes the Jaumann or co-rotational stress-rate and coth is the hyperbolic cotangent. Thus, with reference to Cartesian base vectors coaxial with the principal stress axes, we can express the above as

$$\overset{*}{\sigma}_{ij} = L_{ijkl}\dot{\varepsilon}_{kl} + \dot{p}\delta_{ij}, \tag{2.7}$$

where

$$L_{ijkl} = \frac{2}{3} E_s \left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) - \frac{1}{3}\delta_{ij}\delta_{kl}\right] - (E_s - E_t)\frac{s_{ij}s_{kl}}{\sigma_e^2} + Q_{ijkl}. \tag{2.8}$$

The tensor Q is symmetric under $i \leftrightarrow j$, $k \leftrightarrow l$, and $ij \leftrightarrow kl$; and its only non-zero components in principal axes are the "shearing" terms

$$Q_{1212} = \frac{1}{3} E_s[(\varepsilon_1 - \varepsilon_2)\coth(\varepsilon_1 - \varepsilon_2) - 1], \quad \text{etc.} \tag{2.9}$$

These quantities are inherently non-negative. Note that the instantaneous moduli L in (2.8) and components of Q share the same indicial symmetries.

Recently, the above law has been extended somewhat to account for a slight degree of compressibility [8, 9]. Again, by analogy with the corresponding small-strain J_2 deformation theory, the total strain is written as the sum of an "elastic" part

$$\varepsilon_i^e = \frac{1+\nu}{E} \tau_i - \frac{\nu}{E} \tau_{kk}, \tag{2.10}$$

plus a "plastic" part

$$\varepsilon_i^p = \frac{3}{2}\left(\frac{1}{E_s} - \frac{1}{E}\right) t_i, \tag{2.11}$$

where the τ_i are the principal values of Kirchhoff stress, $\tau_{kk} = \tau_1 + \tau_2 + \tau_3$ and $t_i = \tau_i - \tau_{kk}/3$. Thus,

$$\begin{aligned} \varepsilon_i &= \frac{\partial W^c}{\partial \tau_i} = \frac{1+\nu_s}{E_s} \tau_i - \frac{\nu_s}{E_s} \tau_{kk}, \\ \tau_i &= \frac{\partial W}{\partial \varepsilon_i} = \frac{E_s}{1+\nu_s}\left[\varepsilon_i + \frac{\nu_s}{1-2\nu_s} \varepsilon_{kk}\right], \end{aligned} \tag{2.12}$$

where

$$\frac{\nu_s}{E_s} = \frac{1}{2}\left[1 - (1-2\nu)\frac{E_s}{E}\right], \tag{2.13}$$

and $\varepsilon_{kk} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. In the above, Poisson's ratio ν and Young's modulus E are assumed to be fixed constants. The secant modulus is now $E_s = \tau_e/\varepsilon_e$ where $\tau_e = (3t_it_i/2)^{1/2}$ denotes the effective Kirchhoff stress. The total effective strain is $\varepsilon_e = \tau_e/E + \varepsilon_e^p$ where

$$\varepsilon_e^p = (2\varepsilon_i^p \varepsilon_i^p/3)^{1/2}. \tag{2.14}$$

For simple tension in the 1-direction, $\tau_e = \tau_1$ and $\varepsilon_e = \varepsilon_1$. Note that the finite "plastic" strains (2.11) satisfy incompressibility, as do the total finite strains when $\nu = 1/2$.

The rate form of the above constitutive law can be obtained as described previously. With reference to the Eulerian ellipsoid axes we have

$$\overset{*}{\tau}_{ij} = L_{ijkl}\dot{\varepsilon}_{kl}, \tag{2.15}$$

with

$$L_{ijkl} = \frac{E_s}{1+\nu_s}\left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) + \frac{\nu_s}{1-2\nu_s}\,\delta_{ij}\delta_{kl}\right]$$

$$- \frac{3}{2(1+\nu_s)}\left[h_s(E_s - E_t)\frac{s_{ij}s_{kl}}{\tau_e^2} - Q_{ijkl}\right],$$

(2.16)

$$h_s = \frac{E_s}{E_s - (1-2\nu_s)E_t/3}\,.$$

The tangent modulus is now defined as $E_t = d\tau_e/d\varepsilon_e$.

