A spatial stress response criterion for the stability of incremental deformations

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Statutory declaration

I declare that I have authored this thesis independently, that I have not used other than the declared sources, and that I have explicitly marked all material which has been quoted either literally or by content from the used sources.

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Contents

1 Introduction

My master's thesis "A spatial stress response criterion for the stability of incremental deformations" discusses stability criterions regarding linearizations. The guiding question related to this is whether Liu [2, 10] has found a new stability criterion for linearizations on an intermediate configuration in nonlinear elasticity which may be better than previous ones. In addition, we are interested in the question of what kind of ellipticity conditions result from it. Furthermore, we want to show limits of Liu's statements. Therefore, we will mainly refer to the sources [10] and [2].

We start by giving a basic overview of elasticity, including important theorems for existence theory, in order to give an insight into the topic. In addition, we derive the Euler-Lagrange equations to find a minimum for variational problems in elasticity. Based on this, we give a brief introduction into the topic of Null-Lagrangians. In the fourth section, we talk about linearizations in general. We refer exclusively to Liu's arguments and transform his linearized equation of equilibrium in the intermediate configuration and the linearized equation of equilibrium in the reference configuration until they match. Therefore, we use three different deformations, one affecting the intermediate configuration and two of them affecting the reference configuration. After that, we use them to specify the equations of equilibrium in the reference and in the intermediate configuration. Then we linearize both and transform them into one new equation. Besides, we briefly point out the special case for homogeneous initial deformations.

The second main part of this master's thesis applies the results to a specific energy function by strictly adhering to Liu's calculations in [2]. We also give an existence and uniqueness statement like Liu does and point out which important cases he ignores, i.e. where the limits of his considerations lie. In this regard, we also discuss Korn's inequality in Section 5 and deficiencies of Liu's argumentation. Moreover, we discuss ellipticity and its connection to existence theory. Therefore, we give verification whether a given linearized energy function in the intermediate configuration or in the reference configuration fullfills the condition of Lagrange-Hadamard ellipticity and compare it to the classical concept of ellipticity. Finally, we give a brief outlook of which questions might still be of interest in the future.

2 Mathematical basics and notations

In the following, we introduce some mathematical basics and notations which are important for elasticity in general and for the problem of this master's thesis in particular.

2.1 Sets in $\mathbb{R}^{3\times 3}$

For further work, we define a number of sets in $\mathbb{R}^{3\times 3}$.

Definitions 2.1.1. The set

$$\mathrm{GL}(3) \coloneqq \{X \in \mathbb{R}^{3\times 3} |\, \det(X) \neq 0\}$$

is called the *group of invertible matrices* or *general linear group*. The *group of invertible matrices with a positive determinant* is a subset of GL(3), defined by

$$\mathrm{GL}^+(3) \coloneqq \{X \in \mathbb{R}^{3\times 3} |\, \det(X) > 0\}.$$

Another subset is the *special linear group*

$$\mathrm{SL}(3) \coloneqq \{X \in \mathbb{R}^{3\times 3} |\, \det(X) = 1\}.$$

Other important sets are the *group of orthogonal matrices*

$$\mathrm{O}(3) \coloneqq \{X \in \mathbb{R}^{3\times 3} | X^T X = \mathbb{1}\}$$

and its subset

$$\mathrm{SO}(3) \coloneqq \{X \in \mathbb{R}^{3\times 3} | X^T X = \mathbb{1}, \det(X) = 1\},$$

called the *set of rotation matrices* or *special orthogonal group*. Also all symmetric matrices depict a set, the *set of symmetric matrices*

$$\mathrm{Sym}(3) \coloneqq \{X \in \mathbb{R}^{3\times 3} | X^T = X\}.$$

Its subset

$$\mathrm{Sym}^+(3) \coloneqq \{X \in \mathbb{R}^{3\times 3} | X^T = X, \langle Xv, v\rangle > 0 \quad \forall\, v \in \mathbb{R}^3 \setminus \{0\}\}$$

is the *set of symmetric positive definite matrices.*

2.2 Elasticity

In elasticity, we consider elastic deformations of solid bodies. These deformations are represented by a mapping φ which is defined on a bounded, open, connected subset $\Omega \subset \mathbb{R}^3$, cf. Fig. 1. In contrast to plasticity, "elastic" means that the deformed body is able to revert its shape back to its original state after releasing inner/outer forces.

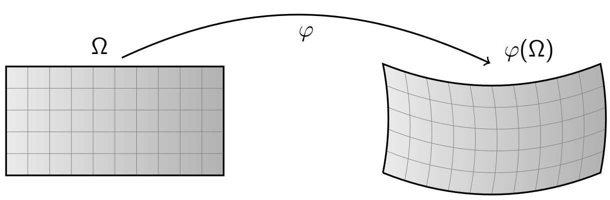

Figure 1: Deformation φ of a solid body with its reference configuration Ω and its current configuration $\varphi(\Omega)$.

We assume the solid body to be continuous and call Ω the stress-free *reference configuration.* The deformed state $\varphi(\Omega) := \widehat{\Omega} \subset \mathbb{R}^3$ is termed the *current configuration.* In addition, we describe the coordinates $x \in \Omega$ in the reference configuration as *Lagrangian coordinates* and the coordinates $\varphi(x) \in \widehat{\Omega}$ in the current configuration as *Eulerian coordinates.* We define the *deformation gradient* as

$$\nabla\varphi := \begin{pmatrix} \partial_1 \varphi_1 & \partial_2 \varphi_1 & \partial_3 \varphi_1 \\ \partial_1 \varphi_2 & \partial_2 \varphi_2 & \partial_3 \varphi_2 \\ \partial_1 \varphi_3 & \partial_2 \varphi_3 & \partial_3 \varphi_3 \end{pmatrix} \in \mathrm{GL}^+(3). \tag{2.2.1}$$

Furthermore, we assume that the deformation gradient $\nabla\varphi$ satisfies

$$\det \nabla\varphi > 0 \text{ for all } x \in \Omega, \tag{2.2.2}$$

since $\nabla\varphi$ needs to be orientation-preserving by definition. In particular, $\nabla\varphi$ is invertible.
Moreover, changes of area, length and volume of Ω are governed by $\mathrm{Cof}(\nabla\varphi)$[1], $\det\nabla\varphi$ and $\nabla\varphi$. More precisely, $\mathrm{Cof}(\nabla\varphi)$ is responsible for the change in area, $\det\nabla\varphi$ for the change in volume and $\nabla\varphi$ for the change in length. We want to illustrate these statements with some calculations; for this we define $F := \nabla\varphi$.
The deformation gradient is responsible for change in length L, because for a vector $\xi \in \Omega$ we can compute the vector's length in the current configuration as

$$L(F\xi) = \|F\xi\| = \sqrt{\langle F\xi, F\xi\rangle} = \sqrt{\langle F^T F\xi, \xi\rangle}. \tag{2.2.3}$$

Besides for two vectors $\xi, \eta \in \Omega$ which describe an area in Ω we obtain $\|\eta \times \xi\|$. This implies $\|F\eta \times F\xi\|$ for the deformed area. We compute

$$\begin{aligned}
F\eta \times F\xi &= \begin{pmatrix} F_{11}\eta_1 + F_{12}\eta_2 + F_{13}\eta_3 \\ F_{21}\eta_1 + F_{22}\eta_2 + F_{23}\eta_3 \\ F_{31}\eta_1 + F_{32}\eta_2 + F_{33}\eta_3 \end{pmatrix} \times \begin{pmatrix} F_{11}\xi_1 + F_{12}\xi_2 + F_{13}\xi_3 \\ F_{21}\xi_1 + F_{22}\xi_2 + F_{23}\xi_3 \\ F_{31}\xi_1 + F_{32}\xi_2 + F_{33}\xi_3 \end{pmatrix} \\
&= \begin{pmatrix} (F_{21}\eta_1 + F_{22}\eta_2 + F_{23}\eta_3)(F_{31}\xi_1 + F_{32}\xi_2 + F_{33}\xi_3) \\ -(F_{31}\eta_1 + F_{32}\eta_2 + F_{33}\eta_3)(F_{21}\xi_1 + F_{22}\xi_2 + F_{23}\xi_3) \\ (F_{31}\eta_1 + F_{32}\eta_2 + F_{33}\eta_3)(F_{11}\xi_1 + F_{12}\xi_2 + F_{13}\xi_3) \\ -(F_{11}\eta_1 + F_{12}\eta_2 + F_{13}\eta_3)(F_{31}\xi_1 + F_{32}\xi_2 + F_{33}\xi_3) \\ (F_{11}\eta_1 + F_{12}\eta_2 + F_{13}\eta_3)(F_{21}\xi_1 + F_{22}\xi_2 + F_{23}\xi_3) \\ -(F_{21}\eta_1 + F_{22}\eta_2 + F_{23}\eta_3)(F_{11}\xi_1 + F_{12}\xi_2 + F_{13}\xi_3) \end{pmatrix} \\
&= \begin{pmatrix} (F_{22}F_{33} - F_{23}F_{32})(\eta_2\xi_3 - \eta_3\xi_2) + (F_{31}F_{23} - F_{21}F_{33})(\eta_3\xi_1 - \eta_1\xi_3) \\ +(F_{21}F_{32} - F_{31}F_{22})(\eta_1\xi_2 - \eta_2\xi_1) \\ (F_{32}F_{13} - F_{12}F_{33})(\eta_2\xi_3 - \eta_3\xi_2) + (F_{11}F_{33} - F_{31}F_{13})(\eta_3\xi_1 - \eta_1\xi_3) \\ +(F_{31}F_{12} - F_{11}F_{32})(\eta_1\xi_2 - \eta_2\xi_1) \\ (F_{12}F_{23} - F_{22}F_{13})(\eta_2\xi_3 - \eta_3\xi_2) + (F_{21}F_{13} - F_{11}F_{23})(\eta_3\xi_1 - \eta_1\xi_3) \\ +(F_{11}F_{22} - F_{21}F_{12})(\eta_1\xi_2 - \eta_2\xi_1) \end{pmatrix} \\
&= \begin{pmatrix} F_{22}F_{33} - F_{23}F_{32} & F_{31}F_{23} - F_{21}F_{33} & F_{21}F_{32} - F_{31}F_{22} \\ F_{32}F_{13} - F_{12}F_{33} & F_{11}F_{33} - F_{31}F_{13} & F_{31}F_{12} - F_{11}F_{32} \\ F_{12}F_{23} - F_{22}F_{13} & F_{21}F_{13} - F_{11}F_{23} & F_{11}F_{22} - F_{21}F_{12} \end{pmatrix} \cdot \begin{pmatrix} \eta_2\xi_3 - \eta_3\xi_2 \\ \eta_3\xi_1 - \eta_1\xi_3 \\ \eta_1\xi_2 - \eta_2\xi_1 \end{pmatrix} \\
&= \mathrm{Cof}\begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \cdot \left[\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} \times \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}\right] \\
&= \mathrm{Cof}(F) \cdot (\eta \times \xi),
\end{aligned} \tag{2.2.4}$$

[1] $\mathrm{Cof}(X) = \det(X)X^{-T}$ for $X \in \mathrm{GL}(n)$.

with

$$F = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \in \mathrm{GL}(3), \quad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}, \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \in \mathbb{R}^3.$$

Thus we obtain

$$\|F\eta \times F\xi\| = \|\mathrm{Cof}(F) \cdot (\eta \times \xi)\| \tag{2.2.5}$$

and therefore, $\mathrm{Cof}(F)$ describes the change in area. In addition, with some transformations and an arbitrary subset $C \subset \Omega$ we get

$$\mathrm{Vol}(\varphi(C)) = \int_{\xi \in \varphi(C)} 1 \, \mathrm{d}\xi = \int_{\eta \in C} 1 \cdot |\det(\nabla\varphi)| \, \mathrm{d}\eta = |\det(\nabla\varphi)| \cdot \mathrm{Vol}(C). \tag{2.2.6}$$

We have to consider that the last step only holds if $\nabla\varphi$ is independent of η. Therefore, we assume the deformation to be infinitesimal (C to be an infinitesimal cube). Thus, $\det(\nabla\varphi)$ is responsible for change in volume.
The *displacement* u is defined as

$$u : \Omega \to \mathbb{R}^3 \quad \text{with} \quad u(x) := \varphi(x) - x. \tag{2.2.7}$$

Hence, the relation between the deformation gradient $\nabla\varphi$ and the *displacement gradient*

$$\nabla u := \begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 & \partial_3 u_1 \\ \partial_1 u_2 & \partial_2 u_2 & \partial_3 u_2 \\ \partial_1 u_3 & \partial_2 u_3 & \partial_3 u_3 \end{pmatrix} \tag{2.2.8}$$

is expressed by the equation

$$\nabla\varphi = \mathbb{1} + \nabla u \tag{2.2.9}$$

with $\mathbb{1} \in \mathrm{GL}^+(3)$ as the identity matrix. Typically, we only know the deformation φ for a part $\Gamma \subset \partial\Omega$ of the boundary and have to find the deformation of the whole body Ω, see Fig. 2.

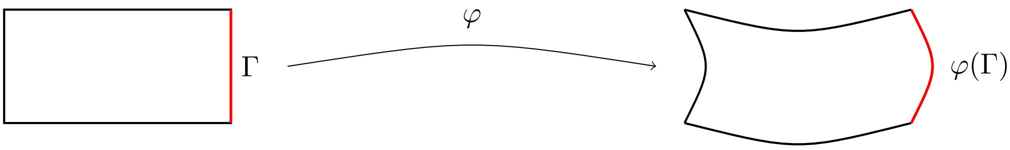

Figure 2: Deformation φ of the elastic body Ω induced by a displacement of the boundary.

The main idea of nonlinear hyperelasticity is to describe the deformation φ as a solution of a minimum problem concerning an elastic energy.
We define the *elastic energy* function

$$W : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathrm{GL}^+(3) \to \mathbb{R} \qquad \text{with} \qquad (x, y, X) \mapsto W(x, y, X), \tag{2.2.10}$$

which is assumed to be twice continuously differentiable. It represents the elastic energy of an infinitesimal cube as shown in the picture below (Fig. 3).

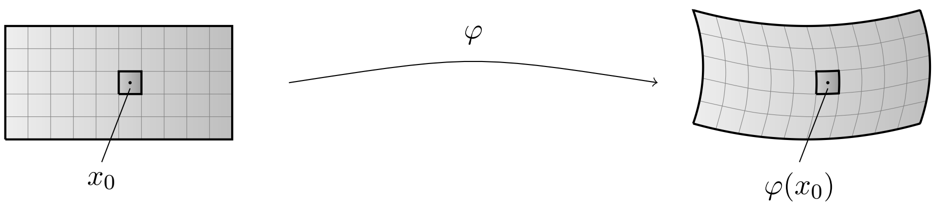

Figure 3: Deformation of an infinitesimal cube.

The energy W of the infinitesimal cube at the point $x_0 \in \Omega$ depends on the place $x_0 \in \Omega$ in the reference configuration, on the deformation $\varphi(x_0)$ at this point and on the deformation gradient $\nabla\varphi(x_0)$; higher orders of $\mathrm{D}^n\varphi(x)$ are omitted. In the following, we assume the body Ω to be homogeneous, which means the elastic energy of the infinitesimal cubes should not depend on the place. Furthermore, for a point x in an arbitrary cube in the reference configuration we obtain

$$\begin{aligned}\varphi(x) = \varphi(x_0 + h) &= \varphi(x_0) + \nabla\varphi(x_0).h + \frac{1}{2}D^2\varphi(x_0).(h,h) + ... \\ &= \varphi(x_0) + \nabla\varphi(x_0).h + O\left(\|h\|^2\right),\end{aligned}$$

i. e. the position $\varphi(x_0)$ in the deformed configuration only represents an "affine movement of the infinitesimal cube" which does not change the shape of the cube. Moreover, for an infinitesimal cube $(h \to 0)$ higher order terms $O\left(\|h\|^2\right)$ vanish. Thus, we can define the energy value of any infinitesimal cube as a function

$$\widehat{W}\colon \mathrm{GL}^+(3) \to \mathbb{R} \quad \widehat{W}(\nabla\varphi(x)) =: W(x, \varphi(x), \nabla\varphi(x)). \tag{2.2.11}$$

In addition, the energy function $\widehat{W}$ should be *frame-indifferent*, which means that a rotation after the actual deformation X should not change the elastic energy, i.e.

$$\widehat{W}(QX) = \widehat{W}(X), \quad \forall Q \in \mathrm{SO}(3). \tag{2.2.12}$$

Furthermore, the following standardizations may apply:

$$\widehat{W}(\nabla\varphi) \geq 0 \quad \forall\, \nabla\varphi \in \mathrm{GL}^+(3) \qquad \text{and} \qquad \widehat{W}(\mathbb{1}) = 0.$$

Thereby, for the set $\mathcal{M} = \{\varphi \in C^2(\overline{\Omega}) | \nabla\varphi \in \mathrm{GL}^+(3) \text{ and } \varphi_{|\Gamma} = g\}$, we get an energy functional

$$I\colon \mathcal{M} \to \mathbb{R} \qquad \text{with} \qquad I(\varphi) := \int_\Omega \widehat{W}(\nabla\varphi)\,\mathrm{d}x \tag{2.2.13}$$

which describes the total elastic energy of the body Ω.[2]

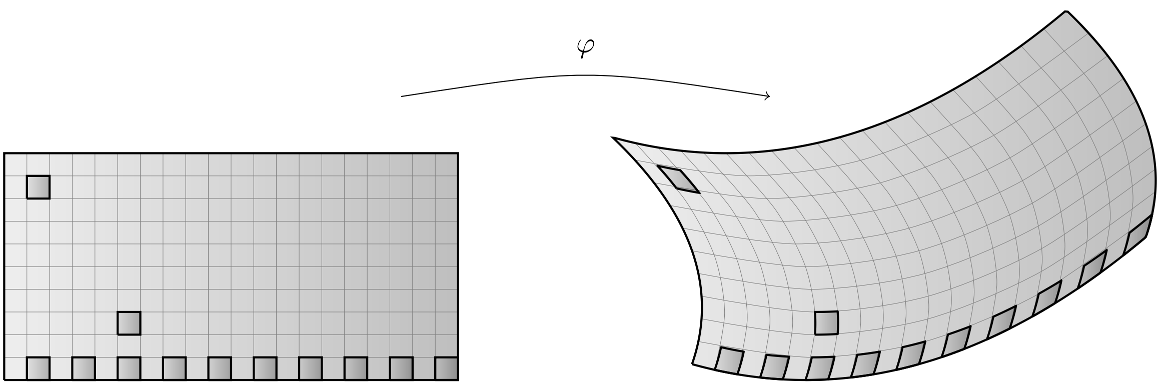

Figure 4: Visualization of the energy functional $I(\varphi)$ of the whole body Ω through approximation with infinitesimal cubes and the energy function $\widehat{W}$.

Furthermore, a deformation has to be energy optimal, i.e. it has to minimize the elastic energy under given boundary conditions. Therefore we get a variational problem

$$I(\varphi) = \int_\Omega \widehat{W}(\nabla\varphi)\,\mathrm{d}x \to \min, \quad \varphi \in \mathcal{M}. \tag{2.2.14}$$

[2]Note that the integral in (2.2.13) can be obtained as the limit of sums of the energies of infinitesimal cubes.

Because of the relation between the deformation gradient $\nabla\varphi$ and the displacement field ∇u in (2.2.9), we can consider the equivalent variational problem

$$I(u) = \int_\Omega \widehat{W}(\mathbb{1} + \nabla u)\,\mathrm{d}x \to \min\,, \quad u \in \mathcal{M} \tag{2.2.15}$$

instead of problem (2.2.14).

2.2.1 Polar decomposition

In elasticity we deal with energy functionals which we assume to be *isotropic*[3]. For this, we need the *polar decomposition* [21].

Definition 2.2.1. Let $F \in \mathrm{GL}^+(3)$. The decomposition

$$F = R \cdot U \quad \text{with} \quad R \in \mathrm{SO}(3),\, U \in \mathrm{Sym}^+(3) \tag{2.2.16}$$

is called *left polar decomposition* (Fig.5) and the decomposition

$$F = V \cdot R \quad \text{with} \quad R \in \mathrm{SO}(3),\, V \in \mathrm{Sym}^+(3) \tag{2.2.17}$$

is termed *right polar decomposition* (Fig. 6).

Here, R is a rotation, U is a right stretch and V is a left stretch.
Moreover, the polar decomposition is uniquely determined.

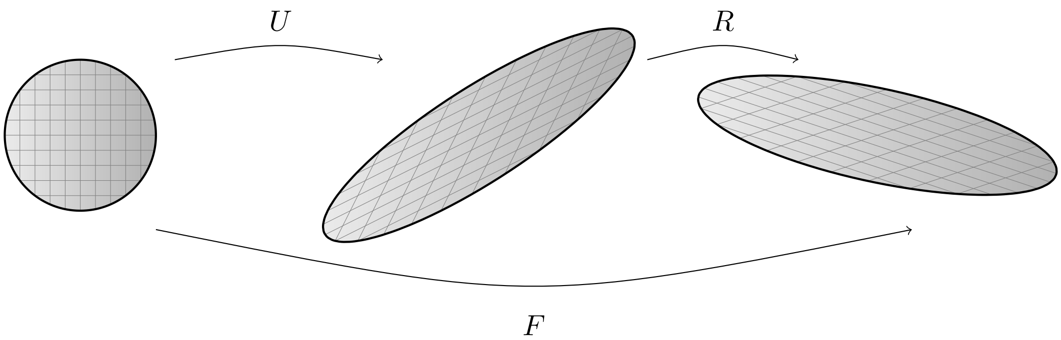

Figure 5: Visualization of the left polar decomposition $F = R \cdot U$.

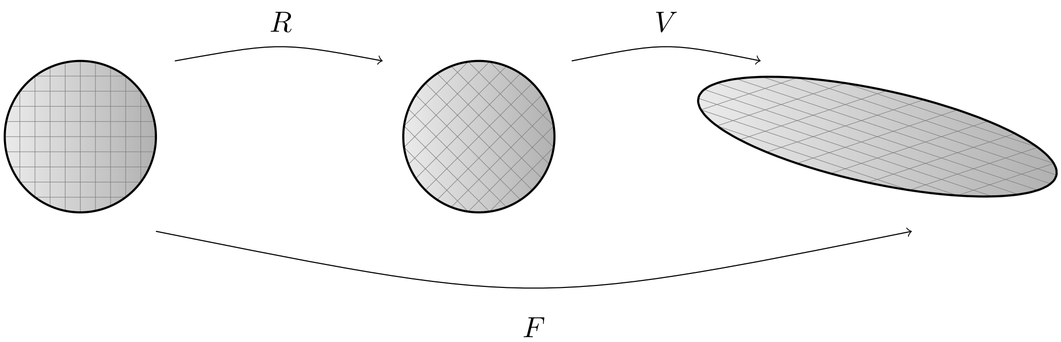

Figure 6: Visualization of the right polar decomposition $F = V \cdot R$.

[3] *Isotropic* means that the material does not have any preferred direction, i.e. applying a rotation $Q \in \mathrm{SO}(3)$ before the actual deformation $X = \nabla\varphi$ should not change the energy, $\forall\, Q \in \mathrm{SO}(3)\colon W(XQ) = W(X)$. Together with *frame-indifference* we get $\forall\, Q \in \mathrm{SO}(3) : W(QXQ^T) = W(X)$, see Münch and Neff in [13].

Theorem 2.2.2 (Existence and uniqueness)**.** *For each $F \in \mathrm{GL}^+(3)$ there exists an uniquely determined left and an uniquely determined right polar decomposition. Furthermore, the rotation $R \in \mathrm{SO}(3)$ in both decompositions is equal.*

Proof. We define $B \coloneqq FF^T$ and $C \coloneqq F^TF$. Let $v \in \mathbb{R}^3 \setminus \{0_{\mathbb{R}^3}\}$. Then[4]

$$\langle v, Cv \rangle = \left\langle v, F^TFv \right\rangle = \langle Fv, Fv \rangle = \|Fv\|^2 > 0$$

and

$$\langle v, Bv \rangle = \left\langle v, FF^Tv \right\rangle = \left\langle F^Tv, F^Tv \right\rangle = \|F^Tv\|^2 > 0.$$

In addition $C^T = \left(F^TF\right)^T = F^TF = C$, $B^T = \left(FF^T\right)^T = FF^T = B$, thus $B, C \in \mathrm{Sym}^+(3)$.
Now let $U \coloneqq \sqrt{C}$ and $V \coloneqq \sqrt{B}$ with $U, V \in \mathrm{Sym}^+(3)$. For $R \coloneqq FU^{-1}$,
$\det(R) = \det(F) \cdot \det(U^{-1}) > 0$ and $R^TR = (FU^{-1})^T(FU^{-1}) = U^{-T}F^TFU^{-1} = U^{-1}CU^{-1} = U^{-1}U^2U^{-1} = \mathbb{1}$. Hence, $R \in \mathrm{SO}(3)$.
To show that R is identical in both decompositions we compute $(FUF^{-1})^2 = (FUF^{-1})(FUF^{-1}) = FU^2F^{-1} = FF^TFF^{-1} = FF^T = V^2$. Since, $U^2 = F^TF \iff F^TFU = UF^TF$ and $U \in \mathrm{Sym}^+(3) \implies U^{-1} \in \mathrm{Sym}^+(3)$, we get $FUF^{-1} = F^{-T}UF^T = (FUF^{-1})^T$ and

$$\left\langle FUF^{-1}v, v \right\rangle = \left\langle UF^{-1}v, F^Tv \right\rangle = \left\langle U^{-1}F^Tv, F^Tv \right\rangle \overset{F^Tv =: w}{=} \left\langle U^{-1}w, w \right\rangle \geq 0.$$

So, $FUF^{-1} \in \mathrm{Sym}^+(3)$. Thus $V = FUF^{-1} \iff V^{-1} = FU^{-1}F^{-1} \iff FU^{-1} = V^{-1}F$.

