# I.A. Kunin on the problems related to extended continuum mechanical formulations. Adapted from the introduction to the book: "Elastic Media with Microstructure. Three Dimensional Models." Springer 1983

"Usually, such theories are derived on phenomenological grounds, roughly according to the following scheme. It is postulated, that the Lagrangian depends on the strain and its derivatives and on a set of additional kinematic variables such as microrotations and microstrain and their derivatives. By analogy with the classical theory of elasticity the new kinematic variables and their derivatives are put into correspondence with moment stresses of various orders, for which equations of equilibrium (conservation laws) and equations of state of the Hooke's law type, are postulated. This enables us to obtain a closed system of equations and a set of boundary conditions. Further formulation of boundary-value problems usually does not differ from the formulation of analogous problems in the classical theory of elasticity. For example, many papers are devoted to investigations, in the scope of the couple-stress theory of elasticity, concerning stress concentration around holes. In this situation, one would expect that significant differences from the classical theory of elasticity arises for holes with diameters of the order of or less than $L_c$.

Let us consider some of the difficulties which arise in such an approach.

1) All the couple-stress theories approximately account for the nonlocal effects, and the parameter $L_c$ which is contained in them has to be considered to be small. Therefore, when extrapolating the results to the regions of the order of or much smaller than $L_c$, one should be very cautious.

2) The presence of a small parameter in terms with higher-order derivatives, as is well-known, leads to effects of the boundary-layer type. In connection with this, it is hardly reasonable to consider boundary-value problems for complicated regions in the scope of couple-stress theories. Moreover, it is also unknown how to interpret the boundary conditions physically and from where to take the values of the moments on the boundary.

3) The theories contain new constants (sometimes in large numbers) but usually it is not clear, from what sort of experiments these could be determined. It is necessary to know at least the qualitative relationship between these constants and the parameters of the micromodel.

4) In an axiomatic construction of the theories without taking into account their approximate nature, the equations of motion are often written down with different degrees of accuracy with respect to different variables.

5) The connection between the couple-stress theories and the classical theory of elasticity is usually discussed in a simplified manner: in order to proceed to the theory of elasticity, it is necessary to set the constants contained in the equation equal to zero. As we have observed in the example of the one-dimensional model of a medium of complex structure, the actual connection is more involved: for sufficiently long wavelengths and for sufficiently low frequencies the internal degrees of freedom are excluded from the equations of motion, but they do make some contribution to the effective macroparameters. In the next section this question will be considered for the three-dimensional medium.

6) We observed that, in the nonlocal theory of a medium of simple structure, the stress tensor is not a physically uniquely determined quantity. This holds even more strongly for the couple stress in the medium of complex structure. For its formally unique determination, it is necessary to postulate conservation laws, which do not have any group basis. As is well-known in field theory, the conservation laws are a result of the invariance of the Lagrangian with respect to the corresponding groups and in our case they are the groups of translations and rotations. However, only displacements and microrotations are essentially with this fundamental group; this leads to a special divergence form of the equations of motion, which ensures the equilibrium of forces and moments. The remaining internal degrees of freedom (microdeformations) are invariant with respect to the fundamental group, and additional laws for couple stress have to be postulated rather arbitrarily.

Note also that couple stresses do not satisfy the natural principle of correspondence, i.e. they do not become the usual stresses in the limiting case of long waves, when a transition to the conventional theory of elasticity should take place.

Taking into account these arguments, one should not be surprised that the rare attempts of experimental determination of couple stress in an nonhomogeneous medium have not led to any results.

Let us sum up. The couple-stress theories of elasticity and, generally, models of a medium with weak dispersion approximately take into account comparatively delicate effects of nonlocality and of internal degrees of freedom. Their application is expedient, first of all, when they yield not merely small quantitative corrections, but describe qualitatively new phenomena. Therefore the physically justified field of applicability of these theories is narrower than that of the classical theory of elasticity, though the latter is the limiting case.

## Modifications

I have replaced $\ell$ with the internal length scale parameter $L_c > 0$. Moreover, I have skipped references inside book.