Both the incompressible and compressible versions of J_2 finite-strain deformation theory assume identical effective stress–effective strain relations in tension and compression. Furthermore, the only stress invariant affecting plastic response is J_2. These assumptions are generally considered to be good first-order approximations in metal plasticity. For the special case where the principal axes remain fixed relative to the material *and* proportional loading (i.e., when the principal stress components increase monotonically and in fixed proportion to one another), the finite-strain J_2 deformation theory is exactly the integrated result of finite-strain J_2 flow theory. As in the small strain theory, applicability of the theory to polycrystalline metals must be suspect when significant deviations from proportional plastic straining arise. A path-dependent version of finite strain J_2 deformation theory has been proposed by Stören and Rice [10] with the primary purpose of modeling nearly proportional responses of a material which develops a yield surface vertex. That theory has the form (2.16) but with $\boldsymbol{Q}$ deleted. For histories in which the principal stress axes remain fixed relative to the material the two versions of the theory coincide.

An alternative approach for incorporating the logarithmic strain measure $\boldsymbol{\varepsilon}$ in finite elasticity constitutive laws has recently been developed by Fitzgerald [11], who has presented a general tensorial formulation for the logarithmic strain components in arbitrary axes. A strain energy function depending on the invariants of $\boldsymbol{\varepsilon}$ can then be assumed to give a hyperelastic law in arbitrary axes, thus eliminating the need for principal axes methods. Fitzgerald's formulation is completely general and includes our J_2 law as a special case.

3. Loss of ellipticity for incompressible J_2 deformation theory

Conditions for ellipticity of the equations governing incremental deformations superimposed on finite homogeneous deformations have been given by a number of authors. Recent studies include those by Hill [12, 13], Hill and Hutchinson [14], Knowles and Sternberg [15], Rice [16], and Sawyers and

Rivlin [17–19]. Here we quickly rederive a set of necessary conditions for strong ellipticity of incompressible, isotropic hyperelastic solids which was originally obtained by Sawyers and Rivlin [17, 18]. These conditions are then specialized to the J_2 deformation theory material. With the aid of numerical calculations for a specific family of materials, it is noted that the necessary conditions may also be sufficient to guarantee strong ellipticity for J_2 deformation theory, although this has not been shown.

In the sequel, Cartesian axes x_i are chosen to coincide with the axes of the principal stresses and strains of the underlying homogeneous state. Strong ellipticity of the equations governing quasi-static, superimposed incremental deformations requires

$$c_{ijkl}\nu_i\eta_j\nu_k\eta_l > 0 \tag{3.1}$$

for *all* mutually orthogonal unit vectors $\boldsymbol{\nu}$ and $\boldsymbol{\eta}$. The so-called acoustic tensor of moduli $\boldsymbol{c}$ is related to the moduli tensor $\boldsymbol{L}$ in (2.7) by

$$c_{ijkl} = L_{ijkl} + \frac{1}{2}\sigma_{ik}\delta_{lj} - \frac{1}{2}\sigma_{il}\delta_{kj} - \frac{1}{2}\sigma_{jk}\delta_{il} - \frac{1}{2}\sigma_{jl}\delta_{ik}, \tag{3.2}$$

and both tensors share the indicial symmetries $c_{ijkl} = c_{klij}$ for hyperelastic solids. Condition (3.1) excludes quasi-static shearing discontinuities characteristic of a planar shear band with normal $\boldsymbol{\nu}$ and shearing direction $\boldsymbol{\eta}$ in the plane of the band. It also ensures that all plane waves with propagation direction $\boldsymbol{\nu}$ and particle velocity parallel to $\boldsymbol{\eta}$ have real wave speeds.