Finally, the polar decomposition is uniquely determined, because for $\widehat{R} \in \mathrm{SO}(3)$ and for $\widehat{U} \in \mathrm{Sym}^+(3)$ with $F = \widehat{R}\widehat{U}$, we find $U^2 = F^TF = \widehat{U}^T\widehat{R}^T\widehat{R}\widehat{U} = \widehat{U}^2$ and thus, $U = \widehat{U}$, since $U = \sqrt{C} \in \mathrm{Sym}^+(3)$ is uniquely determined by the square root. In addition, $R = \widehat{R}$, because $\widehat{R}$ is uniquely determined by $\widehat{U}$. With the same arguments we get that V is uniquely defined. ■

Due to the previous theorem, we can split our deformation gradient $\nabla\varphi \in \mathrm{GL}^+(3)$ uniquely into

$$\nabla\varphi \coloneqq F = VR = RU \quad \text{with } V, U \in \mathrm{Sym}^+(3) \text{ and } R \in \mathrm{SO}(3). \tag{2.2.18}$$

Moreover, as stated in the proof of Theorem 2.2.2, we can diagonalise U and V and due to the equality $V = RUR^T$ they have the same eigenvalues $s_1^+, s_2^+, s_3^+ \in \mathbb{R}_+$. These eigenvalues are called the *singular values* of the deformation gradient F.

2.2.2 Invariants

With the help of $C \coloneqq F^TF = U^2$, called the *right Cauchy-Green-tensor*, and $B \coloneqq FF^T = V^2$, called the *left Cauchy-Green-tensor*, as well as certain invariants, we can simplify our isotropic energy function $\widehat{W}$.
"Invariant" in this case means invariant under orthogonal operations, i.e. a function

$$I\colon \mathbb{R}^{3\times3} \to \mathbb{R} \quad \text{with } B \mapsto I(B) \text{ and } B \in \mathbb{R}^{3\times3} \tag{2.2.19}$$

[4]The inner product in $\mathbb{R}^{3\times3}$ is called *Frobenius scalar product*, defined by $\langle\cdot,\cdot\rangle : \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \to \mathbb{R}$ with $\langle X, Y \rangle \coloneqq \sum_{i=1}^{3}\sum_{j=1}^{3} x_{ij}y_{ij}$. It induces the *Frobenius norm* $\|\cdot\| : \mathbb{R}^{3\times3} \to \mathbb{R}$ with $\|X\| \coloneqq \sqrt{\langle X, X \rangle}$. For more information, see [21].

is *invariant* if $I(QBQ^T) = I(B)$ for all $Q \in \mathrm{O}(3)$. In isotropic nonlinear elasticity, we define three different matrix invariants I_1, I_2 and I_3 by $I_i \colon \mathbb{R}^{3\times 3} \to \mathbb{R}$

$$I_1(B) = \operatorname{tr}(B), \tag{2.2.20}$$

$$I_2(B) = \frac{1}{2}\left[(\operatorname{tr}(B))^2 - \operatorname{tr}\left(B^2\right)\right] = \operatorname{tr}(\operatorname{Cof}(B)), \tag{2.2.21}$$

$$I_3(B) = \det(B). \tag{2.2.22}$$

For $B \in \mathrm{Sym}^+(3)$, there exists $Q \in \mathrm{O}(3)$ with $B = Q^T \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) Q$, where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}_+$ are the eigenvalues of B. Thus

$$I_1(B) = \operatorname{tr}(B) = \operatorname{tr}\left(Q^T \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) Q\right) = \lambda_1 + \lambda_2 + \lambda_3, \tag{2.2.23}$$

$$I_2(B) = \operatorname{tr}(\operatorname{Cof}(B)) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \tag{2.2.24}$$

$$I_3(B) = \det(B) = \lambda_1\lambda_2\lambda_3. \tag{2.2.25}$$

In terms of the deformation gradient F,

$$I_1(B) = \operatorname{tr}(B) = \langle B, \mathbb{1}\rangle = \left\langle FF^T, \mathbb{1}\right\rangle = \langle F, F\rangle = \|F\|^2, \tag{2.2.26}$$

$$\begin{aligned} I_2(B) &= \frac{1}{2}\left[(\operatorname{tr}(B))^2 - \operatorname{tr}\left(B^2\right)\right] = \operatorname{tr}(\operatorname{Cof}(B)) = \operatorname{tr}\left(\det(B)\cdot B^{-T}\right) \\ &= \det\left(FF^T\right)\cdot \operatorname{tr}\left(B^{-T}\right) = \det\left(FF^T\right)\left\langle \left(FF^T\right)^{-T}, \mathbb{1}\right\rangle \\ &= \det\left(FF^T\right)\left\langle F^{-T}\left(F^{-T}\right)^T, \mathbb{1}\right\rangle = \left\langle \det(F)\det(F)F^{-T}, F^{-T}\right\rangle \\ &= \left\langle \det(F)F^{-T}, \det(F)F^{-T}\right\rangle = \langle \operatorname{Cof}(F), \operatorname{Cof}(F)\rangle \\ &= \|\operatorname{Cof}(F)\|^2, \end{aligned} \tag{2.2.27}$$

$$I_3(B) = \det(B) = \det(F)\cdot\det\left(F^T\right) = \det(F)^2. \tag{2.2.28}$$

Note that I_1 , I_2 , $I_3 > 0$.
Due to Marsden and Hughes [12], isotropic energy functions can be represented in dependence of the invariants I_1, I_2, I_3,[5] so we can define a new energy function

$$\widetilde{W} : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \quad \text{with} \quad \widetilde{W}(I_1, I_2, I_3) := \widehat{W}(F). \tag{2.2.29}$$

2.2.3 Matrix derivation

In Section 4, we will need some matrix derivatives to linearize important equations in nonlinear elasticity. For differentiation in $\mathbb{R}$, we use the Taylor series to find a derivative of a function $f \colon \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x)$. More precisely,

$$f(x+h) = f(x) + f'(x)\cdot h + O\left(|h|^2\right), \tag{2.2.30}$$

where $h \in \mathbb{R}$ indicates an arbitrary direction. Nearly the same formula applies for differentiation in $\mathbb{R}^{3\times 3}$. Here, we have to distinguish between scalar valued functions $T \colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ and tensor functions $\widehat{T} \colon \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$. In these respective cases, we get

$$T(X+H) = T(X) + \langle DT[X], H\rangle + O\left(\|H\|^2\right) \tag{2.2.31}$$

and

$$\widehat{T}(X+H) = \widehat{T}(X) + D\widehat{T}[X].H + O\left(\|H\|^2\right) \tag{2.2.32}$$

[5]Note that if we assume F_1, $F_2 \in \mathrm{GL}^+(3)$ with $I_1(F_1) = I_1(F_2)$, $I_2(F_1) = I_2(F_2)$, $I_3(F_1) = I_3(F_2)$, then the corresponding B_1 and B_2 are similar. Hence, $V_1 = \sqrt{B_1}$ and $V_2 = \sqrt{B_2}$ are similar as well. Thus $\widehat{W}(F_1) = \widehat{W}(F_2)$, since we assume $\widehat{W}$ to be objective and isotropic.

for an arbitrary direction $H \in \mathbb{R}^{3\times 3}$, where $\langle DT[X], H\rangle \in \mathbb{R}$, $D\widehat{T}[X].H \in \mathbb{R}^{3\times 3}$ and $O\left(|h|^2\right)$, $O\left(\|H\|^2\right)$ represent terms of higher order.
For further considerations we need the expansion of the *Cofactor mapping* $\mathrm{Cof} : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ in the direction of $H \in \mathbb{R}^{3\times 3}$. First we have to compute $\det(X+H) \in \mathbb{R}$ and $(X+H)^{-T} \in \mathbb{R}^{3\times 3}$. We calculate

$$\begin{aligned}
\det(\mathbb{1}+H) &= \det\begin{pmatrix} 1+H_{11} & H_{12} & H_{13} \\ H_{21} & 1+H_{22} & H_{23} \\ H31 & H_{32} & 1+H_{33} \end{pmatrix} \\
&= (1+H_{11})(1+H_{22})(1+H_{33}) + H_{12}H_{23}H_{31} + H_{13}H_{21}H_{32} - H_{13}(1+H_{22})H_{31} \\
&\quad - H_{23}H_{32}(1+H_{11}) - (1+H_{33})H_{21}H_{12} \\
&= (1+H_{22}+H_{11}+H_{11}H_{22})(1+H_{33}) + H_{12}H_{23}H_{31} + H_{13}H_{21}H_{32} \\
&\quad - H_{13}H_{31} - H_{13}H_{31}H_{22} - H_{23}H_{32} - H_{23}H_{32}H_{11} - H_{21}H_{12} - H_{33}H_{21}H_{12} \\
&= 1 + H_{33} + H_{22} + H_{22}H_{33} + H_{11} + H_{11}H_{33} + H_{11}H_{22} + H_{11}H_{22}H_{33} \\
&\quad + H_{12}H_{23}H_{31} + H_{13}H_{21}H_{32} - H_{13}H_{31} - H_{13}H_{31}H_{22} - H_{23}H_{32} \\
&\quad - H_{23}H_{32}H_{11} - H_{21}H_{12} - H_{33}H_{21}H_{12} \\
&= 1 + (H_{11}+H_{22}+H_{33}) + (H_{11}H_{22} + H_{11}H_{33} + H_{22}H_{33} - H_{13}H_{31} - H_{23}H_{32} - H_{12}H_{21}) \\
&\quad + (H_{11}H_{22}H_{33} + H_{12}H_{23}H_{31} + H_{13}H_{21}H_{32} - H_{22}H_{31}H_{13} - H_{33}H_{21}H_{12} - H_{33}H_{32}H_{23}) \\
&= 1 + \mathrm{tr}(H) + \mathrm{tr}(\mathrm{Cof}(H)) + \det(H) \\
&= \det(\mathbb{1}) + \langle \mathbb{1}, H\rangle + O(\|H\|^2). && (2.2.33)
\end{aligned}$$

Thus

$$\begin{aligned}
\det(X+H) &= \det\left(X\left(\mathbb{1}+X^{-1}H\right)\right) = \det(X)\cdot\det\left(\mathbb{1}+X^{-1}H\right) \\
&= \det(X)\cdot\left(\det(\mathbb{1}) + \left\langle \mathbb{1}, X^{-1}H\right\rangle + O\left(\|H\|^2\right)\right) \\
&= \det(X) + \det(X)\left\langle X^{-T}, H\right\rangle + O\left(\|H\|^2\right) \\
&= \det(X) + \langle \mathrm{Cof}(X), H\rangle + O\left(\|H\|^2\right). && (2.2.34)
\end{aligned}$$

Now, we look at the expansion of $(X+H)^{-T}$, using the *Neumann series*:

$$(\mathbb{1}-X)^{-1} = \mathbb{1} + X + X^2 + X^3 + \ldots \quad \text{for} \quad \|X\| < 1. \tag{2.2.35}$$

We compute

$$\begin{aligned}
(X+H)^{-1} &= \left(X\left(\mathbb{1}+X^{-1}H\right)\right)^{-1} = \left(\mathbb{1} - \left(-X^{-1}H\right)\right)^{-1}\cdot X^{-1} \\
&= \left(\mathbb{1} + \left(-X^{-1}H\right) + O\left(\|H\|^2\right)\right)\cdot X^{-1} \\
&= X^{-1} - X^{-1}HX^{-1} + O\left(\|H\|^2\right) && (2.2.36)
\end{aligned}$$

and

$$(X+H)^T = X^T + H^T. \tag{2.2.37}$$

Thus

$$(X+H)^{-T} = \left((X+H)^T\right)^{-1} = \left(X^T + H^T\right)^{-1} = X^{-T} - X^{-T}H^TX^{-T} + O\left(\|H\|^2\right). \tag{2.2.38}$$

Alltogether, we get

$$\begin{aligned}
\mathrm{Cof}(X+H) &= \det(X+H)\cdot(X+H)^{-T} \\
&= \left(\det(X) + \langle\mathrm{Cof}(X), H\rangle + O\left(\|H\|^2\right)\right)\cdot\left(X^{-T} - X^{-T}H^TX^{-T} + O\left(\|H\|^2\right)\right) \\
&= \det(X)\cdot X^{-T} - \det(X)\cdot X^{-T}H^TX^{-T} + \langle\mathrm{Cof}(X), H\rangle\cdot X^{-T} + O\left(\|H\|^2\right) \\
&= \mathrm{Cof}(X) - \det(X)\cdot\left(X^{-1}HX^{-1}\right)^T + \det(X)\left\langle X^{-T}, H\right\rangle X^{-T} + O\left(\|H\|^2\right) \\
&= \mathrm{Cof}(X) - \det(X)\cdot\left(X^{-1}HX^{-1}\right)^T + \mathrm{tr}\left(X^{-1}H\right)\cdot\mathrm{Cof}(X) + O\left(\|H\|^2\right) && (2.2.39)
\end{aligned}$$

for the Cofactor mapping.

2.2.4 Stress tensors

In a solid body there are two types of external forces, the body force and the surface force. The surface force acts upon every element of the body's surface and the body force acts upon every volume element of the body. Furthermore, there are also internal forces induced by mutual interactions of inner parts of the body. The internal forces are the forces which lead us to define stresses in relation to deformed bodies [11]. To understand how a stress tensor is defined, we need to consider the concept of stress vectors.

Physically a *stress vector* t in a point P of an intersection is defined by

$$t = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \frac{\partial F}{\partial A}, \tag{2.2.40}$$

with the force ΔF which acts upon the area ΔA, see Fig. 7.

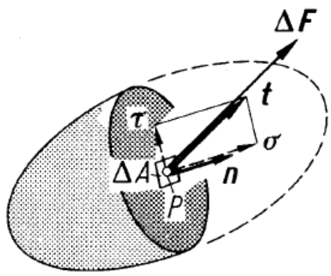

Figure 7: Visualization of a stress vector t in a point P of an area ΔA. Source: [6, p. 71]

The stress vector t depends on the orientation of the intersection through P, i.e. $t \coloneqq t(\vec{n})$. Based on mechanical reasoning it can be shown that for a body in equilibrium[6] the mapping $\vec{n} \mapsto t(\vec{n})$ must be linear, i. e. $t(\vec{n}) = \sigma.\vec{n}$ for some $\sigma \in \mathbb{R}^{3\times 3}$.

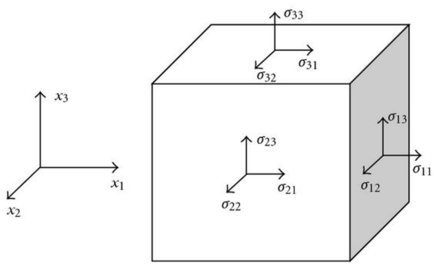

Figure 8: Visualization of the stress tensor σ.

Furthermore, the relations $\sigma_{yz} \coloneqq \tau_{yz} = \tau_{zy}$, $\sigma_{xz} \coloneqq \tau_{xz} = \tau_{zx}$, $\sigma_{xy} \coloneqq \tau_{xy} = \tau_{yx}$ must hold. Thus, we get

$$\sigma \coloneqq \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix} \in \mathrm{Sym}(3) \tag{2.2.41}$$

with six independent components [6].

The second-order tensor[7] σ, called the *Cauchy stress tensor*[8], represents the stress in the current

[6] A body is in a static equilibrium if the *fundamental stress principle of Euler and Cauchy* is satisfied, see [1].

[7] A second-order tensor is expressed by a square matrix.

[8] The Cauchy stress tensor is also called the *true* stress, because it describes the stress which is measured in the actual deformed state.

configuration per deformed surface and can be defined even without an elastic energy W. Note that in the context of hyperelasticity, the existence and properties of the Cauchy stress tensor follow from frame-indifference of the energy W and the Piola transform (2.2.43).
Other important stress tensors are the *first Piola-Kirchhoff stress tensor* and the *second Piola-Kirchhoff stress tensor*. In the hyperelastic framework, the former is defined by

$$S_1(F) := D_F\widehat{W}(F), \tag{2.2.42}$$

with the elastic energy function $\widehat{W}$ defined in (2.2.11), where $D_F\widehat{W}(F)$ is the derivate of $\widehat{W}$ with respect to F. It represents the stress per current surface.
Furthermore, there is a relation between the Cauchy stress tensor and the first Piola-Kirchhoff stress tensor which is expressed by the equation

$$\sigma(F) = S_1(F) \cdot \operatorname{Cof}(F)^{-1}. \tag{2.2.43}$$

In general, this relation is also called *Piola transform* which expresses a particular transformation between tensors defined in the reference configuration Ω and tensors defined in the deformed configuration $\varphi(\Omega) = \widehat{\Omega}$. One important property of this transformation will be discussed in Section 2.3.
The second Piola-Kirchhoff stress tensor is defined by

$$S_2(F) := F^{-1}S_1(F). \tag{2.2.44}$$

While we consider many other stress tensors, as well, they are not important for the present work, so if the reader is interested in them they can find more information in [1, 20].

2.2.5 Linear elasticity

We distinguish between linear and nonlinear elasticity. The latter was discussed in the previous sections. In linear elasticity, we deal with infinitesimal strains, i.e. "small" deformations. It is used in the field of continuum mechanics. "Linear" in this case means that we consider the relation between stress and strain to be linear. This relation is expressed by the isotropic *Hooke's law* [9]

$$\sigma(\varepsilon) = 2\mu\,\varepsilon + \lambda\,\operatorname{tr}(\varepsilon)\,\mathbb{1}, \tag{2.2.45}$$

where μ and λ are the *Lamé (elastic) moduli* and $\varepsilon := \operatorname{sym}(F - \mathbb{1}) = \operatorname{sym}(\nabla u) = \frac{1}{2}[\nabla u + (\nabla u)^T]$ is the *infinitesimal strain tensor* with the displacement field ∇u. Together with the equation $\varepsilon = \varepsilon - \frac{1}{3}\operatorname{tr}(\varepsilon)\,\mathbb{1} + \frac{1}{3}\operatorname{tr}(\varepsilon)\,\mathbb{1} = \operatorname{dev}(\varepsilon) + \frac{1}{3}\operatorname{tr}(\varepsilon)\,\mathbb{1}$, Hooke's law generates the equivalent stress strain law

$$\sigma(\varepsilon) = 2\mu\operatorname{dev}(\varepsilon) + \kappa\operatorname{tr}(\varepsilon)\,\mathbb{1} \tag{2.2.46}$$

for isotropic linear elastic materials, where $\kappa = \frac{2\mu+3\lambda}{3}$ is called the *bulk modulus*.

2.3 Fundamental principles of functional analysis and the calculus of variations

In addition to the previous sections, we need some fundamental principles from the field of functional analysis. We presuppose terms like metric, metric space, norm and normed space[9]. This leads us to the following definitions.

Definition 2.3.1. A metric space X in which every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ converges to a $x \in X$ is called *complete*. A complete and normed space is called *Banach space*.

[9]For repetition see [8, 22].

Definition 2.3.2. A Banach space H is called a *Hilbert space*, if there exists an inner product[10] $\langle\cdot,\cdot\rangle$ which generates the norm on H with $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$.

Other important spaces are the Lebesgue spaces.

Definition 2.3.3. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\Omega \subset \mathbb{R}^p$ measurable, $0 < p < \infty$, λ^n the n-dimensional Lebesgue measure, $\mathcal{L}^p(\Omega) := \{f : \Omega \to \mathbb{K}, \text{f measurable}, \int_\Omega |f|^p \, d\lambda^n < \infty\}$ and $\mathcal{N} := \{f \in \mathcal{L}^p \,|\, \|f\|_{\mathcal{L}^p} = 0\}$, we call the quotient space $L^p := \mathcal{L}^p/\mathcal{N}$ *p-Lebesgue space* with the norm

$$\|f\|_{L^p} = \|f\|_{\mathcal{L}^p} := \left(\int_\Omega |f|^p \, \mathrm{d}\lambda^n\right)^{\frac{1}{p}}, \tag{2.3.1}$$

for more detailed considerations see [22].

To define certain inequalities like the Poincaré inequality and the Korn inequality[11], we need to define the so-called Sobolev spaces as well [19].

Definition 2.3.4. Let $\Omega \subset \mathbb{R}^n$ be an open subset, $k \in \mathbb{N}$ and $p \in [1, \infty]$. Then we call

$$W^{k,p}(\Omega) = W^{k,p}(\Omega, \mathbb{R}) := \{u \in L^p(\Omega, \mathbb{R}) \,|\, D^\alpha u \in L^p(\Omega) \quad \forall\, \alpha \in \mathbb{N}^n \quad \text{with} \quad |\alpha| \leq k\} \tag{2.3.2}$$

Sobolev space with the norm

$$\|u\|_{W^{k,p}} := \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p}^p\right)^{\frac{1}{p}}, \tag{2.3.3}$$

where $D^\alpha u(x) := \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_n} x_n} u(x)$ is the weak derivative and $|\alpha| := \sum_{i=1}^n \alpha_i$.

Remark 2.3.5. For $p = 2$, we write

$$H^k(\Omega) = W^{k,2}(\Omega), \quad (k = 0, 1, 2, \ldots),$$

to emphasize that $W^{k,2}(\Omega)$ defines a Hilbert space, see [4].

Definition 2.3.6. Let $U \subset \mathbb{R}^n$ be an open subset. The space

$$C_0^\infty(U) := \{v \in C^\infty(\mathbb{R}^n) : \operatorname{supp}(v) \subset U \text{ compact}\} \tag{2.3.4}$$

is called the *space of test functions*, where the support is defined by

$$\operatorname{supp}(v) := \overline{\{x \in U : v(x) \neq 0\}}. \tag{2.3.5}$$

Now we are able to define Sobolev spaces with zero boundary conditions.

Definition 2.3.7. Let $1 \leq p < \infty$. Then

$$W_0^{k,p}(\Omega) = \{u \in W^{k,p}(\Omega) \,|\, \exists\, (u_j)_{j \in \mathbb{N}} \subset C_0^\infty(\Omega), \|u - u_j\|_{W^{k,p}} \to 0\} \subset W^{k,p}(\Omega) \tag{2.3.6}$$

is called *Sobolev space with zero boundary conditions*.

[10] A mapping $\langle\,.\,,\,.\,\rangle : H \times H \to \mathbb{R}$ is called an *inner product* if it is linear in both of its arguments and if the following conditions apply:

i) $\langle u, v\rangle = \langle v, u\rangle \quad \forall\, u, v \in H$,

ii) $\langle u, u\rangle \geq 0 \quad \forall\, u \in H$,

iii) $\langle u, u\rangle = 0 \iff u = 0$.

[11] See Section 2.3.3.

Remark 2.3.8. For $p = 2$, we write

$$H_0^k(\Omega) = W_0^{k,2}(\Omega),$$

as well.

For the space $W_0^{1,p}(\Omega)$, we can specify the Poincaré inequality, see [19], which we state here without proof.

Theorem 2.3.9 (Poincaré inequality)**.** *Let $1 \leq p < \infty$, $\Omega \subset \mathbb{R}^n$ be an open bounded subset and $V \subset W_0^{1,p}(\Omega)$. Then there exists a constant $c > 0$ which depends only on p and Ω but not on u such that*

$$\|u\|_{W^{1,p}(\Omega)} \leq c \cdot \|\nabla u\|_{L^p(\Omega)} \quad \forall u \in V. \tag{2.3.7}$$

We rewrite this inequality so that the L^p-norm of a Sobolev function with zero boundary conditions is controlled by the L^p-norm of its gradient:

$$\begin{aligned}
\|u\|_{W^{1,p}} \leq c \cdot \|\nabla u\|_{L^p} \iff& \left(\sum_{|\alpha|\leq 1} \|D^\alpha u\|_{L^p}^p\right)^{\frac{1}{p}} \leq c \cdot \left(\int_\Omega \|\nabla u\|^p \,\mathrm{d}V\right)^{\frac{1}{p}} \\
\iff& \sum_{|\alpha|\leq 1} \int_\Omega \|D^\alpha u\|^p \,\mathrm{d}V \leq c^p \cdot \int_\Omega \|\nabla u\|^p \,\mathrm{d}V \\
\iff& \int_\Omega \|u\|^p \,\mathrm{d}V + \int_\Omega \|\nabla u\|^p \,\mathrm{d}V \leq c^p \cdot \int_\Omega \|\nabla u\|^p \,\mathrm{d}V \\
\iff& \int_\Omega \|u\|^p \,\mathrm{d}V \leq (c^p - 1) \cdot \int_\Omega \|\nabla u\|^p \,\mathrm{d}V \\
\iff& \|u\|_{L^p}^p \leq (c^p - 1) \cdot \|\nabla u\|_{L^p}^p.
\end{aligned} \tag{2.3.8}$$

For the computations for example in (3.1.7) and (3.1.9), we will also require partial integration in the multi-dimensional case. It follows from Gauß's theorem, which is a special case of Stokes' theorem[12] and describes the connection between the divergence of a vector field in an arbitrary volume and the flow of this field through the surface of the volume. Furthermore, its importance is attributed to energy conservation, shown in Fig. 9.

Theorem 2.3.10 (Gauß's theorem)**.** *Let $V \subset \mathbb{R}^n$ be a subset which is compact and has a piecewise smooth boundary. In addition, let the vector field F be continuously differentiable on an open set U with $V \subset U$. Then the following equation holds:*

$$\int_V \operatorname{div} F \,\mathrm{d}V = \int_{\partial V} \langle F, \vec{n} \rangle \,\mathrm{d}S. \tag{2.3.9}$$

Proof. See [4]. ■

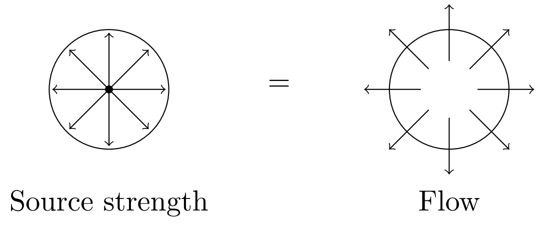

Figure 9: Principle of energy conservation.

[12] See [7].

Thus, for $\operatorname{div}\left(A^T v\right) = \langle A, \nabla v\rangle + \langle v, \operatorname{Div} A\rangle$ with $A \in \mathbb{R}^{3\times 3}, v \in \mathbb{R}^3$, we find

$$\begin{aligned}\int_\Omega \langle \operatorname{Div} A, v\rangle \,\mathrm{d}x &= -\int_\Omega \langle A, \nabla v\rangle \,\mathrm{d}x + \int_\Omega \operatorname{div}\left(A^T v\right)\,\mathrm{d}x \\ &= -\int_\Omega \langle A, \nabla v\rangle \,\mathrm{d}x + \int_{\partial\Omega} \left\langle A^T v, \vec{n}\right\rangle \mathrm{d}x \\ &= -\int_\Omega \langle A, \nabla v\rangle \,\mathrm{d}x + \int_{\partial\Omega} \langle v, A\vec{n}\rangle \,\mathrm{d}x. \end{aligned} \tag{2.3.10}$$

With some application of Green's identities[13], equation (2.3.9) also applies for tensor fields T and is given by

$$\boxed{\int_\Omega \operatorname{Div} T \,\mathrm{d}V = \int_{\partial\Omega} T\cdot\vec{n}\,\mathrm{d}S}\,. \tag{2.3.11}$$

2.3.1 Fundamental lemma of the calculus of variations and the Piola transform

Moreover, we need one important theorem of calculus of variations for the Euler-Lagrange equations in Section 3.1.

Lemma 2.3.11 (Fundamental lemma of the calculus of variations)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open subset and $g \in L^{1,loc}(\Omega)$ with*

$$\int_\Omega g(x)\,v(x)\,\mathrm{d}x = 0 \tag{2.3.12}$$

for all $v \in C_0^\infty(\Omega)$. Then $g(x) \equiv 0$ almost everywhere in Ω.

Proof. First we consider the case $g \in C(\Omega)$. We assume the above prerequisite $\int_\Omega g(x)v(x)\,\mathrm{d}x = 0$ for all $v \in C_0^\infty(\Omega)$. Now let $\eta \in C_0^\infty(\mathbb{R}^n)$, $\eta \geq 0$ with $\operatorname{supp}(\eta) = \overline{\{x : \eta(x) \neq 0\}} \subset B_1(0)$, $\int_{\mathbb{R}^n} \eta(x)\,\mathrm{d}x = 1$ and $\eta(x) = \eta(-x)$. We choose the standard mollifier [4]

$$\eta(x) := \begin{cases} c\cdot\exp\left(\frac{1}{|x|^2-1}\right) & : \quad |x| < 1, \\ 0 & : \quad |x| \geq 1. \end{cases} \tag{2.3.13}$$

For $\varepsilon > 0$, let $\eta_\varepsilon(x) := \varepsilon^{-n}\eta\left(\frac{x}{\varepsilon}\right)$. Then $\operatorname{supp}(\eta_\varepsilon) \subset B_\varepsilon(0)$. If $0 < \varepsilon < \operatorname{dist}(x, \partial\,\Omega)$, then

$$(\eta_\varepsilon * g)(x) = \int_\Omega \eta_\varepsilon(x-y)g(y)\,\mathrm{d}y = 0$$

applies for the convolution[14] of g. Hence,

$$\begin{aligned}|g(x)| = |g(x) - (\eta_\varepsilon * g)(x)| &= |\int_\Omega \eta_\varepsilon(x-y)\,[g(x)-g(y)]\,\mathrm{d}y| \leq \int_\Omega |\eta_\varepsilon(x-y)||g(x)-g(y)|\,\mathrm{d}y \\ &\leq 1\cdot \sup_{y\in\overline{B_\varepsilon}} |g(x)-g(y)| \underset{\varepsilon\to 0}{\to} 0,\end{aligned}$$

[13] For an open and bounded subset $\Omega \subset \mathbb{R}^n$, let $u, v \in C^2(\overline{\Omega})$. Then

i) $\int_\Omega \left(\langle u, \Delta v\rangle + \langle \nabla u, \nabla v\rangle\right)\mathrm{d}V = \int_{\partial\,\Omega} \langle u, \frac{\partial\,v}{\partial\,n}\rangle\,\mathrm{d}S$,

ii) $\int_\Omega \left(\langle u, \Delta v\rangle - \langle v, \Delta u\rangle\right)\mathrm{d}V = \int_{\partial\,\Omega} \left(\langle u, \frac{\partial\,v}{\partial\,n}\rangle - \langle v, \frac{\partial\,u}{\partial\,n}\rangle\right)\mathrm{d}S$,

with the Laplace operator Δ and $\frac{\partial\,v}{\partial\,n} := \nabla v \cdot \vec{n}$.

[14] If $f : \Omega \to \mathbb{R}$ is locally integrable, we define the *convolution* of f by

$$(\eta_\varepsilon * f)(x) := \int_\Omega \eta_\varepsilon(y)f(y)\,\mathrm{d}y = \int_{B_\varepsilon(0)} \eta_\varepsilon(y)f(x-y)\,\mathrm{d}y$$

for $x \in \{x \in \Omega |\operatorname{dist}(x, \partial\Omega) > \varepsilon\}$.

because g is continuous in Ω.
Now consider the case $g \in L^{1,\,\mathrm{loc}}(\Omega)$. Again, assume condition (2.3.12). Let $v \in C_0^\infty(\Omega)$, $v \geq 0$ and $0 < \varepsilon < \mathrm{dist}(\mathrm{supp}(v), \partial\,\Omega)$. Then with the use of Fubini[15], and since $(\eta_\varepsilon * v) \in C_0^\infty(\Omega)$, the following calculation applies:

$$\begin{aligned}\int_\Omega (\eta_\varepsilon * g)\,(x)\,v(x)\,\mathrm{d}x &= \int_\Omega \int_\Omega \eta_\varepsilon(x-y)\,g(y)\,\mathrm{d}y\,v(x)\,\mathrm{d}x = \int_\Omega g(y) \int_\Omega \eta_\varepsilon(x-y)\,v(x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_\Omega g(y)\ (\eta_\varepsilon * v)\,(y)\,\mathrm{d}y = 0.\end{aligned}$$

Therefore, together with the first case and the properties of the convolution[16], we obtain $\eta_\varepsilon * g = 0$. Thus, we get $g(x) \equiv 0$ almost everywhere in Ω.[4] ■

Now we can introduce the following theorem which expresses one important property of the Piola transform mentioned in Section 2.2.4.
We define $x \in \Omega \subset \mathbb{R}^n$ as an element in the reference configuration and $\xi \coloneqq \varphi(x) \in \widehat{\Omega}$ as an element in the deformed configuration, as shown in Fig. 10.

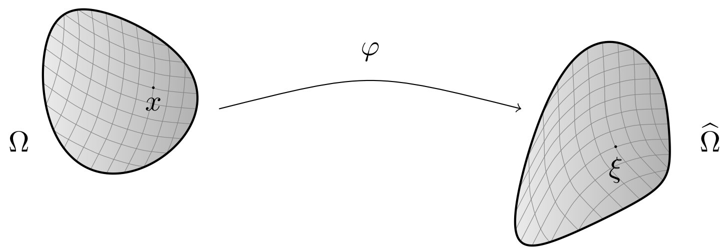

Figure 10: Piola transformation.

[15] If $f : \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m \to [0,\infty]$ is measurable, then $f_y \coloneqq f(\cdot, y) : \mathbb{R}^n \to [0,\infty]$ and $f_x \coloneqq f(x,\cdot) : \mathbb{R}^m \to [0,\infty]$ are measurable. Furthermore the following equation applies

$$\int_{\mathbb{R}^{n+m}} f\,\mathrm{d}\lambda_{n+m} = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} f(x,y)\,\mathrm{d}\lambda_n(x) \right) \mathrm{d}\lambda_m(y) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x,y)\,\mathrm{d}\lambda_m(y) \right) \mathrm{d}\lambda_n(x).$$

[16] The convolution $\eta_\varepsilon * f$ of f has the following properties [4]:

i) $(\eta_\varepsilon * f) \in C^\infty(\Omega_\varepsilon)$ with $\Omega_\varepsilon \coloneqq \{x \in \Omega |\ \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$,

ii) $(\eta_\varepsilon * f) \to f$ almost everywhere as $\varepsilon \to 0$,

iii) If $f \in C(\Omega)$, then $(\eta_\varepsilon * f) \to f$ uniformly on compact subsets of Ω,

iv) If $1 \leq p < \infty$ and $f \in L^{p,loc}(\Omega)$, then $(\eta_\varepsilon * f) \to f$ in $L^{p,\,\mathrm{loc}}(\Omega)$.

Theorem 2.3.12 (Piola-Transform). *Let $\Omega \subset \mathbb{R}^n$ be bounded, open and connected. Assume that φ is a mapping which is defined on Ω with $\varphi : \Omega \to \widehat{\Omega}$ and $\varphi(x) =: \xi$. Moreover, let σ be a tensor field on the deformed configuration $\widehat{\Omega}$ and let S be a tensor field on the reference configuration Ω with the relation (Piola transformation)*

$$S(x) = \sigma(\xi) \cdot \mathrm{Cof}(\nabla_x \varphi(x)). \tag{2.3.14}$$

Then[17]

$$\mathrm{Div}_x(S(x)) = \det(\nabla_x \varphi(x)) \cdot \mathrm{Div}_\xi\, \sigma(\xi). \tag{2.3.15}$$

Proof. Together with equation (4.4.3), the Piola identity[18] and $\delta_{kl} = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases}$, we find

$$\begin{aligned}
[\mathrm{Div}_x\, S(x)]_i &= \sum_{j=1}^{n} \partial_{x_j}\, S(x)_{ij} = \sum_{j=1}^{n} \partial_{x_j} \left[\sigma(\xi) \cdot \mathrm{Cof}(\nabla_x \varphi(x))\right]_{ij} \\
&= \sum_{j,k=1}^{n} \partial_{x_j} \left(\sigma(\xi)_{ik}\right) \cdot \mathrm{Cof}(\nabla_x \varphi(x))_{kj} + \left(\sigma(\xi) \cdot \mathrm{Div}(\mathrm{Cof}(\nabla_x \varphi(x))\right)_i \\
&= \sum_{j,k,l=1}^{n} \partial_{\xi_l}\, \sigma_{ik}(\xi)\, \partial_{x_j \xi_l}\, \varphi(x) \cdot \det(\nabla_x \varphi(x)) \cdot \left[\nabla_x \varphi(x)^{-T}\right]_{kj} \qquad (2.3.16)\\
&= \sum_{k,l=1}^{n} \partial_{\xi_l}\, \sigma_{ik}(\xi) \cdot \det(\nabla_x \varphi(x)) \cdot \sum_{j=1}^{n} \left[\nabla_x \varphi(x)\right]_{lj} \cdot \left[\nabla_x \varphi(x)^{-T}\right]_{kj} \\
&= \sum_{k,l=1}^{n} \partial_{\xi_l}\, \sigma_{ik}(\xi) \cdot \det(\nabla_x \varphi(x)) \cdot \delta_{lk} = \det(\nabla_x \varphi(x)) \cdot \sum_{l=1}^{n} \partial_{\xi_l}\, \sigma_{il}(\xi) \\
&= \det(\nabla_x \varphi(x)) \cdot \left[\mathrm{Div}_\xi\, \sigma(\xi)\right]_i .
\end{aligned}$$

■

[17] Note that

$$\int_\Omega \langle \mathrm{Div}_x\, S(x), v(x) \rangle\, dx = - \int_\Omega \langle S(x), \nabla v(x) \rangle\, dx,$$

for all $v \in C_0^\infty(\Omega)$ and

$$\int_{\widehat{\Omega}} \langle \mathrm{Div}_\xi\, \sigma(\xi), \widehat{v}(\xi) \rangle\, d\xi = - \int_{\widehat{\Omega}} \langle \sigma(\xi), \nabla \widehat{v}(\xi) \rangle\, d\xi,$$

for all $\widehat{v} \in C_0^\infty(\widehat{\Omega})$.

[18] Let $\nabla\varphi = \begin{pmatrix} \varphi_{1,x_1} & \varphi_{1,x_2} & \varphi_{1,x_3} \\ \varphi_{2,x_1} & \varphi_{2,x_2} & \varphi_{2,x_3} \\ \varphi_{3,x_1} & \varphi_{3,x_2} & \varphi_{3,x_3} \end{pmatrix}$ and $\mathrm{Cof}(\nabla\varphi)_{i\,j} = \nabla\varphi_{i+1\,j+1} \cdot \nabla\varphi_{i+2\,j+2} - \nabla\varphi_{i+1\,j+2} \cdot \nabla\varphi_{i+2\,j+1}$ with all indices counted modulo 3. Then

$$\begin{aligned}
\left[\mathrm{Div}\,(\mathrm{Cof}(\nabla\varphi))\right]_i &= \mathrm{div}\,((\mathrm{Cof}(\nabla\varphi)_i)) = \sum_{j=1}^{3} \frac{\partial}{\partial\, x_j} \left(\mathrm{Cof}(\nabla\varphi)\right)_{i\,j} = \sum_{j=1}^{3} \frac{\partial}{\partial\, x_j} \left(\nabla\varphi_{i+1\,j+1} \cdot \nabla\varphi_{i+2\,j+2} - \nabla\varphi_{i+1\,j+2} \cdot \nabla\varphi_{i+2\,j+1}\right) \\
&= \sum_{j=1}^{3} \frac{\partial}{\partial\, x_j} \left(\varphi_{i+1,x_{j+1}} \cdot \varphi_{i+2,x_{j+2}} - \varphi_{i+1,x_{j+2}} \cdot \varphi_{i+2,x_{j+1}}\right) \\
&= \sum_{j=1}^{3} \left(\varphi_{i+1,x_{j+1},x_j} \cdot \varphi_{i+2,x_{j+2}} + \varphi_{i+1,x_{j+1}} \cdot \varphi_{i+2,x_{j+2},x_j} - \varphi_{i+1,X_{j+2},x_j} \cdot \varphi_{i+2,x_{j+1}} - \varphi_{i+1,x_{j+2}} \cdot \varphi_{i+2,x_{j+1},x_j}\right) \\
&= 0.
\end{aligned}$$

Thus, $\mathrm{Div}(\mathrm{Cof}(\nabla\varphi)) = 0$.

2.3.2 Theorem of Lax-Milgram

We are mainly interested in the existence and uniqueness of a solution of a given boundary value problem, especially of the corresponding variational problem. For this we need one of the most important theorems in linear functional analysis, the theorem of Lax-Milgram.

Theorem 2.3.13 (Theorem of Lax-Milgram)**.** *Let $\mathcal{L} : H \times H \to \mathbb{R}$ be a bilinear mapping on a real Hilbert space H with norm $\|.\|_H$ and inner product $\langle\, .\, ,\, .\, \rangle_H$. Additionally, let $f : H \to \mathbb{R}$ be a bounded linear functional on H.*
If there exist constants α, $\gamma > 0$ with

$$\begin{array}{llcll} i) & |\mathcal{L}(u,w)| & \leq & \gamma \cdot \|u\|_H \|w\|_H & \forall\, u, w \in H, \\ ii) & \mathcal{L}(u,u) & \geq & \alpha \cdot \|u\|_H^2 & \forall\, u \in H, \end{array} \tag{2.3.17}$$

then there exists a unique element $u \in H$ with

$$\mathcal{L}(u,w) = \langle f, w \rangle \qquad \forall\, w \in H. \tag{2.3.18}$$

Proof. See [4, 8]. ■

2.3.3 Korn's inequalities

Korn's inequalities play an important role in linear elasticity. They replace the Poincaré inequality if it is only possible to estimate the symmetric part of the gradient of u, which is a measure for the deformation of an elastic body. We will need these equations in Section 5 in order to be able to make certain ellipticity statements regarding linearizations.

Theorem 2.3.14 (Korn's inequalities for $p = 2$)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected subset with a smooth boundary $\partial\Omega$. Then there exist constants c_0, $c_1 \in \mathbb{R}$ such that*

i) $\|\nabla u\|^2_{L^2(\Omega)} \leq c_0 \|\mathrm{sym}(\nabla u)\|^2_{L^2(\Omega)} \quad \forall\, u \in H^1_0(\Omega)$,

ii) $\|u\|^2_{H^1(\Omega)} \leq c_1 \left(\|u\|_{L^2(\Omega)} + \|\mathrm{sym}(\nabla u)\|_{L^2(\Omega)}\right)^2 \quad \forall\, u \in H^1(\Omega)$.

Furthermore, it is possible to generalize Korn's first inequality, which is stated in *i)*. For example, the following generalization is applicable to elasto-plasticity at large deformations [15].

Theorem 2.3.15 (Generalization of Korn's first inequality)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded, connected domain with Lipschitz boundary $\partial\,\Omega$ and let $\widetilde{F}$, $\widetilde{F}^{-1} \in C^1(\overline{\Omega}, \mathbb{R}^{3\times 3})$ with $\det(\widetilde{F}) \geq c > 0$, $c \in \mathbb{R}^+$. Moreover, assume Curl$(\widetilde{F}) \in C^1(\overline{\Omega}, \mathbb{R}^{3\times 3})$. Then there exists a constant $c^+ > 0$ such that*

$$\|\nabla u \cdot \widetilde{F}^{-1}(x) + \widetilde{F}^{-T}(x) \cdot (\nabla u)^T\|^2_{L^2(\Omega)} \geq c^+ \|u\|^2_{H^1(\Omega)} \quad \forall\, u \in H^1_0(\Omega).$$

Remark 2.3.16. For $\widetilde{F} = \mathbb{1}$ we obtain

$$\|\nabla u + (\nabla u)^T\|^2_{L^2(\Omega)} \geq c^+ \|u\|^2_{H^1(\Omega)} \quad \forall\, u \in H^1_0(\Omega),$$

which is another form of Korn's first inequality [15].

For further generalizations, see e.g. [17].

3 Null-Lagrangians in nonlinear elasticity

This section is a preliminary chapter for the guiding question of this masterthesis, "the equivalence of linearized equations of equilibrium". To understand what we want to linearize and why, we have to clarify what is meant by *Null-Lagrangians* and *Euler-Lagrange-equations.*

3.1 Euler-Lagrange equations

First we assume $\Omega \subset \mathbb{R}^n$ to be a bounded, open subset with smooth boundary $\partial\Omega$ and a smooth function

$$L : \overline{\Omega} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}, \quad (x, u, \xi) \mapsto L(x, u, \xi), \tag{3.1.1}$$

which is termed *Lagrangian.* Now we define

$$I(w) := \int_\Omega L(x, w(x), \nabla w(x))\mathrm{d}x \tag{3.1.2}$$

for a smooth function $w : \overline{\Omega} \to \mathbb{R}^n$ satisfying the boundary condition

$$w = g \quad \text{on} \quad \partial\Omega. \tag{3.1.3}$$

We consider a function u with $u = g$ on $\partial\Omega$ to be a minimizer of (3.1.2). Let $v \in C_0^\infty(\Omega)$ be an arbitrary function and define

$$f(t) := I(u + t\, v) \quad \text{with} \quad t \in \mathbb{R}. \tag{3.1.4}$$

Then f has a minimum at $t = 0$ and thus the equation

$$f'(0) = \frac{\partial}{\partial t} I(u + t\, v)|_{t=0} = 0 \tag{3.1.5}$$

holds. Now we form the first variation:

$$f(t) = \int_\Omega L(x, u + t\, v, \nabla u + t\, \nabla v)\, \mathrm{d}x. \tag{3.1.6}$$

Hence,

$$\begin{aligned} f'(t)|_{t=0} &= \int_\Omega \left[\langle D_u\, L(x, u + t\, v, \nabla u + t\, \nabla v), v\rangle_{\mathbb{R}^n} + \langle D_\xi\, L(x, u + t\, v, \nabla u + t\, \nabla v), \nabla v\rangle_{\mathbb{R}^{n\times n}}\right] |_{t=0}\, \mathrm{d}x \\ &= \int_\Omega \langle D_u\, L(x, u, \nabla u), v\rangle + \langle D_\xi\, L(x, u, \nabla u), \nabla v\rangle\, \mathrm{d}x \\ &\underset{v|_{\partial\Omega}=0}{\overset{P.I.}{=}} \int_\Omega D_u\, L(x, u, \nabla u), v\rangle\, \mathrm{d}x - \int_\Omega \langle \mathrm{Div}(D_\xi\, L(x, u, \nabla u)), v\rangle\, \mathrm{d}x \\ &= \int_\Omega \langle D_u\, L(x, u, \nabla u) - \mathrm{Div}(D_\xi\, L(x, u, \nabla u)), v\rangle\, \mathrm{d}x. \end{aligned} \tag{3.1.7}$$

Therefore, together with the *Fundamental lemma of calculus of variations* 2.3.11, we obtain the *Euler-Lagrange-equation*[4]

$$D_u\, L(x, u, \nabla u) - \mathrm{Div}(D_\xi\, L(x, u, \nabla u)) = 0. \tag{3.1.8}$$

Now we calculate the Euler-Lagrange equation for the energy functional I defined in (2.2.15). As before, we get the Euler-Lagrange equations by forming the first variation in an arbitrary direction $v \in C_0^\infty(\Omega)$. For $\mathcal{M}$, defined in (2.2.13), let $u \in \mathcal{M}$. By using the dominated convergence theorem[19] we compute

$$
\begin{aligned}
0 &= \frac{d}{dt} I(u + t \cdot v)\,|_{t=0} = \frac{d}{dt} \int_\Omega W(\nabla u + t \cdot \nabla v)\,\mathrm{d}x\,|_{t=0} = \int_\Omega \frac{d}{dt}[W(\nabla u + t \cdot \nabla v)]\,\mathrm{d}x\,|_{t=0} \\
&= \int_\Omega \langle D_F W(\nabla u + t \cdot \nabla v), \nabla v\rangle\,\mathrm{d}x\,|_{t=0} = \int_\Omega \langle D_F W(\nabla u), \nabla v\rangle\,\mathrm{d}x \\
&= \int_{\partial\Omega} \langle v, D_F W(\nabla u)\rangle\,\mathrm{d}x - \int_\Omega \langle \mathrm{Div}(D_F W(\nabla u)), v\rangle\,\mathrm{d}x = -\int_\Omega \langle \mathrm{Div}(D_F W(\nabla u), v\rangle\,\mathrm{d}x \\
&\implies \mathrm{Div}(D_F W(\nabla u)) = 0.
\end{aligned} \tag{3.1.9}
$$

Hence, the Euler-Lagrange equation in nonlinear elasticity is given by

$$\mathrm{Div}[S_1(F)] = 0 \tag{3.1.10}$$

with the deformation gradient $F \coloneqq \nabla\varphi$. At this point, it is important to mention that it makes no difference whether we consider the Cauchy stress tensor or the Piola-Kirchhoff stress tensor to be in equilibrium, because with the properties of the Piola transformation in (2.3.15), we get

$$\mathrm{Div}[S_1(F)] = \mathrm{Div}[\sigma(F) \cdot \mathrm{Cof}(F)] = \det(F) \cdot \mathrm{Div}[\sigma(F)], \tag{3.1.11}$$

where $\det(F) \neq 0$. That implies

$$\mathrm{Div}[S_1(F)] = 0 \qquad \Longleftrightarrow \qquad \mathrm{Div}[\sigma(F)] = 0. \tag{3.1.12}$$

These statements lead us to define Null-Lagrangians.