Following Sawyers and Rivlin [17, 18], we can obtain necessary conditions for strong ellipticity by restricting $\boldsymbol{\nu}$ to lie in one of the planes of the principal stress axes. Let $\boldsymbol{\nu}$ lie in the x_1x_2-plane at angle ψ from the x_1-axis so that

$$\boldsymbol{\nu} = (\cos\psi, \sin\psi, 0) \quad \text{and} \quad \boldsymbol{\eta} = (-a\sin\psi, a\cos\psi, b) \tag{3.3}$$

where $a^2 + b^2 = 1$. The shearing direction $\boldsymbol{\eta}$ does not lie in the x_1x_2-plane unless $b = 0$. With (3.3), condition (3.1) becomes

$$\begin{aligned} &a^2\{c_{1212}\cos^4\psi + [c_{1111} + c_{2222} - 2c_{1122} - 2c_{1221}]\cos^2\psi\sin^2\psi + c_{2121}\sin^4\psi\} \\ &\quad + b^2\{c_{1313}\cos^2\psi + c_{2323}\sin^2\psi\} > 0 \end{aligned} \tag{3.4}$$

for all ψ and all a such that $|a| \leqq 1$ with $b^2 = 1 - a^2$. In arriving at (3.4) we have used the fact that components such as c_{1113} vanish in the principal axes.

With $b = 0$, (3.4) is satisfied for all ψ if and only if

$$c_{1212} > 0 \quad \text{and} \quad c_{2121} > 0, \tag{3.5}$$

and

$$c_{1111} + c_{2222} - 2c_{1122} > 2c_{1221} - 2\sqrt{c_{1212}c_{2121}}. \tag{3.6}$$

Conditions (3.5) and (3.6) are necessary and sufficient for strong ellipticity of incremental plane strain deformation in the x_1x_2-plane — i.e., for restricted displacement increments of the form $v_1(x_1, x_2)$, $v_2(x_1, x_2)$ and $v_3 = 0$. The choice $a = 0$ in (3.4) requires

$$c_{1313} > 0 \quad \text{and} \quad c_{2323} > 0. \tag{3.7}$$

Conditions (3.5)–(3.7) are equivalent to (3.4). One notes immediately that satisfaction of (3.5) and (3.6) and their equivalents for each of the other two principal planes renders the third set of conditions (3.7) extraneous. In other words, strong ellipticity for incremental plane strain deformations parallel to each of the principal planes ensures that (3.1) is satisfied for any $\boldsymbol{\nu}$ which lies in a principal plane whether or not $\boldsymbol{\eta}$ lies in one of the principal planes. This was established by Sawyers and Rivlin [17, 18]. As they noted [18], these conditions are sufficient for strong ellipticity when the underlying state has two equal principal strains since then any $\boldsymbol{\nu}$ necessarily lies in a principal plane.

At this point it is convenient to introduce two shearing moduli μ and μ^*, used by Hill and Hutchinson [14], governing incremental plane strain deformations in the x_1x_2-plane. For such deformations (2.7) can be written as

$$\overset{*}{\sigma}_{11} - \overset{*}{\sigma}_{22} = 2\mu^*(\dot{\varepsilon}_{11} - \dot{\varepsilon}_{22}), \qquad \overset{*}{\sigma}_{12} = 2\mu\dot{\varepsilon}_{12}, \qquad (\dot{\varepsilon}_{11} + \dot{\varepsilon}_{22} = 0), \tag{3.8}$$

where

$$4\mu^* = L_{1111} + L_{2222} - 2L_{1122} \quad \text{and} \quad \mu = L_{1212}. \tag{3.9}$$

With the aid of (3.2), conditions (3.5) and (3.6) can be expressed as

$$\mu > |\Delta\sigma|/2, \tag{3.10}$$

and

$$2\mu^* > \mu - (\mu^2 - \Delta\sigma^2/4)^{1/2}, \tag{3.11}$$

where $\Delta\sigma = \sigma_1 - \sigma_2$. This is the form of the ellipticity conditions given by Hill and Hutchinson [14] for incremental plane strain deformations of a broad class of incrementally linear materials satisfying (3.8).