Definition 3.1.1. The Lagrangian L in (3.1.1) is called a *Null-Lagrangian* if all smooth functions $u : \Omega \to \mathbb{R}^n$ satisfy the Euler-Lagrange-equation (3.1.8).

In elasticity, Null-Lagrangians are important because the corresponding energy functional I in (2.2.13) with

$$I(u) = \int_\Omega W(x, \varphi, \nabla\varphi)\,\mathrm{d}x \tag{3.1.13}$$

only depends on the boundary conditions [4]:

Theorem 3.1.2. *If L is a Null-Lagrangian and there are two functions u, $\overline{u} \in C^2(\overline{\Omega}, \mathbb{R}^n)$ which are identical on the boundary of Ω, i.e. $u \equiv \overline{u}$ on $\partial\Omega$, then the energy functional I for u is equal to the energy functional I for $\overline{u}$ on the whole set, i.e.*

$$I(u) = I(\overline{u}). \tag{3.1.14}$$

Proof. Like in (3.1.4), we define a function f with

$$f(\tau) \coloneqq I(\tau u + (1 - \tau)\overline{u}), \qquad 0 \leq \tau \leq 1. \tag{3.1.15}$$

[19]If we assume the functions $\{f_k\}_{k=0}^\infty$ to be integrable and $f_k \to f$ and $|f_k| \leq g$ almost everywhere, for some summable function g. Then

$$\int_{\mathbb{R}^n} f_k\,\mathrm{d}x \to \int_{\mathbb{R}^n} f\,dx.$$

Now,

$$\begin{aligned} f'(\tau) &= \frac{d}{d\tau} I(\tau u + (1-\tau)\overline{u}) = \frac{d}{d\tau} \int_\Omega W(x, \tau u + (1-\tau)\overline{u}, \tau \nabla u + (1-\tau)\nabla \overline{u})\, \mathrm{d}x \\ &= \int_\Omega \langle D_\xi W(x, \tau u + (1-\tau)\overline{u}, \tau\nabla u + (1-\tau)\nabla\overline{u}), \nabla u - \nabla \overline{u}\rangle \\ &\quad + \langle D_u W(x, \tau u + (1-\tau)\overline{u}, \tau \nabla u + (1-\tau)\nabla\overline{u}), u - \overline{u}\rangle \, \mathrm{d}x \\ &\underset{u|_{\partial\Omega} = \overline{u}|_{\partial\Omega}}{\overset{P.I.}{=}} \int_\Omega \langle -\operatorname{Div}(D_\xi W(x, \tau u + (1-\tau)\overline{u}, \tau\nabla u + (1-\tau)\nabla \overline{u}), u - \overline{u}\rangle \\ &\quad + \langle D_u W(x, \tau u + (1-\tau)\overline{u}, \tau\nabla u + (1-\tau)\nabla \overline{u}), u - \overline{u}\rangle\, \mathrm{d}x \\ &= 0. \end{aligned} \tag{3.1.16}$$

The last relation applies because with $u - \overline{u} = 0$ on the boundary $\partial\Omega$, the Euler-Lagrange-equation (3.1.8) is valid. Thus f is constant on $[0, 1]$. Furthermore, for $\tau = 0$ and for $\tau = 1$, we get $f(0) = I(\overline{u})$ and $f(1) = I(u)$, so $I(\overline{u}) = I(u)$. ■

3.2 Examples of Null-Lagrangians

We want to give some important examples of Null-Lagrangians in nonlinear elasticity.

Example 3.2.1. The first one is the Jacobi determinant. Consider $L(F) \coloneqq \det(F)$ with the deformation gradient F. We compute $\operatorname{Div}(D_F \det(F))$ and, together with $D_F(\det(F)) = \operatorname{Cof}(F)$ in (2.2.34) and the Piola-identity, obtain

$$\operatorname{Div}(D_F \det(F)) = \operatorname{Div}(\operatorname{Cof}(F)) = 0. \tag{3.2.1}$$

So the Euler-Lagrange-equation (3.1.10) holds and the Jacobi determinant is a Null-Lagrangian.

Example 3.2.2. Another important example depends on the deformation gradient F itself, in particular $L(F) \coloneqq \operatorname{tr}(F)$. For this we also calculate

$$\operatorname{Div}(D_F \operatorname{tr}(F)) = \operatorname{Div}(\mathbb{1}) = 0, \tag{3.2.2}$$

where the fist equality applies because of

$$\operatorname{tr}(F + H) = \langle F + H, \mathbb{1}\rangle = \langle F, \mathbb{1}\rangle + \langle H, \mathbb{1}\rangle \tag{3.2.3}$$

for an arbitrary direction $H \in \mathbb{R}^{3\times 3}$. So the trace is a Null-Lagrangian as well.

Example 3.2.3. The last example we want to mention is $L(F) \coloneqq \operatorname{tr}(\operatorname{Cof}(F))$. First, we have to calculate $\operatorname{tr}(\operatorname{Cof}(F + H))$. For this we use the identity

$$\operatorname{tr}(\operatorname{Cof}(X)) = \frac{1}{2}\left[(\operatorname{tr}(X))^2 - \operatorname{tr}\left(X^2\right)\right], \qquad \forall X \in \mathbb{R}^{n\times n}. \tag{3.2.4}$$

Then[20]

$$\begin{aligned} \operatorname{tr}(\operatorname{Cof}(F+H)) &= \frac{1}{2}\left[(\operatorname{tr}(F+H))^2 - \operatorname{tr}\left((F+H)^2\right)\right] \\ &= \frac{1}{2}\left[(\operatorname{tr}(F)^2 + 2\operatorname{tr}(F)\cdot\operatorname{tr}(H) + \operatorname{tr}(H)^2 - \operatorname{tr}(F^2) - \operatorname{tr}(2FH) - \operatorname{tr}(H^2)\right] \\ &= \frac{1}{2}\left[\operatorname{tr}(F)^2 - \operatorname{tr}\left(F^2\right)\right] + \operatorname{tr}(F)\operatorname{tr}(H) - \operatorname{tr}(FH) + \frac{1}{2}\left[\operatorname{tr}(H)^2 - \operatorname{tr}\left(H^2\right)\right] \\ &= \operatorname{tr}(\operatorname{Cof}(F)) + \operatorname{tr}(F)\langle H, \mathbb{1}\rangle - \langle FH, \mathbb{1}\rangle + O\left(\|H\|^2\right) \\ &= \operatorname{tr}(\operatorname{Cof}(F)) + \langle \operatorname{tr}(F)\mathbb{1}, H\rangle - \langle F^T, H\rangle + O\left(\|H\|^2\right) \\ &= \operatorname{tr}(\operatorname{Cof}(F)) + \langle \operatorname{tr}(F)\mathbb{1} - F^T, H\rangle + O\left(\|H\|^2\right). \end{aligned} \tag{3.2.5}$$

[20]For the calculation we have to consider that $\operatorname{tr}(HF) = \langle HF, \mathbb{1}\rangle = \left\langle F, H^T\right\rangle = \langle FH, \mathbb{1}\rangle = \operatorname{tr}(FH)$.

Hence,

$$D_F(\operatorname{tr}(\operatorname{Cof}(F))) = \operatorname{tr}(F)\,\mathbb{1} - F^T \tag{3.2.6}$$

and thus

$$\operatorname{Div}(D_F \operatorname{tr}(\operatorname{Cof}(F))) = \operatorname{Div}\left(\operatorname{tr}(F)\mathbb{1} - F^T\right). \tag{3.2.7}$$

Now, we want to show that this equality turns to zero for the two-dimensional case as well as for the three-dimensional case. Let us start with

$$F = \nabla\varphi = \begin{pmatrix} \partial_1\varphi_1 & \partial_2\varphi_1 \\ \partial_1\varphi_2 & \partial_2\varphi_2 \end{pmatrix} \in \mathbb{R}^{2\times 2}. \tag{3.2.8}$$

Then

$$\begin{aligned} \operatorname{Div}\left(\operatorname{tr}(F)\mathbb{1} - F^T\right) &= \operatorname{Div}\left[\begin{pmatrix} \partial_1\varphi_1 + \partial_2\varphi_2 & 0 \\ 0 & \partial_1\varphi_1 + \partial_2\varphi_2 \end{pmatrix} - \begin{pmatrix} \partial_1\varphi_1 & \partial_1\varphi_2 \\ \partial_2\varphi_1 & \partial_2\varphi_2 \end{pmatrix}\right] \\ &= \operatorname{Div}\begin{pmatrix} \partial_2\varphi_2 & -\partial_1\varphi_2 \\ -\partial_2\varphi_1 & \partial_1\varphi_1 \end{pmatrix} = \begin{pmatrix} \partial_1\partial_2\varphi_2 - \partial_2\partial_1\varphi_2 \\ -\partial_1\partial_2\varphi_1 + \partial_2\partial_1\varphi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{3.2.9}$$

Now, consider

$$F = \nabla\varphi = \begin{pmatrix} \partial_1\varphi_1 & \partial_2\varphi_1 & \partial_3\varphi_1 \\ \partial_1\varphi_2 & \partial_2\varphi_2 & \partial_3\varphi_2 \\ \partial_1\varphi_3 & \partial_2\varphi_3 & \partial_3\varphi_3 \end{pmatrix} \in \mathbb{R}^{3\times 3}. \tag{3.2.10}$$

We find

$$\begin{aligned} \operatorname{Div}\left(\operatorname{tr}(F)\mathbb{1} - F^T\right) &= \operatorname{Div}\left[\begin{pmatrix} \sum_{i=1}^3 \partial_i\,\varphi_i & 0 & 0 \\ 0 & \sum_{i=1}^3 \partial_i\,\varphi_i & 0 \\ 0 & 0 & \sum_{i=1}^3 \partial_i\,\varphi_i \end{pmatrix} - \begin{pmatrix} \partial_1\varphi_1 & \partial_1\varphi_2 & \partial_1\varphi_3 \\ \partial_2\varphi_1 & \partial_2\varphi_2 & \partial_2\varphi_3 \\ \partial_3\varphi_1 & \partial_3\varphi_2 & \partial_3\varphi_3 \end{pmatrix}\right] \\ &= \operatorname{Div}\begin{pmatrix} \partial_2\varphi_2 + \partial_3\varphi_3 & -\partial_1\varphi_2 & -\partial_1\varphi_3 \\ -\partial_2\varphi_1 & \partial_1\varphi_1 + \partial_3\varphi_3 & -\partial_2\varphi_3 \\ -\partial_3\varphi_1 & -\partial_3\varphi_2 & \partial_1\varphi_1 + \partial_2\varphi_2 \end{pmatrix} \\ &= \begin{pmatrix} \partial_1\partial_2\varphi_2 + \partial_1\partial_3\varphi_3 - \partial_2\partial_1\varphi_2 - \partial_3\partial_1\varphi_3 \\ -\partial_1\partial_2\varphi_1 + \partial_2\partial_1\varphi_1 + \partial_2\partial_3\varphi_3 - \partial_3\partial_2\varphi_3 \\ -\partial_1\partial_3\varphi_1 - \partial_2\partial_3\varphi_2 + \partial_3\partial_1\varphi_1 + \partial_3\partial_2\varphi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{3.2.11}$$

So the Euler-Lagrange equation in (3.1.10) holds for $F \in \mathbb{R}^{2\times 2}$ and for $F \in \mathbb{R}^{3\times 3}$. Thus, $L(F) = \operatorname{tr}(\operatorname{Cof}(F))$ is a Null-Lagrangian.

4 Linearisation in elasticity

In nonlinear elasticity we deal with difficult problems[21]. So we want to simplify these problems. Therefore, we use the method of linearization applied to important equations in nonlinear elasticity. First we want to define what is meant by the linearization of an equation, see also [12].

Definition 4.0.1. Let X and Y be Banach spaces and let $U \subset X$ be open. Additionally, let $f : U \to Y$ be a C^1 map and let $x_0 \in U$. The *linearization* of the equation $f(x) = 0$ about x_0 is the equation:

$$L_{x_0}(v) = 0 \tag{4.0.1}$$

where $L_{x_0}(v) = f(x_0) + Df(x_0) \cdot v$ with an arbitrary direction $v \in X$.

So with the help of the linearization method, we can approximate nonlinear mappings with linear mappings. For this reason, Marsden and Hughes call it the *keylink* between linear and nonlinear elasticity.
At this point, we have to notice the connection between the previous definition and the Taylor series, which is also mentioned in Section 2.2.3. For simplification, we use the linearization method/the Taylor series about $x_0 = \mathbb{1} + \nabla u$, where u is the displacement of a given deformation φ, which we assume to be small. For example, we can linearize equation (2.2.43) for the Cauchy stress tensor.
But above all, we are interested in linearization of equations of equilibrium and stability. More precisely, we are interested in the linearization of the Euler-Lagrange equation for the energy functional I defined in (2.2.15).
With this in mind, our intention is to find out if Liu generates some new stability requirements in [10].
He considers a back transformation of the Cauchy stress tensor to an intermediate configuration which he calls “the (first) Piola-Kirchhoff stress tensor relative to the updated reference configuration”. For this transformation, he enforces incremental linear approximations. Now we want to clarify if these approximations differ from the incremental linear approximations regarding to the usual Piola transformation in (2.3.15) with respect to the reference configuration. First we consider the mappings $\varphi\colon \Omega \to \Omega'$, $\varphi'\colon \Omega' \to \Omega''$ and $\widetilde{\varphi}\colon \Omega \to \Omega''$, as illustrated in Figures 11-14.

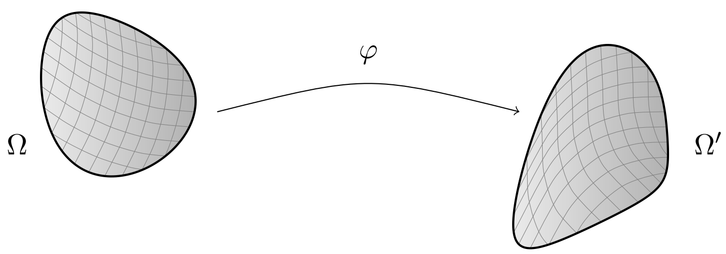

Figure 11: Deformation φ affecting the reference configuration Ω.

[21] An example is the main problem in elastostatics, i.e. finding a deformation φ with $\varphi : \Omega \to \mathbb{R}^{3\times 3}$ such that $\operatorname{Div} S_1(\nabla\varphi) + \rho_{\text{ref}} f = 0$, where S_1 is the first Piola-Kirchhoff stress tensor, ρ_{ref} is the mass density in the reference configuration Ω and f expresses the body force, see [12].

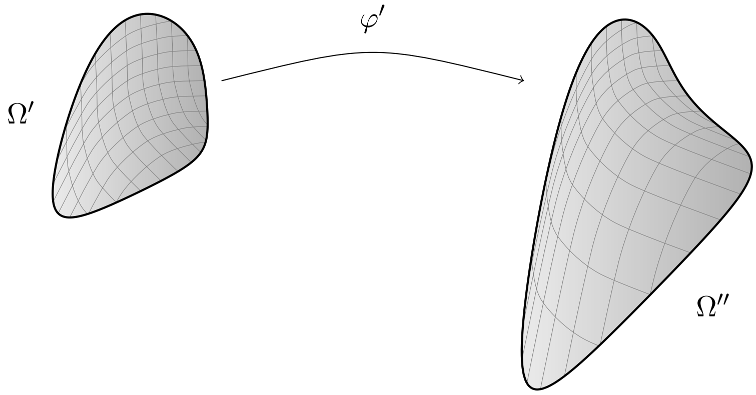

Figure 12: Deformation φ' affecting the intermediate configuration Ω'.

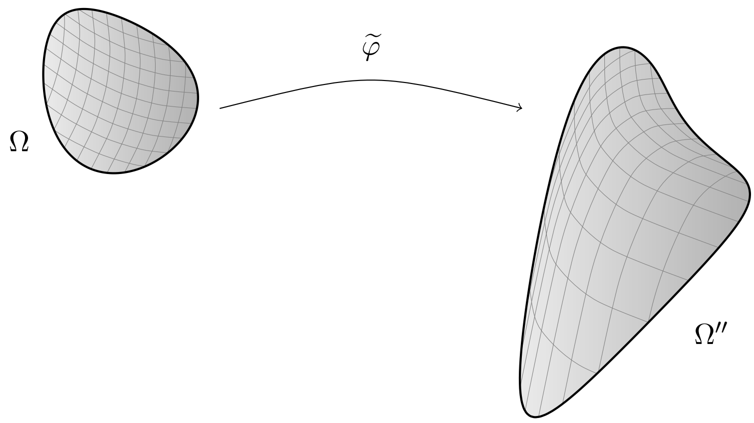

Figure 13: Deformation $\widetilde{\varphi}$ affecting the reference configuration Ω.

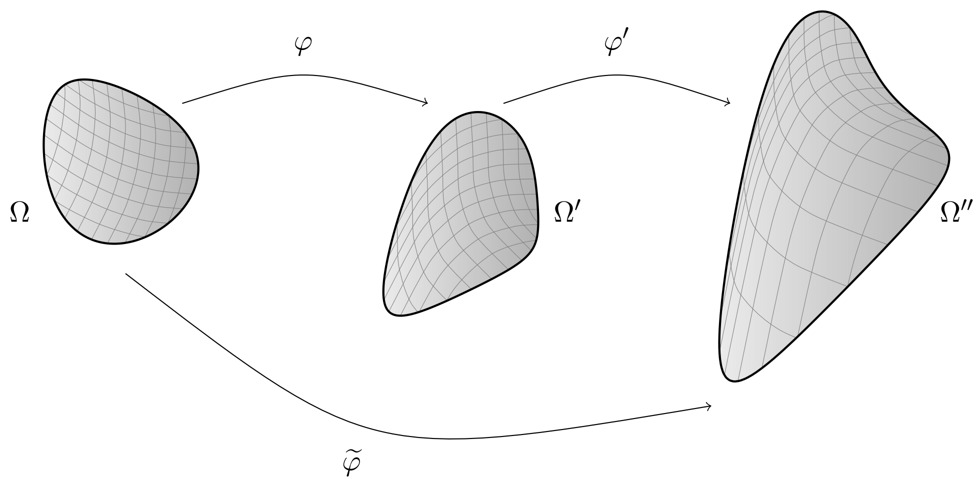

Figure 14: Multiplicative composition $\widetilde{\varphi} = \varphi' \circ \varphi$ of mappings.

Here, $\Omega \subset \mathbb{R}^n$ is the reference configuration, $\Omega' \subset \mathbb{R}^n$ is the intermediate configuration and $\Omega'' \subset \mathbb{R}^n$ is the final/current configuration. We employ the following notation: $\sigma\colon \mathrm{GL}^+(n) \to \mathrm{Sym}(n)$ denotes the *(constitutive) mapping* $F \mapsto \sigma(F)$ of the deformation gradient F to the corresponding Cauchy stress tensor, $\Sigma'\colon \Omega' \to \mathrm{Sym}(n)$ denotes the *Cauchy stress tensor field* induced by φ on the intermediate configuration Ω' and $\widetilde{\Sigma}\colon \Omega'' \to \mathrm{Sym}(n)$ denotes the Cauchy stress tensor field induced by $\widetilde{\varphi}$ on Ω''; more precisely,

$$\Sigma'(x') = \sigma\left(\nabla\varphi(\varphi^{-1}(x'))\right) \qquad \text{for all } \; x' \in \Omega' \tag{4.0.2}$$

$$\text{and} \quad \widetilde{\Sigma}(x'') = \sigma\left(\nabla\widetilde{\varphi}(\widetilde{\varphi}^{-1}(x''))\right) \qquad \text{for all } \; x'' \in \Omega''\,. \tag{4.0.3}$$

4.1 Equations of equilibrium and stability

According to (3.1.10) and (3.1.12) the homogeneous equations of equilibrium (Euler-Lagrange-equations) for Σ' and for $\widetilde{\Sigma}$ are given by

$$\operatorname{Div}_{x'} \Sigma'(x') = 0 \qquad \text{for all } \; x' \in \Omega'\,, \tag{4.1.1}$$

$$\operatorname{Div}_{x''} \widetilde{\Sigma}(x'') = 0 \qquad \text{for all } \; x'' \in \Omega''\,. \tag{4.1.2}$$

Now we propose the following basic stability criterion in the spatial configuration: if

i) φ and $\widetilde{\varphi}$ both satisfy the equations of equilibrium (4.1.1) and (4.1.2),

ii) φ' is sufficiently close to the identity mapping,

iii) φ' does not change the boundary of the configuration,

then $\varphi' = \operatorname{id}$ or, equivalently, $\widetilde{\varphi} = \varphi$. This condition ensures a "local uniqueness" of the displacement boundary problem.

4.1.1 The equations of equilibrium in the intermediate configuration

In the following, we assume that φ and $\widetilde{\varphi}$ satisfy the equations of equilibrium (4.1.1) and (4.1.2). Furthermore, we assume that φ and $\widetilde{\varphi}$ satisfy the same displacement boundary conditions, i.e. $\varphi|_{\partial\Omega} = \widetilde{\varphi}|_{\partial\Omega}$; note that this identity can equivalently be expressed as $\varphi'|_{\partial\Omega'} = \operatorname{id}$ and implies $\Omega' = \Omega''$.

Due to the assumption that "φ' is sufficiently close to the identity mapping", we write φ' as

$$\varphi'(x') = x' + h(x') \qquad \text{for all } \; x' \in \Omega' \tag{4.1.3}$$

with $h\colon \Omega' \to \mathbb{R}^n$. We also define $H := \nabla h$, thus

$$\nabla\varphi'(x') = \mathbb{1} + H(x') \qquad \text{for all } \; x' \in \Omega'\,. \tag{4.1.4}$$

Note that

$$\begin{aligned}\nabla\widetilde{\varphi}(x) = \nabla\left(\varphi'(\varphi(x))\right) &= \nabla\varphi'(\varphi(x)) \cdot \nabla\varphi(x) = (\mathbb{1} + H(\varphi(x))) \cdot \nabla\varphi(x) \\ &= \nabla\varphi(x) + H(\varphi(x)) \cdot \nabla\varphi(x)\end{aligned} \tag{4.1.5}$$

for all $x \in \Omega$.

Due to the Piola transformation in (2.3.15) and $\det(\nabla\varphi'(x')) \neq 0$, equation (4.1.2) can equivalently be written as

$$\begin{aligned}\operatorname{Div}_{x''} \widetilde{\Sigma}(x'') = 0 &\iff \operatorname{Div}_{x''}\left[\sigma\left(\nabla\widetilde{\varphi}(\widetilde{\varphi}^{-1}(x''))\right)\right] = 0 \\ &\iff \det(\nabla\varphi'(x')) \cdot \operatorname{Div}_{x''}\left[\sigma\left(\nabla\widetilde{\varphi}(\widetilde{\varphi}^{-1}(x''))\right)\right] = 0 \\ &\iff \operatorname{Div}_{x'}\left[\sigma\left(\nabla\widetilde{\varphi}(\widetilde{\varphi}^{-1}(\varphi'(x')))\right) \cdot \operatorname{Cof}(\nabla\varphi'(x'))\right] = 0 \\ &\iff \operatorname{Div}_{x'}\left[\widetilde{\Sigma}(\varphi'(x')) \cdot \operatorname{Cof} \nabla\varphi'(x')\right] = 0.\end{aligned} \tag{4.1.6}$$

The last equation in (4.1.6) corresponds to Liu's boundary value problem of an elastic body in equilibrium [10, eq. (6)], which is given by

$$\begin{cases} -\operatorname{Div}(S) &= 0 \quad \text{on} \quad \Omega' \times (t_0, t], \\ S \cdot \vec{n} &= f \quad \text{on} \quad \Gamma_1 \times (t_0, t], \\ \langle u, \vec{n} \rangle &= g \quad \text{on} \quad \Gamma_2 \times (t_0, t]. \end{cases} \tag{4.1.7}$$

Here, S denotes the back transformation to the intermediate configuration Ω' which Liu calls the (first) Piola-Kirchhoff stress tensor relative to the updated reference configuration, $\vec{n}$ is the normal vector on the surface of Ω', u is the displacement vector relative to the intermediate configuration and $(t_0, t]$ denotes a period of time. Furthermore, $\partial\Omega' = \Gamma_1 \cup \Gamma_2$, f is the prescribed surface traction and g is the displacement which Liu assumes to be time dependent. Note that in our case, we assume the surface traction f and the displacement g to be constant.
Now continue with relation (4.1.6). Due to (4.0.3) and (4.1.5), we get

$$\begin{aligned} \widetilde{\Sigma}(x'') &= \sigma\left(\nabla\widetilde{\varphi}(\widetilde{\varphi}^{-1}(x''))\right) \\ &= \sigma\left(\nabla\varphi(\widetilde{\varphi}^{-1}(x'')) + H\left(\varphi(\widetilde{\varphi}^{-1}(x''))\right) \cdot \nabla\varphi\left(\widetilde{\varphi}^{-1}(x'')\right)\right) \end{aligned} \tag{4.1.8}$$

for all $x'' \in \Omega''$. Thus, together with $\widetilde{\varphi}(x) = \varphi'(\varphi(x))$, we find

$$\begin{aligned} \widetilde{\Sigma}(\varphi'(x')) &= \sigma\left(\nabla\varphi(\widetilde{\varphi}^{-1}(\varphi'(x'))) + H\left(\varphi(\widetilde{\varphi}^{-1}(\varphi'(x')))\right) \cdot \nabla\varphi\left(\widetilde{\varphi}^{-1}(\varphi'(x'))\right)\right) \\ &= \sigma\left(\nabla\varphi(\varphi^{-1}(x')) + H\left(\varphi(\varphi^{-1}(x'))\right) \cdot \nabla\varphi\left(\varphi^{-1}(x')\right)\right) \\ &= \sigma\left(\nabla\varphi(\varphi^{-1}(x')) + H(x') \cdot \nabla\varphi\left(\varphi^{-1}(x')\right)\right). \end{aligned} \tag{4.1.9}$$

Combining (4.1.4), (4.1.6) and (4.1.9), we find that the equation of equilibrium (4.1.2) is equivalent to

$$\begin{aligned} 0 &= \operatorname{Div}_{x'}\left[\sigma\left(\nabla\varphi(\varphi^{-1}(x')) + H(x') \cdot \nabla\varphi\left(\varphi^{-1}(x')\right)\right) \cdot \operatorname{Cof}\nabla\varphi'(x')\right] \\ &= \operatorname{Div}_{x'}\left[\sigma\left(\nabla\varphi(\varphi^{-1}(x')) + H(x') \cdot \nabla\varphi\left(\varphi^{-1}(x')\right)\right) \cdot \operatorname{Cof}(\mathbb{1} + H(x'))\right]. \end{aligned} \tag{4.1.10}$$

4.1.2 The equation of equilibrium in the reference configuration

Under the assumptions of Section 4.1.1, we now consider the equation of equilibrium in terms of the reference configuration. Again, we employ the Piola transformation (2.3.15) to find

$$\begin{aligned} \operatorname{Div}_{x''}\widetilde{\Sigma}(x'') = 0 \iff& \operatorname{Div}_{x''}\left[\sigma\left(\nabla\widetilde{\varphi}(\widetilde{\varphi}^{-1}(x''))\right)\right] = 0 \\ \iff& \det\left(\nabla\widetilde{\varphi}(x)\right) \cdot \operatorname{Div}_{x''}\left[\sigma\left(\nabla\widetilde{\varphi}(\widetilde{\varphi}^{-1}(x''))\right)\right] = 0 \\ \iff& \operatorname{Div}_{x}\left[\sigma\left(\nabla\widetilde{\varphi}(\widetilde{\varphi}^{-1}(\widetilde{\varphi}(x)))\right) \cdot \operatorname{Cof}\left(\nabla\widetilde{\varphi}(x)\right)\right] = 0 \\ \iff& \operatorname{Div}_{x}\left[\widetilde{\Sigma}(\widetilde{\varphi}(x)) \cdot \operatorname{Cof}\left(\nabla\widetilde{\varphi}(x)\right)\right] = 0 \\ (\iff& \operatorname{Div}_{x}\left[S_1\left(\nabla\widetilde{\varphi}(x)\right)\right] = 0\,)\,. \end{aligned} \tag{4.1.11}$$

4.2 Linearisation in the intermediate configuration

In order to linearize equation (4.1.10), we first observe that together with (2.2.39), i.e.

$$\operatorname{Cof}(\mathbb{1} + X) = \mathbb{1} + \operatorname{tr}(X) \cdot \mathbb{1} - X^T + O\left(\|X\|^2\right), \tag{4.2.1}$$

we find

$$\operatorname{Cof}\left(\mathbb{1} + H(x')\right) = \left(1 + \operatorname{tr}(H(x'))\right) \cdot \mathbb{1} - H(x')^T + O\left(\|H(x')\|^2\right) \tag{4.2.2}$$

and thus, for sufficiently small H,

$$\begin{aligned}
&\sigma\left(\nabla\varphi(\varphi^{-1}(x'))+H(x')\cdot\nabla\varphi(\varphi^{-1}(x'))\right)\cdot\operatorname{Cof}\left(\mathbb{1}+H(x')\right)\\
&=\Big[\sigma\left(\nabla\varphi(\varphi^{-1}(x'))\right)+D\sigma\left(\nabla\varphi(\varphi^{-1}(x'))\right).\left[H(x')\cdot\nabla\varphi(\varphi^{-1}(x'))\right]\\
&\quad+O\left(\|H(x')\|^2\right)\Big]\cdot\Big[\left(1+\operatorname{tr}(H(x'))\right)\cdot\mathbb{1}-H(x')^T+O\left(\|H(x')\|^2\right)\Big]\\
&=\sigma\left(\nabla\varphi(\varphi^{-1}(x'))\right)\cdot\Big[\left(1+\operatorname{tr}(H(x'))\right)\cdot\mathbb{1}-H(x')^T\Big]\\
&\quad+D\sigma\left(\nabla\varphi(\varphi^{-1}(x'))\right).\left[H(x')\cdot\nabla\varphi(\varphi^{-1}(x'))\right]+O\left(\|H(x')\|^2\right)
\end{aligned}\tag{4.2.3}$$

for all $x'\in\Omega'$. The linearization of the equation of equilibrium with respect to the intermediate configuration is therefore given by

$$\begin{aligned}
0=\operatorname{Div}_{x'}\Big[&\sigma\left(\nabla\varphi(\varphi^{-1}(x'))\right)\cdot\left[\left(1+\operatorname{tr}(H(x'))\right)\cdot\mathbb{1}-H(x')^T\right]\\
&+D\sigma\left(\nabla\varphi(\varphi^{-1}(x'))\right).\left[H(x')\cdot\nabla\varphi(\varphi^{-1}(x'))\right]\Big]
\end{aligned}\tag{4.2.4}$$

for all $x'\in\Omega'$.

4.3 Linearisation in the reference configuration

Now we want to linearize equation (4.1.11). Therefore, with respect to (2.2.39), i.e.

$$\operatorname{Cof}(X+H)=\operatorname{Cof}(X)-\det(X)\left(X^{-1}HX^{-1}\right)^T+\operatorname{tr}\left(X^{-1}H\right)\cdot\operatorname{Cof}(X)+O\left(\|H\|^2\right),$$

we get

$$\operatorname{Cof}(A+X)=\operatorname{Cof}(A)+\operatorname{tr}\left(\operatorname{Cof}(A)^TX\right)\cdot A^{-T}-\det(A)\cdot A^{-T}X^TA^{-T}+O\left(\|X\|^2\right)\tag{4.3.1}$$

$$=\operatorname{Cof}(A)+\operatorname{tr}\left(\operatorname{Cof}(A)^TX\right)\cdot A^{-T}-A^{-T}X^T\operatorname{Cof}(A)+O\left(\|X\|^2\right)\tag{4.3.2}$$

for $A\in\mathrm{GL}^+(n)$ and $X\in\mathbb{R}^{n\times n}$. So together with (4.1.5), we find

$$\begin{aligned}
\operatorname{Cof}\left(\nabla\widetilde{\varphi}(x)\right)&=\operatorname{Cof}\left(\nabla\varphi(x)+H(\varphi(x))\cdot\nabla\varphi(x)\right)\\
&=\operatorname{Cof}(\nabla\varphi(x))+\operatorname{tr}\left(\operatorname{Cof}(\nabla\varphi(x))^TH(\varphi(x))\cdot\nabla\varphi(x)\right)\cdot\nabla\varphi(x)^{-T}\\
&\quad-\nabla\varphi^{-T}\cdot\nabla\varphi^T\cdot H(\varphi(x))^T\cdot\operatorname{Cof}(\nabla\varphi(x))+O\left(\|H(\varphi(x))\|^2\right)\\
&=\operatorname{Cof}(\nabla\varphi(x))+\operatorname{tr}\left(\det(\nabla\varphi(x))\cdot\nabla\varphi(x)^{-1}\cdot H(\varphi(x))\cdot\nabla\varphi(x)\right)\cdot\nabla\varphi(x)^{-T}\\
&\quad-H(\varphi(x))^T\cdot\operatorname{Cof}(\nabla\varphi(x))+O\left(\|H(\varphi(x))\|^2\right)\\
&=\operatorname{Cof}(\nabla\varphi(x))+\operatorname{tr}\left(\det(\nabla\varphi(x))\cdot H(\varphi(x))\right)\cdot\nabla\varphi(x)^{-T}\\
&\quad-H(\varphi(x))^T\cdot\operatorname{Cof}(\nabla\varphi(x))+O\left(\|H(\varphi(x))\|^2\right)\\
&=\operatorname{Cof}(\nabla\varphi(x))+\operatorname{tr}\left(H(\varphi(x))\right)\cdot\operatorname{Cof}(\nabla\varphi(x))\\
&\quad-H(\varphi(x))^T\cdot\operatorname{Cof}(\nabla\varphi(x))+O\left(\|H(\varphi(x))\|^2\right)\\
&=\left[(1+\operatorname{tr}(H(\varphi(x))))\cdot\mathbb{1}-H(\varphi(x))^T\right]\cdot\operatorname{Cof}(\nabla\varphi(x))+O\left(\|H(\varphi(x))\|^2\right).
\end{aligned}\tag{4.3.3}$$

Thus, for the first Piola-Kirchhoff stress tensor we get

$$\begin{aligned}
S_1(\nabla\widetilde{\varphi}(x)) &= \widetilde{\Sigma}(\widetilde{\varphi}(x)) \cdot \operatorname{Cof}\nabla\widetilde{\varphi}(x) \\
&= \sigma\left(\nabla\widetilde{\varphi}(\widetilde{\varphi}^{-1}(\widetilde{\varphi}(x)))\right) \cdot \operatorname{Cof}\left(\nabla\widetilde{\varphi}(x)\right) \\
&= \sigma\left(\nabla\widetilde{\varphi}(x)\right) \cdot \operatorname{Cof}\left(\nabla\widetilde{\varphi}(x)\right) \\
&= \sigma\left(\nabla\varphi(x) + H(\varphi(x)) \cdot \nabla\varphi(x)\right) \cdot \operatorname{Cof}\left(\nabla\widetilde{\varphi}(x)\right) \\
&= \Big[\sigma\left(\nabla\varphi(x)\right) + D\sigma\left(\nabla\varphi(x)\right).\left[H(\varphi(x)) \cdot \nabla\varphi(x)\right]\Big] \cdot \operatorname{Cof}\left(\nabla\widetilde{\varphi}(x)\right) + O\left(\|H(\varphi(x))\|^2\right) \\
&= \sigma\left(\nabla\varphi(x)\right) \cdot \operatorname{Cof}\left(\nabla\widetilde{\varphi}(x)\right) + \Big[D\sigma\left(\nabla\varphi(x)\right).\left[H(\varphi(x)) \cdot \nabla\varphi(x)\right]\Big] \\
&\quad \cdot \operatorname{Cof}(\nabla\widetilde{\varphi}(x)) + O\left(\|H(\varphi(x))\|^2\right) \\
&= \sigma\left(\nabla\varphi(x)\right) \cdot \operatorname{Cof}\left(\nabla\widetilde{\varphi}(x)\right) + \Big[D\sigma\left(\nabla\varphi(x)\right).\left[H(\varphi(x)) \cdot \nabla\varphi(x)\right]\Big] \\
&\quad \cdot \left[(1 + \operatorname{tr}(H(\varphi(x)))) \cdot \mathbb{1} - H(\varphi(x))^T\right] \cdot \operatorname{Cof}\left(\nabla\varphi(x)\right) + O\left(\|H(\varphi(x))\|^2\right) \\
&= \sigma\left(\nabla\varphi(x)\right) \cdot \left[(1 + \operatorname{tr}(H(\varphi(x)))) \cdot \mathbb{1} - H(\varphi(x))^T\right] \cdot \operatorname{Cof}\left(\nabla\varphi(x)\right) \\
&\quad + \Big[D\sigma\left(\nabla\varphi(x)\right).\left[H(\varphi(x)) \cdot \nabla\varphi(x)\right]\Big] \cdot \left[(1 + \operatorname{tr}(H(\varphi(x)))) \cdot \mathbb{1} - H(\varphi(x))^T\right] \\
&\quad \cdot \operatorname{Cof}\left(\nabla\varphi(x)\right) + O\left(\|H(\varphi(x))\|^2\right) \\
&= \sigma\left(\nabla\varphi(x)\right) \cdot \left[(1 + \operatorname{tr}(H(\varphi(x)))) \cdot \mathbb{1} - H(\varphi(x))^T\right] \cdot \operatorname{Cof}\left(\nabla\varphi(x)\right) \\
&\quad + \Big[D\sigma\left(\nabla\varphi(x)\right).\left[H(\varphi(x)) \cdot \nabla\varphi(x)\right]\Big] \cdot \operatorname{Cof}\left(\nabla\varphi(x)\right) + O\left(\|H(\varphi(x))\|^2\right). \qquad (4.3.4)
\end{aligned}$$

The linearized equation of equilibrium with respect to the reference configuration is therefore given by

$$\begin{aligned}
0 = \operatorname{Div}_x \Big[&\left[\sigma\left(\nabla\varphi(x)\right) \cdot \left[(1 + \operatorname{tr}(H(\varphi(x)))) \cdot \mathbb{1} - H(\varphi(x))^T\right]\right. \\
&\left. + D\sigma\left(\nabla\varphi(x)\right).\left[H(\varphi(x)) \cdot \nabla\varphi(x)\right]\right] \cdot \operatorname{Cof}(\nabla\varphi(x))\Big] \qquad (4.3.5)
\end{aligned}$$

for all $x \in \Omega$, which (due to the Piola transformation (2.3.15)) is in turn equivalent to

$$\begin{aligned}
&0 = \det(\nabla\varphi(x)) \cdot \operatorname{Div}_{x'} \Big[\sigma\left(\nabla\varphi(\varphi^{-1}(x'))\right) \cdot \left[\left(1 + \operatorname{tr}(H(x'))\right) \cdot \mathbb{1} - H(x')^T\right] \\
&\qquad\qquad + D\sigma\left(\nabla\varphi(\varphi^{-1}(x'))\right).\left[H(x') \cdot \nabla\varphi(\varphi^{-1}(x'))\right]\Big] \\
\iff\ &0 = \operatorname{Div}_{x'} \Big[\sigma\left(\nabla\varphi(\varphi^{-1}(x'))\right) \cdot \left[\left(1 + \operatorname{tr}(H(x'))\right) \cdot \mathbb{1} - H(x')^T\right] \\
&\qquad + D\sigma\left(\nabla\varphi(\varphi^{-1}(x'))\right).\left[H(x') \cdot \nabla\varphi(\varphi^{-1}(x'))\right]\Big] \qquad (4.3.6)
\end{aligned}$$

for all $x' \in \Omega'$. So, in conclusion, the linearized equation of equilibrium in the reference configuration is equivalent to the linearized equation of equilibrium in the intermediate configuration (4.2.4). This result is also shown in Fig. 15.

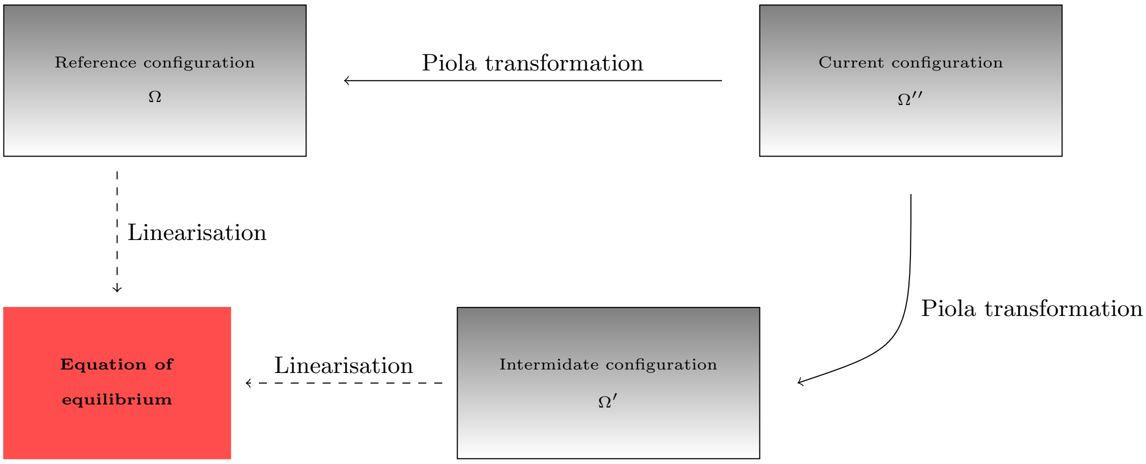

Figure 15: Visualization of previous calculations.

Moreover, equation (4.3.6) corresponds to Liu's linearized boundary value problem [10, eq. (7)], which is given by

$$\begin{cases} -\operatorname{Div}\left[\operatorname{tr} H \cdot \sigma - \sigma H^T + D_F\sigma.[HF]\right] &= \operatorname{Div}\sigma \quad \text{on } \Omega' \times (t_0, t], \\ \left[\operatorname{tr} H \cdot \sigma - \sigma H^T + D_F\sigma.[HF]\right] \cdot \vec{n} &= f - \sigma \cdot \vec{n} \quad \text{on } \Gamma_1 \times (t_0, t], \\ \langle u, \vec{n}\rangle &= g \quad \text{on } \Gamma_2 \times (t_0, t], \end{cases} \tag{4.3.7}$$

with the same conditions as in (4.1.7).

Remark 4.3.1. Note that in (4.3.4) we started with

$$\begin{aligned} S_1\left[\nabla\widetilde{\varphi}(x)\right] &= S_1\left[\nabla\varphi'(\varphi(x)) \cdot \nabla\varphi(x)\right] = S_1\left[(\mathbb{1} + H(\varphi(x))) \cdot \nabla\varphi(x)\right] \\ &= S_1\left[\nabla\varphi(x) + H(\varphi(x)) \cdot \nabla\varphi(x)\right]. \end{aligned}$$

So (4.3.5) is equivalent to

$$\begin{aligned} 0 &= \operatorname{Div}_x\left[S_1\left[\nabla\varphi(x) + H(\varphi(x)) \cdot \nabla\varphi(x)\right]\right] \\ &= \operatorname{Div}_x\left[S_1\left(\nabla\varphi(x)\right) + D_F S_1\left(\nabla\varphi(x)\right).\left[H(\varphi(x)) \cdot \nabla\varphi(x)\right] + O\left(\|H(\varphi(x)) \cdot \nabla\varphi(x)\|^2\right)\right] \\ &\approx \operatorname{Div}_x\left[S_1\left(\nabla\varphi(x)\right) + D_F S_1\left(\nabla\varphi(x)\right).\left[H(\varphi(x)) \cdot \nabla\varphi(x)\right]\right]. \end{aligned}$$

Furthermore, if we assume $\nabla\varphi$ to be a gradient of an equilibrium solution, the equation reduces to

$$0 = \operatorname{Div}_x\left[D_F S_1\left(\nabla\varphi(x)\right).\left[H(\varphi(x)) \cdot \nabla\varphi(x)\right]\right]. \tag{4.3.8}$$

4.4 Restriction to homogeneous initial deformations

If $x \mapsto \varphi(x) = F \cdot x$ with $F \in \mathrm{GL}^+(n)$ is a homogeneous[22] deformation, then the linearization in (4.3.6) immediately reduces to

$$\begin{aligned} &\left(1 + \operatorname{tr} H(x')\right) \cdot \sigma(F) - \sigma(F) \cdot H(x')^T + D\sigma(F).\left[H(x') \cdot F\right] + O\left(\|H(x')\|^2\right) \\ &= \sigma(F) \cdot \left[\left(1 + \operatorname{tr} H(x')\right) \cdot \mathbb{1} - H(x')^T\right] + D\sigma(F).\left[H(x') \cdot F\right] + O\left(\|H(x')\|^2\right). \end{aligned} \tag{4.4.1}$$

In the case $n = 3$, we get

$$\begin{aligned} &\operatorname{Div}_{x'}\left[\left(1 + \operatorname{tr}(H(x'))\right) \cdot \mathbb{1} - H(x')^T\right] \\ &= \operatorname{Div}_{x'}\begin{pmatrix} 1 + \partial_{x_2'}h_2 + \partial_{x_3'}h_3 & -\partial_{x_1'}h_2 & -\partial_{x_1'}h_3 \\ -\partial_{x_2'}h_1 & 1 + \partial_{x_1'}h_1 + \partial_{x_3'}h_3 & -\partial_{x_2'}h_3 \\ -\partial_{x_3'}h_1 & -\partial_{x_3'}h_2 & 1 + \partial_{x_1'}h_1 + \partial_{x_2'}h_2 \end{pmatrix} = 0. \end{aligned}$$

[22] A deformation is homogeneous if the deformation gradient is independent of the place.

The transformation also applies in the n-dimensional case,[23] thus,

$$\operatorname{Div}_{x'}\left[\left(1+\operatorname{tr}(H(x'))\right)\cdot \mathbb{1} - H(x')^T\right] = 0 \tag{4.4.2}$$

for any gradient field H on Ω'.
Moreover, the i-th component of the divergence of a matrix product of $A, B \in \mathbb{R}^{n\times n}$ can be written as

$$\begin{aligned}
(\operatorname{Div}(A\cdot B))_i &= \sum_{j=1}^{n} \partial_{x_j}(A\cdot B)_{ij} \\
&= \sum_{j=1}^{n} \partial_{x_j} \sum_{k=1}^{n} A_{ik}B_{kj} \\
&= \sum_{j,k=1}^{n} \left[(\partial_{x_j}A_{ik})\cdot B_{kj} + A_{ik}\cdot(\partial_{x_j}B_{kj})\right] \\
&= \sum_{j,k=1}^{n} (\partial_{x_j}A_{ik})\cdot B_{kj} + \sum_{j,k=1}^{n} A_{ik}\cdot(\partial_{x_j}B_{kj}) \\
&= \sum_{j,k=1}^{n} (\partial_{x_j}A_{ik})\cdot B_{kj} + \sum_{k=1}^{n}(A_{ik}\cdot\sum_{j=1}^{n}(\partial_{x_j}B_{kj})) \\
&= \sum_{j,k=1}^{n} (\partial_{x_j}A_{ik})\cdot B_{kj} + \sum_{k=1}^{n}(A_{ik}\cdot(\operatorname{Div}B)_k \\
&= \sum_{j,k=1}^{n} (\partial_{x_j}A_{ik})\cdot B_{kj} + (A\cdot\operatorname{Div}B)_i\,.
\end{aligned} \tag{4.4.3}$$

Since F is assumed to be constant, the linearized equation of equilibrium can therefore be stated as

$$\begin{aligned}
0 &= \operatorname{Div}_{x'}\left[\sigma(F)\cdot\left[\left(1+\operatorname{tr}H(x')\right)\cdot\mathbb{1} - H(x')^T\right] + D\sigma(F).\left[H(x')\cdot F\right]\right] \\
&= \operatorname{Div}_{x'}\left[\sigma(F)\cdot\left[\left(1+\operatorname{tr}H(x')\right)\cdot\mathbb{1} - H(x')^T\right]\right] + \operatorname{Div}_{x'}\left[D\sigma(F).\left[H(x')\cdot F\right]\right] \\
&= \operatorname{Div}_{x'}\left[D\sigma(F).\left[H(x')\cdot F\right]\right]
\end{aligned} \tag{4.4.4}$$

for all $x' \in \Omega'$.

4.5 Application to a specific energy

We consider the energy-function

$$\widehat{W}(F) := \mu\cdot\left[\frac{1}{2}\|F\|^2 - \log(\det(F))\right] \tag{4.5.1}$$

with the infinitesimal shear modulus μ. Now, we want to find the linearized equation of equilibrium and subsequently the linearized boundary value problem for $\widehat{W}$ with respect to previous results. First, we define $F_0 := \nabla\varphi$, $F := \nabla\varphi' = \mathbb{1} + H$ with $H = \nabla h$ (see (4.1.3) and (4.1.4)) and $\widetilde{F} := \nabla\widetilde{\varphi}$ with the relation $\widetilde{F} = \nabla\left(\varphi'\circ\varphi\right) = \nabla\varphi'\cdot\nabla\varphi = F\cdot F_0 = (\mathbb{1}+H)F_0$, see Fig. 16.

[23] Because of the considerations of the Null Lagrangian $L(F) = \operatorname{tr}(\operatorname{Cof}(F))$ in Example 3.2.3, we know that $\operatorname{Div}\left((1+\operatorname{tr}(H))\,\mathbb{1} - H^T\right) = \operatorname{Div}\left(D_H(\operatorname{tr}(\operatorname{Cof}(H))\right) = 0$ as well.