The six conditions of the form (3.5) or, equivalently, the three conditions (3.10) are always satisfied by J_2 deformation theory as long as the secant modulus E_s is positive. For example, one can show that

$$c_{2121} = \mu - \frac{1}{2}\Delta\sigma = \frac{1}{3}E_s\frac{\Delta\varepsilon e^{-\Delta\varepsilon}}{\sinh(\Delta\varepsilon)}, \tag{3.12}$$

and

$$c_{1212} = \mu + \frac{1}{2}\Delta\sigma = \frac{1}{3}E_s\frac{\Delta\varepsilon e^{\Delta\varepsilon}}{\sinh(\Delta\varepsilon)}, \tag{3.13}$$

where $\Delta\sigma = \sigma_1 - \sigma_2$ and $\Delta\varepsilon = \varepsilon_1 - \varepsilon_2$.

We have not yet succeeded in showing that the three remaining conditions of the form (3.11) are sufficient for strong ellipticity for J_2 deformation theory. Sawyers and Rivlin [18] have shown that their conditions, which are equivalent to those of the form (3.10) and (3.11), are both necessary and sufficient for strong ellipticity for two special classes of solids, each of which has a strain energy function which depends on only one strain invariant. The energy density of J_2 deformation theory depends on only one strain invariant, ε_e, but it is not included in either of the special classes of solids for which Sawyers and Rivlin have established sufficiency. It is known [19] that incompressible, isotropic hyperelastic materials do exist for which conditions (3.10) and (3.11) are not sufficient for strong ellipticity.

A limited numerical study of the sufficiency of the three conditions of the form (3.11) has been carried out for J_2 deformation theory with a power-law relation between true stress and natural strain of the form

$$\sigma_e = K\varepsilon_e^N, \tag{3.14}$$

where the hardening index N is restricted to the range $0 < N \leqq 1$. The secant and tangent moduli diminish with increasing strain according to

$$E_s = K\varepsilon_e^{N-1} \quad \text{and} \quad E_t = NE_s. \tag{3.15}$$

Using (2.8) and (3.9), one can show that the three conditions of the form (3.11) become

$$\Delta\varepsilon^2 < 4\left[1 - \frac{1}{3}(N-1)\left(\frac{\Delta\varepsilon}{\varepsilon_e}\right)^2\right]\left[(\Delta\varepsilon \coth\Delta\varepsilon - 1) + \frac{1}{3}(N-1)\left(\frac{\Delta\varepsilon}{\varepsilon_e}\right)^2\right], \tag{3.16}$$

where $\Delta\varepsilon$ is identified with the principal strain differences $\varepsilon_1 - \varepsilon_2$, $\varepsilon_1 - \varepsilon_3$, and $\varepsilon_2 - \varepsilon_3$. The conditions (3.16) are first violated by the maximum principal strain difference corresponding to a shear band with normal and shearing direction lying in the plane of the maximum principal strain difference.

The boundary of the region of principal strains states within which the three conditions (3.16) are satisfied is shown in Figure 1 for $N = 0.1, 0.5$ and 1. Since $\varepsilon_3 = -\varepsilon_1 - \varepsilon_2$, the region is fully depicted by its trace in the $\varepsilon_1\varepsilon_2$-plane. The region is symmetric with respect to the 45° lines in Figure 1 and only one quarter of the region is shown. The line on which $\varepsilon_1 = \varepsilon_3 = -\frac{1}{2}\varepsilon_2$ marks the switch in the maximum principal strain difference from $\varepsilon_2 - \varepsilon_1$ to $\varepsilon_2 - \varepsilon_3$. For $N = 1$, (3.16) reduces to

$$\Delta\varepsilon^2 < 4(\Delta\varepsilon \coth\Delta\varepsilon - 1), \tag{3.17}$$