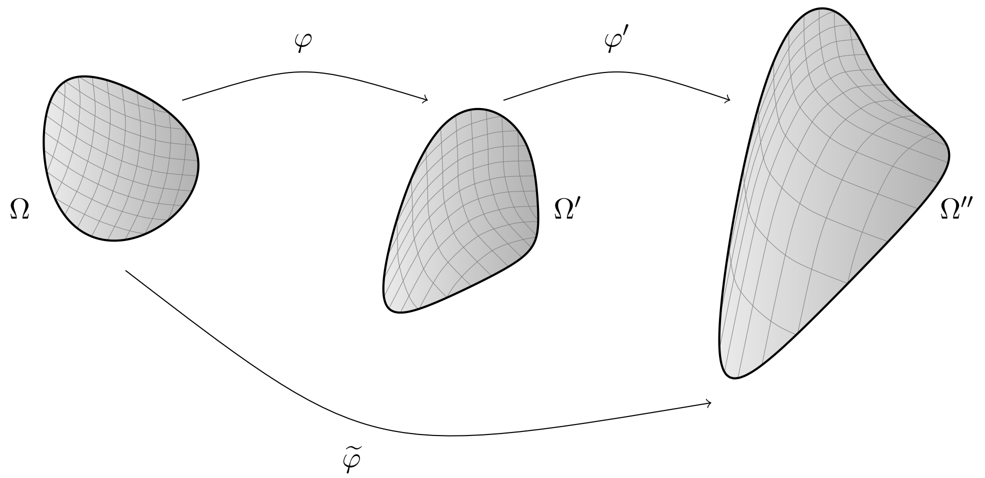

Figure 16: Reminder: Relationships between the deformations φ, φ', $\widetilde{\varphi}$ and the configurations Ω (reference configuration), Ω' (intermediate configuration), Ω'' (current configuration).

In order to find a formula for the Cauchy stress tensor relative to the intermediate configuration, we compute

$$\begin{aligned}\sigma(F_0) &= D_F\widehat{W}(F_0)\cdot \operatorname{Cof}(F_0)^{-1} = \mu\cdot\left[\frac{1}{2}\cdot 2F_0 - \frac{1}{\det(F_0)}\cdot\operatorname{Cof}(F_0)\right]\cdot\operatorname{Cof}(F_0)^{-1}\\ &= \mu\cdot\left[F_0\cdot\operatorname{Cof}(F_0)^{-1} - \frac{1}{\det(F_0)}\cdot\mathbb{1}\right] = \frac{\mu}{\det(F_0)}\left[B_0-\mathbb{1}\right]. \qquad (4.5.2)\end{aligned}$$

Hence,

$$\begin{aligned}D_{F_0}\sigma(F_0).[HF_0] &= \mu\left[-\det(F_0)^{-2}\cdot\langle\operatorname{Cof}(F_0),HF_0\rangle\cdot(B_0-\mathbb{1}) + \det(F_0)^{-1}\cdot(HF_0F_0^T + F_0(HF_0)^T)\right]\\ &= \mu\left[\langle-\det(F_0)^{-1}F_0^{-T},HF_0\rangle\cdot\left(F_0F_0^T-\mathbb{1}\right) + \det(F_0)^{-1}\cdot\left(HF_0F_0^T + F_0F_0^TH^T\right)\right]\\ &= \frac{\mu}{\det(F_0)}\left[-\langle F_0^{-T},HF_0\rangle\cdot\left(F_0F_0^T-\mathbb{1}\right) + \left(HF_0F_0^T+F_0F_0^TH^T\right)\right]\\ &= \frac{\mu}{\det(F_0)}\left[-\operatorname{tr}(H)\cdot(B_0-\mathbb{1}) + \left(HB_0+B_0H^T\right)\right], \qquad (4.5.3)\end{aligned}$$

where $B_0 \coloneqq F_0F_0^T$ denotes the right Cauchy-Green tensor.
Therefore, due to equation (4.2.3), the Piola-Kirchhoff stress tensor is given by

$$\begin{aligned}S_1(\widetilde{F}) &= \sigma(F_0+HF_0)\cdot\operatorname{Cof}(F) = \left[\sigma(F_0) + D_{F_0}\sigma(F_0).[HF_0] + O\left(\|H\|^2\right)\right]\cdot\operatorname{Cof}(\mathbb{1}+H)\\ &= \left[\sigma(F_0) + \frac{\mu}{\det(F_0)}\left(-\operatorname{tr}(H)\cdot(B_0-\mathbb{1}) + \left(HB_0+B_0H^T\right)\right) + O\left(\|H\|^2\right)\right]\cdot\det(\mathbb{1}+H)\cdot(\mathbb{1}+H)^{-T}\\ &= (1+\operatorname{tr}(H))\cdot\left[\sigma(F_0) + \frac{\mu}{\det(F_0)}\left(-\operatorname{tr}(H)(B_0-\mathbb{1}) + \left(HB_0+B_0H^T\right)\right)\right]\cdot\left(\mathbb{1}-H^T\right) + O\left(\|H\|^2\right)\\ &= \sigma(F_0) + \frac{\mu}{\det(F_0)}\left[-\operatorname{tr}(H)(B_0-\mathbb{1}) + \left(HB_0+B_0H^T\right)\right] - \sigma(F_0)\cdot H^T + \operatorname{tr}(H)\cdot\sigma(F_0) + O\left(\|H\|^2\right)\\ &= \sigma(F_0) + \frac{\mu}{\det(F_0)}\left(HB_0+B_0H^T\right) - \sigma(F_0)\cdot H^T + O\left(\|H\|^2\right). \qquad (4.5.4)\end{aligned}$$

Thus, together we get the linearized equation of equilibrium in the intermediate configuration

$$0 = \operatorname{Div}\left[\sigma(F_0) + \frac{\mu}{\det(F_0)}\left(HB_0+B_0H^T\right) - \sigma(F_0)\cdot H^T\right], \qquad (4.5.5)$$

which is equivalent to equation (4.2.4).
From this and together with $H = \nabla u$ we obtain the linearized boundary value problem, which is given by

$$\begin{cases} -\operatorname{Div}\left[\sigma(F_0) + \frac{\mu}{\det(F_0)}\left(HB_0 + B_0H^T\right) - \sigma(F_0)\cdot H^T\right] &= 0 \quad \text{on} \quad \Omega, \\ \left[\sigma(F_0) + \frac{\mu}{\det(F_0)}\left(HB_0 + B_0H^T\right) - \sigma(F_0)\cdot H^T\right]\cdot \vec{n_k} &= f \quad \text{on} \quad \Gamma_1 \\ \langle u, \vec{n_k}\rangle &= 0 \quad \text{on} \quad \Gamma_2 \\ u &= 0 \quad \text{on} \quad \Gamma_3, \end{cases} \tag{4.5.6}$$

where $\vec{n_k}$ is the exterior unit normal on $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and f is a constant surface traction.

This boundary value problem can be formulated as a variational problem, as Liu does for the Cauchy stress tensor

$$\sigma(F) = -p\mathbb{1} + s_1 B + s_2 B^{-1} \tag{4.5.7}$$

with the pressure p and constant material parameters s_1 and s_2 [2]. Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain. For the displacement u, let

$$\mathcal{V} := \{u \in (H^1(\Omega))^3 : \langle u, \vec{n_k}\rangle = 0 \quad \text{on} \quad \Gamma_2,\ u = 0 \quad \text{on} \quad \Gamma_3\}. \tag{4.5.8}$$

Now for $w \in \mathcal{V}$, $H := \nabla u$ and $K(H) := \frac{\mu}{\det(F_0)}\left[\left(HB_0 + B_0H^T\right)\right] - \sigma(F_0)\cdot H^T$, we calculate

$$\begin{aligned} 0 &= \int_\Omega -\operatorname{Div}\left[\sigma(F_0) + K(H)\right]\cdot w \,\mathrm{d}V \\ &\overset{P.I.}{=} -\int_{\partial\Omega} \langle[\sigma(F_0) + K(H)]\cdot \vec{n_k}, w\rangle \, dS + \int_\Omega \langle \sigma(F_0) + K(H), \nabla w\rangle \,\mathrm{d}V \\ &= \int_\Omega \langle \sigma(F_0), \nabla w\rangle \,\mathrm{d}V + \int_\Omega \langle K(H), \nabla w\rangle \,\mathrm{d}V - \int_{\Gamma_1} \langle f, w\rangle \, dS. \end{aligned} \tag{4.5.9}$$

Thus,

$$\begin{aligned} &\int_\Omega \langle K(H), \nabla w\rangle \,\mathrm{d}V = \int_{\Gamma_1} \langle f, w\rangle \,\mathrm{d}S - \int_\Omega \langle \sigma(F_0), \nabla w\rangle \,\mathrm{d}V \\ \Longleftrightarrow \quad &\int_\Omega \operatorname{tr}\left(K(H)\cdot(\nabla w)^T\right) \,\mathrm{d}V = \int_{\Gamma_1} \langle f, w\rangle \,\mathrm{d}S - \int_\Omega \operatorname{tr}\left(\sigma(F_0)\cdot(\nabla w)^T\right) \,\mathrm{d}V. \end{aligned} \tag{4.5.10}$$

Hence, we define a bilinear form $\mathcal{L}(u, w)$ and a linear form $\mathcal{N}(w)$ by

$$\mathcal{L}(u, w) := \int_\Omega \operatorname{tr}\left(K(\nabla u)\cdot(\nabla w)^T\right) \,\mathrm{d}V, \tag{4.5.11}$$

$$\mathcal{N}(w) := \int_{\Gamma_1} \langle f, w\rangle \,\mathrm{d}S - \int_\Omega \operatorname{tr}\left(\sigma(F_0)\cdot(\nabla w)^T\right) \,\mathrm{d}V. \tag{4.5.12}$$

Moreover, $\mathcal{L}$ and $\mathcal{N}$ can be written in coordinates as

$$\begin{aligned}
\mathcal{L}(u,w) &= \int_\Omega \operatorname{tr}\left(K(\nabla u)\cdot(\nabla w)^T\right)\,\mathrm{d}V \\
&= \int_\Omega \langle K(\nabla u), \nabla w\rangle\,\mathrm{d}V \\
&= \int_\Omega \sum_{i=1}^{3}\sum_{j=1}^{3} K(\nabla u)_{ij}\cdot(\nabla w)_{ij}\,\mathrm{d}V \\
&= \int_\Omega \sum_{i=1}^{3}\sum_{j=1}^{3}\left[\frac{\mu}{\det(F_0)}\left[\nabla u B_0 + B_0(\nabla u)^T\right] - \sigma(F_0)(\nabla u)^T\right]_{ij}\cdot(\nabla w)_{ij}\,\mathrm{d}V \\
&= \frac{\mu}{\det(F_0)}\int_\Omega \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\left(\frac{\partial u_i}{\partial x_k}B_{0_{kj}} + B_{0_{ik}}\frac{\partial u_j}{\partial x_k}\right)\cdot\frac{\partial w_i}{\partial x_j}\,\mathrm{d}V \\
&\quad - \int_\Omega \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\left[\sigma(F_0)_{ik}\cdot\frac{\partial u_j}{\partial x_k}\right]\cdot\frac{\partial w_i}{\partial x_j}\,\mathrm{d}V, \qquad (4.5.13)\\
\mathcal{N}(w) &= \int_{\Gamma_1}\langle f, w\rangle\,\mathrm{d}S - \int_\Omega\langle\sigma(F_0),\nabla w\rangle\,\mathrm{d}V \\
&= \int_{\Gamma_1}\sum_{i=1}^{3} f_i\cdot w_i\,\mathrm{d}S - \int_\Omega\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma(F_0)_{ij}\cdot\frac{\partial w_i}{\partial x_j}\,\mathrm{d}V. \qquad (4.5.14)
\end{aligned}$$

Overall, the variational problem corresponding to the boundary value problem in (4.5.6) is to find a solution $u \in \mathcal{V}$ such that

$$\mathcal{L}(u,w) = \mathcal{N}(w), \quad \forall\, w \in \mathcal{V}. \qquad (4.5.15)$$

4.5.1 Existence and uniqueness of a solution

In the following, we will study the existence and uniqueness of a solution u to (4.5.15), i.e. to (4.5.6) and we apply Liu's considerations in [2] to the energy

$$\widehat{W}(F) \coloneqq \mu\cdot\left[\frac{1}{2}\|F\|^2 - \log(\det(F))\right]. \qquad (4.5.16)$$

If $u \in \mathcal{V}$ is a solution of (4.5.15), then u is a weak solution of (4.5.6). This leads us to formulate the following theorem.

Theorem 4.5.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that for $F_0 = \nabla\varphi$, $\frac{\mu}{\det(F_0)} \in L^\infty(\Omega)$ and $\sigma(F_0)$, $B_0 \in L^\infty(\Omega,\mathbb{R}^{3\times 3})$, where $\sigma(F_0)$ denotes the Cauchy stress corresponding to the intermediate configuration and B_0 denotes the right Cauchy-Green tensor.*
If u is a solution of the variational problem (4.5.15)*, i. e.*

$$\int_\Omega \operatorname{tr}\left(K(\nabla u)\cdot(\nabla w)^T\right)\,\mathrm{d}V = \int_{\Gamma_1}\langle f,w\rangle\,\mathrm{d}S - \int_\Omega \operatorname{tr}\left(\sigma(F_0)\cdot(\nabla w)^T\right)\,\mathrm{d}V, \quad \forall\, w\in\mathcal{V}$$

then u is a weak solution of the boundary-value problem (4.5.6)*, i. e.*

$$\begin{cases}
-\operatorname{Div}\left[\sigma(F_0) + \frac{\mu}{\det(F_0)}\left(HB_0 + B_0H^T\right) - \sigma(F_0)\cdot H^T\right] = 0 & \text{on } \Omega,\\
\left[\sigma(F_0) + \frac{\mu}{\det(F_0)}\left(HB_0 + B_0H^T\right) - \sigma(F_0)\cdot H^T\right]\cdot\vec{n_k} = f & \text{on } \Gamma_1\\
\langle u, \vec{n_k}\rangle = 0 & \text{on } \Gamma_2\\
u = 0 & \text{on } \Gamma_3.
\end{cases}$$

Proof. See [2]. ■

In the following, we will concentrate on the two-dimensional case, i.e.

$$\mathcal{V} := \{u \in H^1(\Omega)^2 : \langle u, \vec{n_k}\rangle = 0 \quad \text{on} \quad \Gamma_2 \quad \text{and} \quad u = 0 \quad \text{on} \quad \Gamma_3\}. \tag{4.5.17}$$

For u and $w \in \mathcal{V}$ we define the inner product

$$\langle u, w\rangle_{\mathcal{V}} := \int_\Omega \langle \nabla u_1(x), \nabla w_1(x)\rangle_{\mathbb{R}^2} + \langle \nabla u_2(x), \nabla w_2(x)\rangle_{\mathbb{R}^2}\,\mathrm{d}x \tag{4.5.18}$$

and the norm

$$\|u\|_{\mathcal{V}}^2 := \|\nabla u_1\|_{L^2}^2 + \|\nabla u_2\|_{L^2}^2. \tag{4.5.19}$$

At this point, we want to show briefly that these definitions really define an inner product and a norm, respectively. We already know the Poincaré inequality, which was introduced in Theorem 2.3.9: there exists a constant $\widetilde{c} > 0$, such that $\|u\|_{\mathcal{V}}^2 \geq \widetilde{c} \cdot \|u\|_{L^2}^2$ for all $u \in \mathcal{V}$. So for $\lambda \in \mathbb{R}$, u, $w \in \mathcal{V}$, and with the Poincaré inequality in mind, we get

$$
\begin{aligned}
\text{i)} \quad & \|\lambda \cdot u\|_{\mathcal{V}} = \sqrt{\|\lambda \nabla u_1\|_{L^2}^2 + \|\lambda \nabla u_2\|_{L^2}^2} = \sqrt{|\lambda|^2 \left(\|\nabla u_1\|_{L^2}^2 + \|\nabla u_2\|_{L^2}^2\right)} = |\lambda| \cdot \|u\|_{\mathcal{V}},\\
\text{ii)} \quad & \|u\|_{\mathcal{V}} = 0 \Rightarrow \|u\|_{\mathcal{V}}^2 = 0 \Rightarrow \|u\|_{L^2}^2 \leq 0 \Rightarrow \|u\|_{L^2}^2 = 0 \Rightarrow u \equiv 0; \quad u \equiv 0 \Rightarrow \|u\|_{\mathcal{V}} = 0,\\
\text{iii)} \quad & \|u + w\|_{\mathcal{V}}^2 = \|\nabla u_1 + \nabla w_1\|_{L^2}^2 + \|\nabla u_2 + \nabla w_2\|_{L^2}^2\\
& = \|\nabla u_1\|_{L^2}^2 + \|\nabla u_2\|_{L^2}^2 + 2 \cdot (\|\nabla u_1 \nabla w_1\|_{L^2} + \|\nabla u_2 \nabla w_2\|_{L^2}) + \|\nabla w_1\|_{L^2}^2 + \|\nabla w_2\|_{L^2}^2\\
& \leq \|u\|_{\mathcal{V}}^2 + 2 \cdot \sqrt{\|\nabla u_1 \nabla w_1\|_{L^2}^2 + \|\nabla u_2 \nabla w_2\|_{L^2}^2 + \|\nabla u_2 \nabla w_1\|_{L^2}^2 + \|\nabla u_1 \nabla w_2\|_{L^2}^2} + \|w\|_{\mathcal{V}}^2\\
& = \|u\|_{\mathcal{V}}^2 + 2\|u\|_{\mathcal{V}}\|w\|_{\mathcal{V}} + \|w\|_{\mathcal{V}}^2\\
& = (\|u\|_{\mathcal{V}} + \|w\|_{\mathcal{V}})^2\\
& \Rightarrow \|u + w\|_{\mathcal{V}} \leq \|u\|_{\mathcal{V}} + \|w\|_{\mathcal{V}};\\
\text{I)} \quad & \langle u, w\rangle_{\mathcal{V}} = \int_\Omega \langle \nabla u_1(x), \nabla w_1(x)\rangle + \langle \nabla u_2(x), \nabla w_2(x)\rangle\,\mathrm{d}x\\
& = \int_\Omega \langle \nabla w_1(x), \nabla u_1(x)\rangle + \langle \nabla w_2(x), \nabla u_2(x)\rangle\,\mathrm{d}x\\
& = \langle w, u\rangle_{\mathcal{V}},\\
\text{II)} \quad & \langle u, u\rangle_{\mathcal{V}} = \|u\|_{\mathcal{V}}^2 \geq 0,\\
\text{III)} \quad & \langle u, u\rangle_{\mathcal{V}} = 0 \iff \|u\|_{\mathcal{V}}^2 = 0 \iff u = 0.
\end{aligned}
$$

Therefore, $\langle\cdot,\cdot\rangle_{\mathcal{V}}$ is symmetric, positive definite and linear in both arguments (because $\langle\cdot,\cdot\rangle_{\mathbb{R}^2}$ is an inner product) and hence defines an inner product, thus $\|\cdot\|_{\mathcal{V}}$ is a norm on $\mathcal{V}$ as well.

We return to checking for existence and uniqueness of a solution u to (4.5.15) . In order to use Theorem 4.5.1, we assume $s_1 := \frac{\mu}{\det(F_0)} \in L^\infty(\Omega)$ and $B_0 \in L^\infty(\Omega, \mathrm{Sym}^+(2))$. Moreover, we consider the Cauchy stress tensor corresponding to the intermediate configuration (related to the energy in (4.5.16))

$$\sigma(F_0) = \frac{\mu}{\det(F_0)}(B_0 - \mathbb{1}) \tag{4.5.20}$$

and note that the bilinear form $\mathcal{L}$ in (4.5.11) is continuous on $\mathcal{V} \times \mathcal{V}$ and can be represented by

$$\mathcal{L}(u, w) = \int_\Omega \mathcal{A}(x; \nabla u(x), \nabla w(x))\,\mathrm{d}x \tag{4.5.21}$$

with $\mathcal{A}(x; \nabla u(x), \nabla w(x)) = -\operatorname{tr}\left[\sigma(F_0)(\nabla u)^T(\nabla w)^T\right] + \frac{\mu}{\det(F_0)} \operatorname{tr}\left[(\nabla u B_0 + B_0(\nabla u)^T)(\nabla w)^T\right]$. For $w = u$, we find

$$\mathcal{L}(u, u) = \int_\Omega \mathcal{A}(x; \nabla u, \nabla u)\, \mathrm{d}x. \tag{4.5.22}$$

Now we want to examine the bilinear form $\mathcal{L}$ for coercivity, i.e. we try to find a constant $\alpha > 0$ with

$$\mathcal{L}(u, u) \geq \alpha \cdot \|u\|_{\mathcal{V}}^2 \quad \forall\, u \in \mathcal{V}. \tag{4.5.23}$$

For this purpose, it is sufficient to show the uniform coercivity of $\mathcal{A}$, i.e. to find a constant $\beta > 0$ with

$$\mathcal{A}(x; X, X) \geq \beta \cdot \|X\|_{\mathbb{R}^{2\times 2}}^2 \quad \forall\, x \in \Omega. \tag{4.5.24}$$

First we formulate an equivalent statement to (4.5.46), for which without loss of generality[24] we assume $B_0 \in \operatorname{Sym}^+(2)$ to be a diagonal matrix

$$B_0 \coloneqq \begin{pmatrix} \gamma_1 & o \\ 0 & \gamma_2 \end{pmatrix} \tag{4.5.25}$$

with the eigenvalues γ_1 and γ_2. Then

$$\sigma(F_0) = \frac{\mu}{\det(F_0)} \cdot \begin{pmatrix} \gamma_1 - 1 & 0 \\ 0 & \gamma_2 - 1 \end{pmatrix}. \tag{4.5.26}$$

Moreover, we can divide ∇u in a symmetric and in a skew symmetric part $\nabla u = \operatorname{sym}(\nabla u) + \operatorname{skew}(\nabla u)$ with $\operatorname{sym}(\nabla u) \coloneqq \frac{1}{2}\left[\nabla u + (\nabla u)^T\right]$ and $\operatorname{skew}(\nabla u) \coloneqq \frac{1}{2}\left[\nabla u - (\nabla u)^T\right]$. For simpler calculations, we define

$$\operatorname{sym}(\nabla u) \coloneqq \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad \text{and} \quad \operatorname{skew}(\nabla u) \coloneqq \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix} \tag{4.5.27}$$

with $a, b, c, d \in \mathbb{R}$. We compute

$$(\operatorname{sym}(\nabla u))^2 = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & ab + bc \\ ab + bc & b^2 + c^2 \end{pmatrix}, \tag{4.5.28}$$

$$(\operatorname{skew}(\nabla u))^2 = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix} = \begin{pmatrix} -d^2 & 0 \\ 0 & -d^2 \end{pmatrix}, \tag{4.5.29}$$

$$\operatorname{sym}(\nabla u) \cdot \operatorname{skew}(\nabla u) = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix} = \begin{pmatrix} -bd & ad \\ -cd & bd \end{pmatrix}, \tag{4.5.30}$$

$$\operatorname{skew}(\nabla u) \cdot \operatorname{sym}(\nabla u) = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} bd & cd \\ -ad & -bd \end{pmatrix}. \tag{4.5.31}$$

[24] For any $B_0 \in \operatorname{Sym}^+(2)$ there exists an orthogonal matrix $Q \in \mathrm{O}(2)$ with $B_0 = Q \operatorname{diag}(\gamma_1(B_0), \gamma_2(B_0)) Q^T$. Thus

$$\begin{aligned}
\mathcal{A}(x; \nabla u, \nabla u) =& -\frac{\mu}{\det(F_0)} \left\langle Q(\operatorname{diag}(\gamma_1, \gamma_2) - \mathbb{1})Q^T, (\nabla u)(\nabla u) \right\rangle \\
&+ \frac{\mu}{\det(F_0)} \left[\left\langle \nabla u\, Q \operatorname{diag}(\gamma_1, \gamma_2) Q^T, \nabla u \right\rangle + \left\langle Q \operatorname{diag}(\gamma_1, \gamma_2) Q^T (\nabla u)^T, \nabla u \right\rangle \right] \\
=& \frac{\mu}{\det(F_0)} \left[-\langle \operatorname{diag}(\gamma_1, \gamma_2) - \mathbb{1}, (\nabla \widetilde{u})\,(\nabla \widetilde{u}) \rangle + \left\langle \operatorname{diag}(\gamma_1, \gamma_2), (\nabla \widetilde{u})^T (\nabla \widetilde{u})^T \right\rangle + \langle \operatorname{diag}(\gamma_1, \gamma_2), (\nabla \widetilde{u})\,(\nabla \widetilde{u}) \rangle \right] \\
=& \mathcal{A}(x; \nabla \widetilde{u}, \nabla \widetilde{u})
\end{aligned}$$

with $\nabla \widetilde{u} \coloneqq Q^T \nabla u\, Q$.

Therefore,

$$\mathcal{A}(x;\nabla u,\nabla u)=\mathcal{A}_1(x;\nabla u,\nabla u)+\mathcal{A}_2(x;\nabla u,\nabla u) \tag{4.5.32}$$

with

$$\begin{aligned}
\mathcal{A}_1(x;\nabla u,\nabla u) &\coloneqq -\operatorname{tr}\left[\sigma(F_0)\cdot(\nabla u)^T(\nabla u)^T\right]\\
&=-\operatorname{tr}\left[\sigma(F_0)\cdot\left(\frac{1}{2}(\nabla u)^2+\frac{1}{2}\left(\nabla u^T\right)^2-\frac{1}{2}(\nabla u)^2+\frac{1}{2}\left(\nabla u^T\right)^2\right)\right]\\
&=-\operatorname{tr}\left[\sigma(F_0)\cdot\left(\frac{1}{4}\left((\nabla u)^2+2\nabla u\nabla u^T+\left(\nabla u^T\right)^2\right)+\frac{1}{4}\left((\nabla u)^2-2\nabla u\,(\nabla u)^T+\left(\nabla u^T\right)^2\right)\right)\right]\\
&\quad+\operatorname{tr}\left[\sigma(F_0)\cdot\left(\frac{1}{4}\left((\nabla u)^2-\left(\nabla u^T\right)^2\right)+\frac{1}{4}\left((\nabla u)^2-\left(\nabla u^T\right)^2\right)\right)\right]\\
&=-\operatorname{tr}\left[\sigma(F_0)\left(\operatorname{sym}(\nabla u)^2+\operatorname{skew}(\nabla u)^2\right)\right]\\
&\quad+\operatorname{tr}\left[\sigma(F_0)\left(\operatorname{sym}(\nabla u)\operatorname{skew}(\nabla u)+\operatorname{skew}(\nabla u)\operatorname{sym}(\nabla u)\right)\right]\\
&=-\frac{\mu}{\sqrt{\gamma_1\gamma_2}}(\gamma_1-1)\left(a^2+b^2\right)-\frac{\mu}{\sqrt{\gamma_1\gamma_2}}(\gamma_2-1)\left(b^2+c^2\right)\\
&\quad+\frac{\mu}{\sqrt{\gamma_1\gamma_2}}(\gamma_1-1)d^2+\frac{\mu}{\sqrt{\gamma_1\gamma_2}}(\gamma_2-1)d^2\\
&=-\frac{\mu}{\sqrt{\gamma_1\gamma_2}}(\gamma_1-1)\left(a^2+b^2-d^2\right)-\frac{\mu}{\sqrt{\gamma_1\gamma_2}}(\gamma_2-1)\left(b^2+c^2-d^2\right)
\end{aligned} \tag{4.5.33}$$

and

$$\mathcal{A}_2(x;\nabla u,\nabla u)\coloneqq\frac{\mu}{\det(F_0)}\cdot\operatorname{tr}\left[\left(\nabla uB_0+B_0(\nabla u)^T\right)(\nabla u)^T\right]. \tag{4.5.34}$$

Since

$$\begin{aligned}
B_0(\nabla u)^T(\nabla u)^T &= B_0\left[\operatorname{sym}(\nabla u)^2+\operatorname{skew}(\nabla u)^2\right]\\
&\quad-B_0\left[\operatorname{sym}(\nabla u)\operatorname{skew}(\nabla u)+\operatorname{skew}(\nabla u)\operatorname{sym}(\nabla u)\right],
\end{aligned} \tag{4.5.35}$$

$$\begin{aligned}
\nabla u\,B_0(\nabla u)^T &= \left[\operatorname{sym}(\nabla u)+\operatorname{skew}(\nabla u)\right]B_0\left[(\operatorname{skew}(\nabla u))^T+(\operatorname{sym}(\nabla u))^T\right]\\
&=-\operatorname{sym}(\nabla u)B_0\operatorname{skew}(\nabla u)+\operatorname{sym}(\nabla u)B_0\operatorname{sym}(\nabla u)\\
&\quad-\operatorname{skew}(\nabla u)B_0\operatorname{skew}(\nabla u)+\operatorname{skew}(\nabla u)B_0\operatorname{sym}(\nabla u)\\
&=\left[\operatorname{sym}(\nabla u)B_0\operatorname{sym}(\nabla u)-\operatorname{skew}(\nabla u)B_0\operatorname{skew}(\nabla u)\right]\\
&\quad+\left[\operatorname{skew}(\nabla u)B_0\operatorname{sym}(\nabla u)-\operatorname{sym}(\nabla u)B_0\operatorname{skew}(\nabla u)\right],
\end{aligned} \tag{4.5.36}$$

with Definitions (4.5.25) and (4.5.27), we get

$$\begin{aligned}
B_0\cdot\operatorname{sym}(\nabla u)^2 &= \begin{pmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{pmatrix}\cdot\begin{pmatrix}a & b\\ b & c\end{pmatrix}\cdot\begin{pmatrix}a & b\\ b & c\end{pmatrix}\\
&=\begin{pmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{pmatrix}\cdot\begin{pmatrix}a^2+b^2 & ab+bc\\ ba+bc & b^2+c^2\end{pmatrix}\\
&=\begin{pmatrix}\gamma_1(a^2+b^2) & \gamma_1(ab+bc)\\ \gamma_2(ba+bc) & \gamma_2(b^2+c^2)\end{pmatrix},
\end{aligned} \tag{4.5.37}$$

$$\begin{aligned}
B_0\cdot\operatorname{skew}(\nabla u)^2 &= \begin{pmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{pmatrix}\cdot\begin{pmatrix}0 & d\\ -d & 0\end{pmatrix}\cdot\begin{pmatrix}0 & d\\ -d & 0\end{pmatrix}\\
&=\begin{pmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{pmatrix}\cdot\begin{pmatrix}-d^2 & 0\\ 0 & -d^2\end{pmatrix}\\
&=\begin{pmatrix}-\gamma_1d^2 & 0\\ 0 & -\gamma_2d^2\end{pmatrix},
\end{aligned} \tag{4.5.38}$$

$$\begin{aligned}
\operatorname{sym}(\nabla u)B_0\operatorname{sym}(\nabla u) &= \begin{pmatrix} a & b \\ b & c \end{pmatrix}\cdot\begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}\cdot\begin{pmatrix} a & b \\ b & c \end{pmatrix} \\
&= \begin{pmatrix} a\gamma_1 & b\gamma_2 \\ b\gamma_1 & c\gamma_2 \end{pmatrix}\cdot\begin{pmatrix} a & b \\ b & c \end{pmatrix} \\
&= \begin{pmatrix} a^2\gamma_1+b^2\gamma_2 & ab\gamma_1+bc\gamma_2 \\ ab\gamma_1+bc\gamma_2 & b^2\gamma_1+c^2\gamma_2 \end{pmatrix}, && (4.5.39)\\
\operatorname{skew}(\nabla u)B_0\operatorname{skew}(\nabla u) &= \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}\cdot\begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}\cdot\begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & d\gamma_2 \\ -d\gamma_1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix} \\
&= \begin{pmatrix} -d^2\gamma_2 & 0 \\ 0 & -d^2\gamma_1 \end{pmatrix}, && (4.5.40)\\
\operatorname{skew}(\nabla u)B_0\operatorname{sym}(\nabla u) &= \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}\cdot\begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}\cdot\begin{pmatrix} a & b \\ b & c \end{pmatrix} \\
&= \begin{pmatrix} 0 & d\gamma_2 \\ -d\gamma_1 & 0 \end{pmatrix}\cdot\begin{pmatrix} a & b \\ b & c \end{pmatrix} \\
&= \begin{pmatrix} db\gamma_2 & dc\gamma_2 \\ -ad\gamma_1 & -db\gamma_1 \end{pmatrix}, && (4.5.41)\\
\operatorname{sym}(\nabla u)B_0\operatorname{skew}(\nabla u) &= \begin{pmatrix} a & b \\ b & c \end{pmatrix}\cdot\begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}\cdot\begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix} \\
&= \begin{pmatrix} a & b \\ b & c \end{pmatrix}\cdot\begin{pmatrix} 0 & \gamma_1 d \\ -\gamma_2 d & 0 \end{pmatrix} \\
&= \begin{pmatrix} -bd\gamma_2 & ad\gamma_1 \\ -cd\gamma_2 & bd\gamma_1 \end{pmatrix}. && (4.5.42)
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathcal{A}_2(x;\nabla u,\nabla u) &= \frac{\mu}{\det(F_0)}\left(a^2\gamma_1+b^2\gamma_2+b^2\gamma_1+c^2\gamma_2+d^2\gamma_2+d^2\gamma_1+db\gamma_2-db\gamma_1+db\gamma_2-bd\gamma_1\right)\\
&\quad+\frac{\mu}{\det(F_0)}\left(\gamma_1(a^2+b^2)+\gamma_2(b^2+c^2)-\gamma_1 d^2-\gamma_2 d^2\right)\\
&= 2\frac{\mu}{\det(F_0)}\left[a^2\gamma_1+b^2(\gamma_1+\gamma_2)+c^2\gamma_2+db(\gamma_2-\gamma_1)\right] && (4.5.43)
\end{aligned}$$

and thus,

$$\mathcal{A}(x;\nabla u,\nabla u) = X^T A(x) X \tag{4.5.44}$$

with $X := (a,c,b,d)^T$, $s_1 := \frac{\mu}{\sqrt{\gamma_1\gamma_2}}$ and

$$\begin{aligned}
A(x) &:= \begin{pmatrix} -s_1(\gamma_1-1)+2s_1\gamma_1 & 0 & 0 & 0 \\ 0 & -s_1(\gamma_2-1)+2s_1\gamma_2 & 0 & 0 \\ 0 & 0 & 2s_1\operatorname{tr}(B_0)-s_1\operatorname{tr}(\sigma(F_0)) & s_1(\gamma_2-\gamma_1) \\ 0 & 0 & s_1(\gamma_2-\gamma_1) & s_1\operatorname{tr}(\sigma(F_0)) \end{pmatrix}\\
&= s_1\cdot\begin{pmatrix} -(\gamma_1-1)+2\gamma_1 & 0 & 0 & 0 \\ 0 & -(\gamma_2-1)+2\gamma_2 & 0 & 0 \\ 0 & 0 & 2\operatorname{tr}(B_0)-\operatorname{tr}(\sigma(F_0)) & (\gamma_2-\gamma_1) \\ 0 & 0 & (\gamma_2-\gamma_1) & \operatorname{tr}(\sigma(F_0)) \end{pmatrix}.
\end{aligned}\tag{4.5.45}$$

The positive semidefiniteness of the matrix $A - \beta\,\mathbb{1}$ implies the coercivity of $\mathcal{A}$,[25] because for $X := (a, c, b, d)^T \in \mathbb{R}^4$ and $\nabla u = \begin{pmatrix} a & b+d \\ b-d & c \end{pmatrix}$ we get

$$\begin{aligned}
& \langle (A - \beta\,\mathbb{1})\ X, X\rangle \geq 0 \\
\iff & \langle A\,X, X\rangle \geq \beta \cdot \langle X, X\rangle \\
\iff & \langle A\,X, X\rangle \geq \beta \cdot \|X\|^2 \\
\iff & \langle A\,X, X\rangle \geq \beta \cdot \left(a^2 + c^2 + b^2 + d^2\right) \\
\iff & \mathcal{A}(x; \nabla u, \nabla u) \geq \beta \cdot \left(|\frac{\partial u_1}{\partial x_1}|^2 + |\frac{\partial u_2}{\partial x_2}|^2 + \frac{1}{4}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)^2 + \frac{1}{4}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right)^2 \right) \\
& \qquad = \beta \cdot \left(|\frac{\partial u_1}{\partial x_1}|^2 + |\frac{\partial u_2}{\partial x_2}|^2 + \frac{1}{2}|\frac{\partial u_1}{\partial x_2}|^2 + \frac{1}{2}|\frac{\partial u_2}{\partial x_1}|^2 \right) \\
& \qquad \geq \frac{\beta}{2} \cdot \left(2\,|\frac{\partial u_1}{\partial x_1}|^2 + 2\,|\frac{\partial u_2}{\partial x_2}|^2 + |\frac{\partial u_1}{\partial x_2}|^2 + |\frac{\partial u_2}{\partial x_1}|^2 \right) \\
& \qquad \geq \frac{\beta}{2} \cdot \|\nabla u\|^2.
\end{aligned} \tag{4.5.46}$$

So, in order to determine whether the matrix $A(x) - \beta\,\mathbb{1}$ is positive semidefinite, we fix a constant $k > 0$ and take $\varepsilon := s_1$. Moreover, we define functions $a_0 = a_0(x)$ and $b_0 = b_0(x)$ by

$$a_0 := -2s_1\sqrt{\det(B_0)}, \quad b_0 := 2s_1\sqrt{\det(B_0)}. \tag{4.5.47}$$

In addition, we assume $\det(B_0) \geq k > 0$ and get the inequality

$$b_0 - a_0 = 4\,s_1\sqrt{\det(B_0)} = 4\,\varepsilon\sqrt{\det(B_0)} \geq 4\,\varepsilon\sqrt{k}. \tag{4.5.48}$$

Hence, the interval defined by $[a_0(x), b_0(x)]$ is not empty.

Now we formulate the theorem which finally ensures the coercivity of the bilinear form $\mathcal{L}$ [2].

Theorem 4.5.2. *Assume* $\det(B_0) \geq k$. *Let* $\beta > 0$ *and* $\overline{d} := \sup_{x\in\Omega}\left(\frac{\operatorname{tr}(B_0)}{\sqrt{\det(B_0)}}\right)$ *such that*

$$\beta\,\overline{d} < 2\,\varepsilon\,\sqrt{k} \tag{4.5.49}$$

and suppose that $s_1 = \frac{\mu}{\det(F_0)}$ *satisfies the condition*

$$\boxed{a_0(x) + \beta\,\overline{d} < -2s_1 < b_0(x) - \beta\overline{d} \quad \forall x \in \Omega}. \tag{4.5.50}$$

Then there exists a constant $\alpha_0 = \alpha_0(s_1, \beta) \geq 0$ *such that the matrix* $A(x) - \beta\,\mathbb{1}$ *is positive semidefinite in* Ω, *provided that* $\frac{\mu}{\det(F_0)} \geq \alpha_0$.

[25] In [2] Liu assumes the coercivity of $\mathcal{A}(x; \nabla u, \nabla u)$ to be equivalent to the positive semidefiniteness of $A(x) - \beta\,\mathbb{1}$ for all $x \in \Omega$. This statement is given closer consideration in the appendix.

Proof. First, the Hadamard[26] inequality implies

$$
\begin{aligned}
2\cdot\sqrt{\det(B_0)} &= 2\cdot\sqrt{\det(F_0)\det(F_0)^T} = 2\cdot\sqrt{|\det(F_0)|^2} \le 2\cdot\prod_{j=1}^{2}\sqrt{\|F_0 e_j\|^2}\\
&= 2\cdot\prod_{j=1}^{2}\sqrt{\sum_{i=1}^{2}|f_{ij}|^2} = 2\cdot\prod_{j=1}^{2}\sqrt{|f_{1j}|^2+|f_{2j}|^2}\\
&= 2\cdot\sqrt{|f_{11}|^2+|f_{21}|^2}\cdot\sqrt{|f_{12}|^2+|f_{22}|^2}\\
&\le \left(\sqrt{|f_{11}|^2+|f_{21}|^2}\right)^2+\left(\sqrt{|f_{12}|^2+|f_{22}|^2}\right)^2\\
&= |f_{11}|^2+|f_{12}|^2+|f_{21}|^2+|f_{22}|^2\\
&= \langle F_0,F_0\rangle = \|F_0\|^2 = \mathrm{tr}(B_0) \qquad (4.5.51)
\end{aligned}
$$

where $F_0 e_j$ denotes the column vectors of F_0 and f_{ij} denote the entries of the matrix F_0. So $\overline{d} = \sup_{x\in\Omega}\left(\frac{\mathrm{tr}(B_0)}{\sqrt{\det(B_0)}}\right) \ge 2$, and if β satisfies (4.5.49), we get

$$\beta < \varepsilon\sqrt{k}. \qquad (4.5.52)$$

Now we take a look to the nonzero entries of $A(x)$:

$$
\begin{cases}
A_{11} = -s_1(\gamma_1-1)+2s_1\gamma_1,\\
A_{22} = -s_1(\gamma_2-1)+2s_1\gamma_2,\\
A_{33} = 2s_1\,\mathrm{tr}(B_0)-\mathrm{tr}(\sigma(F_0)),\\
A_{34} = s_1(\gamma_2-\gamma_1),\\
A_{44} = \mathrm{tr}(\sigma(F_0)).
\end{cases}
$$

A matrix is positive semidefinite if and only if all leading principal minors are bigger than zero or equal to zero. This statement is equivalent to

$$\min\{A_{11}-\beta,(A_{11}-\beta)(A_{22}-\beta),A_{33}-\beta,(A_{33}-\beta)(A_{44}-\beta)-A_{34}^2\} \ge 0. \qquad (4.5.53)$$

Without loss of generality we assume that $\gamma_1 \ge \gamma_2$. Then

$$A_{11}-\beta = -s_1(\gamma_1-1)+2s_1\gamma_1-\beta = (1+\gamma_1)s_1-\beta. \qquad (4.5.54)$$

So $A_{11}-\beta \ge 0$ holds if and only if

$$\boxed{(1+\gamma_1)s_1 \ge \beta}. \qquad (4.5.55)$$

The required condition (4.5.50) implies this statement, because

$$
\begin{aligned}
2(1+\gamma_1)s_1 &= 2s_1+2s_1\gamma_1 = 2s_1+2s_1\sqrt{\gamma_1^2} \ge 2s_1+2s_1\sqrt{\gamma_1\gamma_2}\\
&= 2s_1+2s_1\sqrt{\det(B_0)} = 2s_1+b_0 > \beta\overline{d} \ge 2\beta.
\end{aligned}
$$

[26] Let $M \in \mathbb{R}^{n\times n}$ and $Me_1, Me_2, ..., Me_n$ denote the column vectors of the matrix M. Then $|\det(M)| \le \prod_{i=1}^{n}\|Me_i\|$ and $(\det(M))^2 \le \prod_{i=1}^{n}\|Me_i\|^2$, with the Euclidean norm $\|\cdot\|$; e.g. $M \in \mathbb{R}^{2\times 2}$, then $|\det(M)| \le \|Me_1\|\cdot\|Me_2\|$.

Moreover,

$$\begin{aligned}(A_{11}-\beta)(A_{22}-\beta) &= [-s_1(\gamma_1-1)+2s_1\gamma_1-\beta]\,[-s_1(\gamma_2-1)+2s_1\gamma_2-\beta]\\ &= s_1^2(\gamma_1-1)(\gamma_2-1)-2s_1^2(\gamma_1-1)\gamma_2+\beta s_1(\gamma_1-1)-2s_1^2\gamma_1(\gamma_2-1)\\ &\quad +4s_1^2\gamma_1\gamma_2-2s_1\beta\gamma_1+\beta s_1(\gamma_2-1)-2s_1\beta\gamma_2+\beta^2\\ &= s_1^2(\gamma_2\gamma_1+\gamma_1+\gamma_2+1)+\beta s_1(-\gamma_1-\gamma_2-2)+\beta^2\\ &= \left[\beta+\frac{s_1}{2}(-\gamma_1-\gamma_2-2)\right]^2-\frac{s_1^2}{4}(-\gamma_1-\gamma_2-2)^2+s_1^2(\gamma_2\gamma_1+\gamma_1+\gamma_2+1)\\ &> \left[\beta+\frac{s_1}{2}(-\gamma_1-\gamma_2-2)\right]^2-\frac{s_1^2}{4}(-\gamma_1-\gamma_2-2)^2\\ &\geq 0\end{aligned}$$

if and only if

$$\left[\beta+\frac{s_1}{2}(-\gamma_1-\gamma_2-2)\right]^2 \geq \frac{s_1^2}{4}(-\gamma_1-\gamma_2-2)^2. \tag{4.5.56}$$

This inequality is satisfied if and only if

$$\beta+\frac{s_1}{2}(-\gamma_1-\gamma_2-2) \geq \frac{s_1}{2}(-\gamma_1-\gamma_2-2) \tag{4.5.57}$$

or

$$\beta+\frac{s_1}{2}(-\gamma_1-\gamma_2-2) \leq -\frac{s_1}{2}(-\gamma_1-\gamma_2-2). \tag{4.5.58}$$

Inequalities (4.5.57) and (4.5.58) can be restated equivalently as

$$\beta \geq 0 \tag{4.5.59}$$

and

$$\begin{aligned} &\beta \leq -s_1(-\gamma_1-\gamma_2-2)\\ \iff\ &-\beta \geq -s_1(\gamma_1+\gamma_2+2)\\ \iff\ &-\beta \geq -2s_1-s_1\,\mathrm{tr}(B_0),\end{aligned} \tag{4.5.60}$$

respectively; (4.5.59) is true by assumption and (4.5.60) implies $(A_{11}-\beta)(A_{22}-\beta)\geq 0$ if

$$\boxed{-2s_1 < -\beta + s_1\,\mathrm{tr}(B_0)}\,. \tag{4.5.61}$$

Furthermore,

$$\begin{aligned}(A_{33}-\beta) &= 2s_1\,\mathrm{tr}(B_0)-\mathrm{tr}(\sigma(F_0))-\beta = 2s_1(\gamma_1+\gamma_2)-s_1(\gamma_1+\gamma_2-2)-\beta\\ &= s_1(\gamma_1+\gamma_2)+2s_1-\beta.\end{aligned}$$

Hence, $(A_{33}-\beta)\geq 0$ if

$$\boxed{-2s_1 \leq -\beta + s_1\,\mathrm{tr}(B_0)}\,. \tag{4.5.62}$$

In addition,

$$(A_{44}-\beta) = \mathrm{tr}(\sigma(F_0))-\beta = s_1(\gamma_1+\gamma_2-2)-\beta = -2s_1+s_1\,\mathrm{tr}(B_0)-\beta \geq 0$$

if and only if

$$\boxed{-2s_1 \geq \beta - s_1\,\mathrm{tr}(B_0)}\,. \tag{4.5.63}$$

Thus, the conditions (4.5.61), (4.5.62) and (4.5.63) can be expressed together by

$$\boxed{\beta - s_1 \operatorname{tr}(B_0) \leq -2s_1 \leq -\beta + s_1 \operatorname{tr}(B_0)}, \tag{4.5.64}$$

which is similar to Liu's condition in [2, eq.(5.13)-(5.14)].
Moreover, the interval

$$[\beta - s_1 \operatorname{tr}(B_0), -\beta + s_1 \operatorname{tr}(B_0)] \tag{4.5.65}$$

is not empty, because

$$\begin{aligned}
-\beta + s_1 \operatorname{tr}(B_0) - \beta + s_1 \operatorname{tr}(B_0) &= -2\,\beta + 2s_1 \operatorname{tr}(B_0) \geq 2\,\varepsilon \operatorname{tr}(B_0) - 2\,\beta \\
&\geq 2\,\varepsilon \cdot 2\sqrt{k} - 2\,\beta = 4\,\varepsilon\sqrt{k} - 2\,\beta \\
&> 4\,\varepsilon\sqrt{k} - 2\,\varepsilon\sqrt{k} = 2\,\varepsilon\sqrt{k} \\
&= 0.
\end{aligned}$$

Furthermore, for the calculation of $(A_{33} - \beta)(A_{44} - \beta) - A_{34}^2$ we define $f_1 := s_1\gamma_1$ and $f_2 := s_1\gamma_2$. Then,

$$\begin{aligned}
(A_{33} - \beta)(A_{44} - \beta) - A_{34}^2 &= (f_1 + f_2 + 2s_1 - \beta)\,(f_1 + f_2 - 2s_1 - \beta) - (f_2 - f_1)^2 \\
&= f_1^2 + f_1 f_2 - 2s_1 f_1 - \beta f_1 + f_2 f_1 + f_2^2 - 2s_1 f_2 - \beta f_2 + 2s_1 f_1 + 2s_1 f_2 \\
&\quad - 4s_1^2 - 2s_1\beta - \beta f_1 - \beta f_2 + 2\,\beta s_1 + \beta^2 - f_2^2 + 2f_2 f_1 - f_1^2 \\
&= 4f_1 f_2 - 2\,\beta f_1 - 2\,\beta f_2 - 4s_1^2 + \beta^2 \\
&= -4s_1^2 + 4f_1 f_2 - 2\,\beta(f_1 + f_2) + \beta^2 \\
&\geq 0
\end{aligned}$$

if and only if

$$\begin{aligned}
&4s_1^2 \leq 4f_1 f_2 - 2\,\beta(f_1 + f_2) + \beta^2 \\
\iff\ & -2s_1 \leq 2\sqrt{f_1 f_2 - \frac{\beta}{2}(f_1 + f_2) + \frac{\beta^2}{4}} \quad \vee \quad -2s_1 \geq -2\sqrt{f_1 f_2 - \frac{\beta}{2}(f_1 + f_2) + \frac{\beta^2}{4}}.
\end{aligned} \tag{4.5.66}$$