which is equivalent to

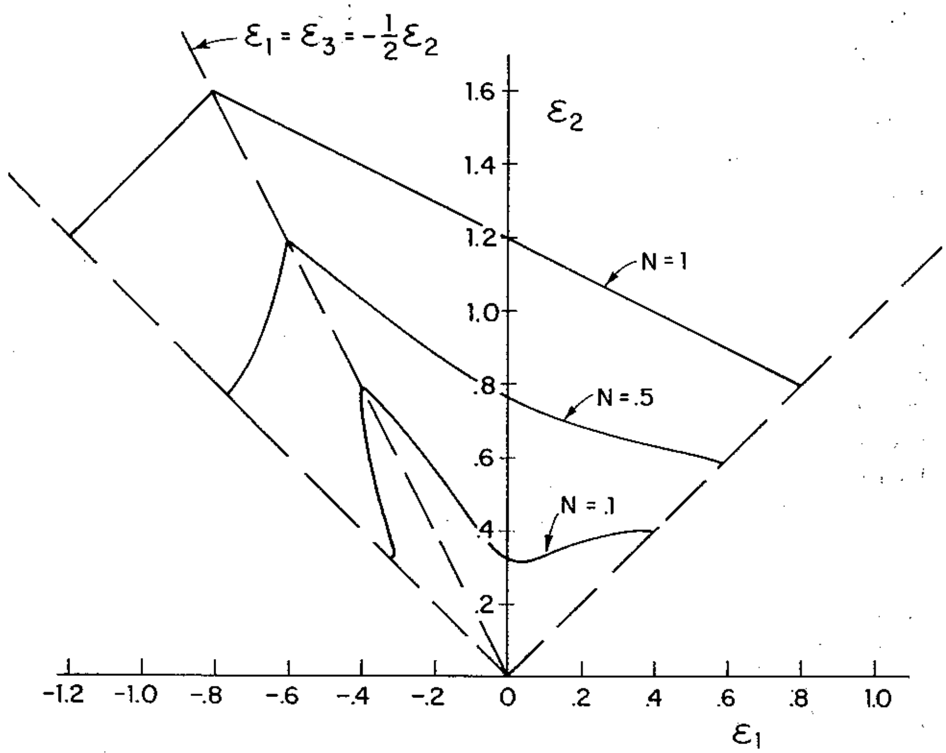

Figure 1. Boundary of the elliptic region for power-law material for three values of the exponent N. The elliptic region is symmetric with respect to the lines $\varepsilon_1 = \varepsilon_2$ and $\varepsilon_1 = -\varepsilon_2$.

(3.18) $$|\Delta\varepsilon| < 2.399,$$

and which leads to the boundary with straight line segments in Figure 1.

Numerical calculations have been carried out to ascertain whether (3.1) is satisfied for all mutually orthogonal $\boldsymbol{\nu}$ and $\boldsymbol{\eta}$ when the strain states lie within the boundaries of Figure 1. Orientations of $\boldsymbol{\nu}$ and $\boldsymbol{\eta}$ with respect to the principal axes were specified by three Euler angles and these angular coordinates were taken to range over their full range consistent with the symmetry of the underlying state. For each strain state considered, (3.1) was checked at more than 10^4 orientations. For each of $N = 0.1$ 0.5 and 1, the strain states considered were those at 5° intervals measured from the 45° line in Figure 1 at strains which were 0.999 times the corresponding values at the boundary as ascertained by (3.16). In no case was (3.1) violated. Although strain states further within the boundaries of Figure 1 were not considered, no violation of (3.1) is expected since the incremental moduli $\boldsymbol{L}$ increase with decreasing strain when $N < 1$.

Our numerical checks of (3.1) for the example of the power-law solid suggest that the three conditions (3.16) may be both necessary and sufficient for strong ellipticity for this material. It is an open question as to whether this actually is the case and whether, more generally, the three conditions of the form (3.11) are sufficient for gauging strong ellipticity of any J_2 deformation theory solid.

ACKNOWLEDGEMENT

The work of J.W.H. was supported in part by the National Science Foundation under Grant ENG78-10756, and by the Division of Applied Sciences, Harvard University. The work was conducted while K.W.N. was on sabbatical leave at Harvard University, and was supported in part by the Faculty of Applied Sciences at the University of Sherbrooke, and by the Division of Applied Sciences, Harvard University.

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