So for $a_\beta := -2\sqrt{f_1 f_2 - \frac{\beta}{2}(f_1 + f_2) + \frac{\beta^2}{4}}$ and $b_\beta := 2\sqrt{f_1 f_2 - \frac{\beta}{2}(f_1 + f_2) + \frac{\beta^2}{4}}$, we find that $(A_{33} - \beta)(A_{44} - \beta) - A_{34}^2 \geq 0$ if and only if

$$\boxed{a_\beta \leq -2s_1 \leq b_\beta}. \tag{4.5.67}$$

The last step is to show whether the derived conditions are true under the given condition in (4.5.50) or not. Hence, we have to verify, if

$$\begin{cases}
\beta - s_1 \operatorname{tr}(B_0) \leq a_0 + \beta\overline{d}, \\
b_0 - \beta\overline{d} \leq -\beta + s_1 \operatorname{tr}(B_0), \\
a_\beta \leq a_0 + \beta\overline{d}, \\
b_0 - \beta\overline{d} \leq b_\beta.
\end{cases} \tag{4.5.68}$$

If these inequalities apply, then all leading main minors of $A - \beta\,\mathbb{1}$ are bigger than zero or equal to zero and therefore, there exists a constant α_0 which depends on s_1 and β, such that $A(x) - \beta\,\mathbb{1}$ is positive semidefinite in Ω.
Due to (4.5.51), we get

$$\beta - s_1 \operatorname{tr}(B_0) \leq \beta - 2s_1\sqrt{\det(B_0)} \leq \overline{d}\beta - 2s_1\sqrt{\det(B_0)} = a_0 + \overline{d}\beta \tag{4.5.69}$$

and

$$b_0 - \beta\overline{d} = 2s_1\sqrt{\det(B_0)} - \beta\overline{d} \leq s_1 \operatorname{tr}(B_0) - \beta\overline{d} \leq s_1 \operatorname{tr}(B_0) - 2\,\beta \leq s_1 \operatorname{tr}(B_0) - \beta. \tag{4.5.70}$$

Moreover,

$$\begin{aligned} a_\beta &= -2\sqrt{s_1^2\det(B_0) - \frac{\beta}{2}s_1\operatorname{tr}(B_0) + \frac{\beta^2}{4}} \leq -2\sqrt{s_1^2\det(B_0) - \frac{\beta}{2}s_1\operatorname{tr}(B_0)} \\ &= -2s_1\sqrt{\det(B_0)}\sqrt{1 - \frac{\beta}{2}\frac{\operatorname{tr}(B_0)}{s_1\det(B_0)}} \overset{(*)}{\leq} -2s_1\sqrt{\det(B_0)} + \beta s_1\sqrt{\det(B_0)}\,\frac{\operatorname{tr}(B_0)}{s_1\det(B_0)} \\ &= -2s_1\sqrt{\det(B_0)} + \beta\cdot\frac{\operatorname{tr}(B_0)}{\sqrt{\det(B_0)}} \leq a_0 + \beta\overline{d}, \end{aligned} \tag{4.5.71}$$

with $(*)\ \sqrt{1-x} \geq 1 - x$ for all $x \in [0,1]$, and analogously

$$\begin{aligned} b_\beta &= 2\sqrt{s_1^2\det(B_0) - \frac{\beta}{2}s_1\operatorname{tr}(B_0) + \frac{\beta^2}{4}} \geq 2\sqrt{s_1^2\det(B_0) - \frac{\beta}{2}s_1\operatorname{tr}(B_0)} \\ &= 2s_1\sqrt{\det(B_0)}\sqrt{1 - \frac{\beta}{2}\frac{\operatorname{tr}(B_0)}{s_1\det(B_0)}} \geq 2s_1\sqrt{\det(B_0)} - \beta s_1\sqrt{\det(B_0)}\frac{\operatorname{tr}(B_0)}{s_1\det(B_0)} \\ &= b_0 - \beta\frac{\operatorname{tr}(B_0)}{\sqrt{\det(B_0)}} \geq b_0 - \beta\overline{d}. \end{aligned} \tag{4.5.72}$$

Therefore, the inequalities hold, which concludes the proof. ■

So if condition (4.5.50) applies, then $A(x) - \beta\,\mathbb{1}$ is positive semidefinite and this in turn implies the coercivity of $\mathcal{A}$.
Thus $\mathcal{A}$ is coercive and together with (4.5.46) the following considerations apply:

$$\begin{aligned} \mathcal{L}(u,u) &= \int_\Omega \mathcal{A}(x;\nabla u,\nabla u)\,\mathrm{d}V \geq \frac{\beta}{2}\int_\Omega \|\nabla u\|^2\,\mathrm{d}V = \frac{\beta}{2}\int_\Omega \|\nabla u_1\|^2_{\mathbb{R}^2} + \|\nabla u_2\|^2_{\mathbb{R}^2}\,\mathrm{d}V \\ &= \frac{\beta}{2}\left(\|\nabla u_1\|^2_{L^2} + \|\nabla u_2\|^2_{L^2}\right) = \frac{\beta}{2}\|u\|^2_{\mathcal{V}}. \end{aligned} \tag{4.5.73}$$

In conclusion,

$$\mathcal{L}(u,u) \geq \alpha\,\|u\|^2_{\mathcal{V}} \tag{4.5.74}$$

with $\alpha := \frac{\beta}{2} > 0$. Thus the bilinear form $\mathcal{L}$ is coercive. Now we want to use the Lemma of Lax-Milgram to get a unique solution $u \in \mathcal{V}$ for the boundary value problem in (4.5.6). Therefore, we have to show that there exists a constant $\delta > 0$ such that $|\mathcal{L}(u,w)| \leq \delta\cdot\|u\|_{\mathcal{V}}\,\|w\|_{\mathcal{V}}$ for all $u,\ w \in \mathcal{V}$. For $B_0 \in L^\infty(\Omega, \operatorname{Sym}^+(2))$, $\frac{\mu}{\det(F_0)} \in L^\infty(\Omega)$ and by means of Cauchy-Schwarz[27],

[27]If v is a vector space with the inner product $\langle\,.\,,\,.\,\rangle_V$ then

$$|\langle x,y\rangle|^2 \leq \langle x,x\rangle\cdot\langle y,y\rangle \quad \forall\, x,y \in V.$$

triangle and Hölder[28] inequalities, we calculate

$$
\begin{aligned}
|\mathcal{L}(u,w)| &= |\int_\Omega \frac{\mu}{\det(F_0)} \operatorname{tr}\left[(\nabla u)^T(\nabla w)^T + \nabla u\, B_0\,(\nabla w)^T\right]\,\mathrm{d}V| \\
&= |\int_\Omega \frac{\mu}{\det(F_0)} \left\langle(\nabla u)^T + \nabla u\, B_0, \nabla w\right\rangle\,\mathrm{d}V| \\
&\leq \int_\Omega |\frac{\mu}{\det(F_0)}||\left\langle(\nabla u)^T + \nabla u\, B_0, \nabla w\right\rangle|\,\mathrm{d}V \\
&\leq \int_\Omega |\frac{\mu}{\det(F_0)}|\|(\nabla u)^T + \nabla u\, B_0\| \cdot \|\nabla w\|\,\mathrm{d}V \\
&\leq \int_\Omega |\frac{\mu}{\det(F_0)}|\,(\|\nabla u\| + \|\nabla u\, B_0\|) \cdot \|\nabla w\|\,\mathrm{d}V \\
&\leq c_1 \int_\Omega (\|\nabla u\| + \|\nabla u\|\|B_0\|) \cdot \|\nabla w\|\,\mathrm{d}V \\
&= c_1 \int_\Omega (1 + \|B_0\|)\,\|\nabla u\|\|\nabla w\|\,\mathrm{d}V \\
&\leq c_2 \int_\Omega \|\nabla u\|\|\nabla w\|\,\mathrm{d}V \\
&\leq c_2 \left(\int_\Omega \|\nabla u\|^2\,\mathrm{d}V\right)^{\frac{1}{2}} \cdot \left(\int_\Omega \|\nabla w\|^2\,\mathrm{d}V\right)^{\frac{1}{2}} \\
&= c_2 \left(\int_\Omega \|\nabla u_1\|^2 + \|\nabla u_2\|^2\,\mathrm{d}V\right)^{\frac{1}{2}} \cdot \left(\int_\Omega \|\nabla w_1\|^2 + \|\nabla w_2\|^2\,\mathrm{d}V\right)^{\frac{1}{2}} \\
&= c_2\,(\|\nabla u_1\|_{L^2} + \|\nabla u_2\|_{L^2})^{\frac{1}{2}} \cdot (\|\nabla w_1\|_{L^2} + \|\nabla w_2\|_{L^2})^{\frac{1}{2}} \\
&= c_2\,\|u\|_{\mathcal{V}} \cdot \|w\|_{\mathcal{V}}
\end{aligned}
\tag{4.5.75}
$$

with constants c_1, $c_2 > 0$. In conclusion,

$$
|\mathcal{L}(u,w)| \leq \delta \cdot \|u\|_{\mathcal{V}} \cdot \|w\|_{\mathcal{V}} \tag{4.5.76}
$$

with $\delta \coloneqq c_2 > 0$. Thus, together with the Lemma of Lax-Milgram[29] we get a unique solution $u \in \mathcal{V}$ for the boundary value problem

$$
\begin{cases}
-\operatorname{Div}\left[\sigma(F_0) + \frac{\mu}{\det(F_0)}\left(HB_0 + B_0H^T\right) - \sigma(F_0)\cdot H^T\right] &= 0 \quad \text{on} \quad \Omega, \\
\left[\sigma(F_0) + \frac{\mu}{\det(F_0)}\left(HB_0 + B_0H^T\right) - \sigma(F_0)\cdot H^T\right]\cdot \vec{n_k} &= f \quad \text{on} \quad \Gamma_1 \\
\langle u, \vec{n_k}\rangle &= 0 \quad \text{on} \quad \Gamma_2 \\
u &= 0 \quad \text{on} \quad \Gamma_3,
\end{cases}
\tag{4.5.77}
$$

stated in (4.5.6).

Liu's argument for the existence and uniqueness of a solution of this linearized boundary value problem seems to be conclusive at first glance, but at this point we have to take a closer look at the condition

$$
\boxed{a_0(x) + \beta\,\overline{d} < -2s_1 < b_0(x) - \beta\overline{d} \quad \forall x \in \Omega}
$$

[28]Let $1 \leq p \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and let $f \in L^p(\Omega)$, $g \in L^q(\Omega)$. Then $fg \in L^1(\Omega)$ and

$$
\|fg\|_{L^1} \leq \|f\|_{L^p}\,\|g\|_{L^q}.
$$

[29]See Section 2.3.2.

in (4.5.50). If B_0 is constant, i.e. the eigenvalues γ_1 and γ_2 do not depend on $x \in \Omega$, then the above condition is equivalent to

$$\begin{array}{rcccc} & -2s_1\sqrt{\det(B_0)} + \beta\,\frac{\operatorname{tr}(B_0)}{\sqrt{\det(B_0)}} & < & -2s_1 & < & 2s_1\sqrt{\det(B_0)} - \beta\,\frac{\operatorname{tr}(B_0)}{\sqrt{\det(B_0)}} \\ \Longleftrightarrow & -2\frac{\mu}{\det(F_0)}\sqrt{\det(B_0)} + \beta\,\frac{\gamma_1+\gamma_2}{\sqrt{\gamma_1\gamma_2}} & < & -2\frac{\mu}{\det(F_0)} & < & 2\frac{\mu}{\det(F_0)}\sqrt{\det(B_0)} - \beta\,\frac{\gamma_1+\gamma_2}{\sqrt{\gamma_1\gamma_2}} \\ \Longleftrightarrow & -2\,\mu + \beta\,\frac{\gamma_1+\gamma_2}{\sqrt{\gamma_1\gamma_2}} & < & -2\frac{\mu}{\det(F_0)} & < & 2\,\mu - \beta\,\frac{\gamma_1+\gamma_2}{\sqrt{\gamma_1\gamma_2}} \\ \Longleftrightarrow & -2\,\mu + \beta\,\frac{\gamma_1+\gamma_2}{\sqrt{\gamma_1\gamma_2}} & < & -2\frac{\mu}{\sqrt{\gamma_1\gamma_2}} & < & 2\,\mu - \beta\,\frac{\gamma_1+\gamma_2}{\sqrt{\gamma_1\gamma_2}}, \end{array}$$

i.e.

$$\begin{array}{rccc} & \beta\,(\gamma_1+\gamma_2) & < & -2\,\mu + 2\,\mu\sqrt{\gamma_1\gamma_2} \\ \Longleftrightarrow & \beta & < & \frac{2\,\mu\left(\sqrt{\gamma_1\gamma_2}-1\right)}{\gamma_1+\gamma_2} \end{array} \tag{4.5.78}$$

and

$$\begin{array}{rccc} & \beta\,(\gamma_1+\gamma_2) & < & 2\,\mu\sqrt{\gamma_1\gamma_2} + 2\,\mu \\ \Longleftrightarrow & \beta & < & \frac{2\,\mu\left(\sqrt{\gamma_1\gamma_2}+1\right)}{\gamma_1+\gamma_2}\,. \end{array} \tag{4.5.79}$$

Let $B_0 := \lambda \cdot \mathbb{1}$, $\lambda > 0$. Then the above conditions are in turn equivalent to

$$\beta < \frac{2\,\mu\,(\lambda-1)}{2\lambda} = \mu\left(1 - \frac{1}{\lambda}\right) \tag{4.5.80}$$

and

$$\beta < \frac{2\,\mu\,(\lambda+1)}{2\lambda} = \mu\left(1 + \frac{1}{\lambda}\right). \tag{4.5.81}$$

Thus the first condition only applies for $\lambda > 1$. Otherwise, if $\lambda \leq 1$ then β is strictly negative. This would be a contradiction to the assumption $\beta > 0$, which means that Liu excludes both compression and the identity. However, homogeneous compressional states as well as the identity play an important role, for example, in the engineering sciences.

5 Deficiencies of Liu's argumentation

Now we want to show explicitly that Liu's pointwise considerations are not sufficient. For this we focus on the *Saint-Venant-Kirchhoff* type energy, which is given by

$$\widehat{W}(F) := \|F^T F - \mathbb{1}\|^2. \tag{5.0.1}$$

With the chain-rule, and for an arbitrary direction $H \in \mathbb{R}^{3\times 3}$, we get

$$D_F \widehat{W}(F).[H] = 2 \cdot \left\langle F^T F - \mathbb{1}, F^T H + H^T F \right\rangle. \tag{5.0.2}$$

Hence

$$\begin{aligned} D_F^2 \widehat{W}(F).[H,H] &= 2 \cdot \left\langle F^T H + H^T F, F^T H + H^T F \right\rangle + 4 \cdot \left\langle F^T F - \mathbb{1}, H^T H \right\rangle \\ &= 2 \cdot \left\langle 2 \cdot \operatorname{sym}(F^T H), 2 \cdot \operatorname{sym}(F^T H) \right\rangle + 4 \cdot \left\langle F^T F - \mathbb{1}, H^T H \right\rangle \\ &= 8 \cdot \|\operatorname{sym}(F^T H)\|^2 + 4 \cdot \left\langle F^T F - \mathbb{1}, H^T H \right\rangle. \end{aligned} \tag{5.0.3}$$

First of all, we use Liu's approach to show the coercivity for the given energy function $\widehat{W}$. We have seen that the assumptions in Liu's theorem 4.5.2 exclude compression and the identity, so we assume $F = \lambda \cdot \mathbb{1}$ with $\lambda > 1$. Thus, for $Q \in \mathrm{O}(3)$ we obtain

$$\begin{aligned} D_F^2 \widehat{W}(F).[H,H] &\geq 4 \cdot \left\langle F F^T - \mathbb{1}, H^T H \right\rangle = 4 \cdot \left\langle Q^T \operatorname{diag}\left(F F^T - \mathbb{1}\right) Q, H^T H \right\rangle \\ &= 4 \cdot \left\langle \operatorname{diag}\left(F F^T - \mathbb{1}\right) Q, Q H^T H \right\rangle = 4 \cdot \left\langle \operatorname{diag}\left(F F^T - \mathbb{1}\right) Q H^T, Q H^T \right\rangle \\ &\geq 4 \cdot \lambda_{\min}\left(F F^T - \mathbb{1}\right) \|QH\|^2 = 4 \cdot \lambda_{\min}\left(F F^T - \mathbb{1}\right) \|H\|^2 \\ &= 4 \cdot \left(\lambda^2 - 1\right) \|H\|^2. \end{aligned} \tag{5.0.4}$$

Consequently,

$$\begin{aligned} \mathcal{L}(u,u) &= \int_\Omega D_F^2 \widehat{W}(F)[\nabla u, \nabla u] \, \mathrm{d}x \geq 4 \cdot \left(\lambda^2 - 1\right) \|\nabla u\|_{L^2}^2 \\ &= c \cdot \|\nabla u\|_{L^2}^2 \end{aligned} \tag{5.0.5}$$

with $c = 4 \cdot \left(\lambda^2 - 1\right) > 0$.
But if we assume $0 < \lambda \leq 1$, then Liu's argumentation does not provide coercivity. In this case, we need Korn's inequality in an expanded version. First we take a look at the part

$$\begin{aligned} 8 \cdot \|\operatorname{sym}(F^T H)\|^2 &= 8 \cdot \|\frac{1}{2}\left(F^T H + H^T F\right)\|^2 = 2 \cdot \|F^T H + H^T F\|^2 \\ &= 2 \cdot \left(\|F^T H\|^2 + 2 \cdot \left\langle F^T H, H^T F \right\rangle + \|H^T F\|^2\right) \\ &= 2 \cdot \left(2 \cdot \|H^T F\|^2 + 2 \cdot \left\langle F^T H, H^T F \right\rangle\right) \\ &= 4 \cdot \|H^T F\|^2 + 4 \cdot \operatorname{tr}\left(\left(H^T F\right)^2\right). \end{aligned} \tag{5.0.6}$$

For $A \in \mathbb{R}^{3\times 3}$, and with $\operatorname{Adj}(A) := \operatorname{Cof}(A)^T$, the Caley-Hamilton theorem yields[30]

$$A^3 - \operatorname{tr}(A) \cdot A^2 + \operatorname{tr}\left(\operatorname{Adj}(A)\right) \cdot A - \det(A) \cdot \mathbb{1} = 0. \tag{5.0.7}$$

This in turn is equivalent to

$$A^2 - \operatorname{tr}(A) \cdot A + \operatorname{tr}\left(\operatorname{Adj}(A)\right) \cdot \mathbb{1} - \det(A) \cdot A^{-1} = 0. \tag{5.0.8}$$

Moreover, by applying the trace operator, this equation implies

$$\operatorname{tr}\left(A^2\right) - \operatorname{tr}(A)^2 + 2 \cdot \operatorname{tr}\left(\operatorname{Adj}(A)\right) = 0. \tag{5.0.9}$$

[30]For additional steps, see the Appendix.

With this in mind we get

$$\begin{aligned} 4\cdot\|H^TF\|^2+4\cdot\operatorname{tr}\left(\left(H^TF\right)^2\right) &= 4\cdot\|H^TF\|^2+4\cdot\left[\operatorname{tr}\left(H^TF\right)^2-2\cdot\operatorname{tr}\left(\operatorname{Adj}\left(H^TF\right)\right)\right] \\ &= 4\cdot\|H^TF\|^2+4\cdot\operatorname{tr}\left(H^TF\right)^2-8\cdot\operatorname{tr}\left(\operatorname{Adj}\left(H^TF\right)\right) \\ &\geq 4\cdot\|H^TF\|^2-8\cdot\operatorname{tr}\left(\operatorname{Adj}\left(H^TF\right)\right). \end{aligned} \tag{5.0.10}$$

So with integration,

$$\begin{aligned} \int_\Omega 8\cdot\operatorname{tr}\left(\operatorname{Adj}\left(H^TF\right)\right)\,\mathrm{d}x &= 8\cdot\int_\Omega\left\langle\operatorname{Adj}\left(H^TF\right),\mathbb{1}\right\rangle\,\mathrm{d}x \\ &= 8\cdot\int_\Omega\left\langle\det\left(H^T\right)\det(F)H^{-T}F^{-1},\mathbb{1}\right\rangle\,\mathrm{d}x \\ &= 8\cdot\int_\Omega\left\langle\operatorname{Adj}(H),\det(F)F^{-1}\right\rangle\,\mathrm{d}x \\ &= 8\cdot\det(F)\int_\Omega\left\langle\operatorname{Adj}(H)^T,F^{-T}\right\rangle\,\mathrm{d}x. \end{aligned} \tag{5.0.11}$$

Furthermore, $F=\nabla\varphi=\lambda\cdot\mathbb{1}$ is invertible and $\nabla\psi=F^{-1}=\frac{1}{\lambda}\cdot\mathbb{1}=(\nabla\psi)^T$ is a gradient, too. Moreover, we assume $u\in C_0^\infty$. Thus, together with the Piola identity, equation (5.0.11) reduces to

$$\begin{aligned} 8\cdot\det(F)\int_\Omega\left\langle\operatorname{Adj}(H)^T,F^{-T}\right\rangle\,\mathrm{d}x &= 8\cdot\det(F)\int_\Omega\left\langle\operatorname{Div}\left(\operatorname{Adj}(H)^T\right),\psi\right\rangle\,\mathrm{d}x \\ &= 8\cdot\det(F)\int_\Omega\left\langle\operatorname{Div}\left(\operatorname{Cof}(H)\right),\psi\right\rangle\,\mathrm{d}x \\ &= 0. \end{aligned} \tag{5.0.12}$$

Hence

$$8\cdot\|\operatorname{sym}(F^TH)\|^2\geq 4\cdot\|H^TF\|^2\geq 4\cdot\lambda_{\min}(FF^T)\cdot\|H\|^2=4\cdot\lambda^2\|H\|^2. \tag{5.0.13}$$

In conclusion,

$$\begin{aligned} \mathcal{L}(u,u) &\geq 4\cdot\lambda^2\|\nabla u\|_{L^2}^2+4\cdot\int_\Omega\left[\left\langle F^TF,(\nabla u)^T\nabla u\right\rangle-\langle\nabla u,\nabla u\rangle\right]\,\mathrm{d}x \\ &\geq 4\cdot\lambda^2\|\nabla u\|_{L^2}^2+4\cdot\int_\Omega\left(\gamma_{\min}(F^TF)-1\right)\|\nabla u\|^2\,\mathrm{d}x \\ &= 4\cdot\lambda^2\|\nabla u\|_{L^2}^2+4\cdot\left(\lambda^2-1\right)\|\nabla u\|_{L^2}^2 \\ &= 4\left(2\lambda^2-1\right)\|\nabla u\|_{L^2}^2 \\ &= c^+\cdot\|\nabla u\|_{L^2}^2 \end{aligned} \tag{5.0.14}$$

with $c^+=4\left(2\lambda^2-1\right)>0$. Therefore, with the same further considerations as in Section 4.5.1, there exists a unique solution for the boundary value problem in (4.5.15) for the Saint-Venant-Kirchhoff energy and for the case $F=\lambda\cdot\mathbb{1}$.
At this point, it should be noted that the constant c^+ is positive only if $\lambda>\frac{1}{\sqrt{2}}$. This is due to the fact that this model is generally not suitable for compression.
Thus Liu's approach is not generally applicable, but with the help of Korn's inequality we still get the uniqueness of the solution u for the boundary value problem for all $\lambda>\frac{1}{\sqrt{2}}$.

6 Ellipticity conditions and convexity

For energy functions $\widehat{W}$ defined in (2.2.11), there exist different notions of ellipticity and convexity conditions. They are important for the existence of a solution of the variational problem in (2.2.15). For convexity, we normally concentrate on *polyconvexity*, *quasiconvexity* and *rank-one convexity*, which are defined as follows.

Definition 6.0.1. $\widehat{W}$ is *polyconvex* if there exists a convex function $P : \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \times \mathbb{R} \to \mathbb{R}$ such that $\widehat{W}(F) = P(F, \mathrm{Cof}(F), \det(F))$.

Definition 6.0.2. $\widehat{W}$ is *quasiconvex* if for all $\Omega \subset \mathbb{R}^3$, $F \in \mathbb{R}^{3\times3}$ and $v \in C_0^\infty(\Omega)$ the following inequality holds:

$$\widehat{W}(F) \cdot |\Omega| = \int_\Omega \widehat{W}(F)\,\mathrm{d}x \leq \int_\Omega \widehat{W}(F + \nabla v(x))\,\mathrm{d}x. \tag{6.0.1}$$

Definition 6.0.3. $\widehat{W}$ is *rank-one convex* if the function $f : \mathbb{R} \to \mathbb{R}$ with $f(t) := \widehat{W}\,(F + t \cdot (\xi \otimes \eta))$ is convex for all $F \in \mathbb{R}^{3\times3}$ with $\xi,\ \eta \in \mathbb{R}^3$.

Furthermore, there is a connection between these properties:

$$\text{convexity} \implies \text{polyconvexity} \implies \text{quasiconvexity} \implies \text{rank-one convexity},$$

see [3]. Dacorogna shows in [3] that the reverse implications are false[31]. Furthermore, rank-one convexity is equivalent to *Legendre-Hadamard ellipticity.*

Definition 6.0.4. $\widehat{W}$ is (*Legendre-Hadamard*) *elliptic* if there exists a positive constant $c^+ \in \mathbb{R}$ such that

$$D^2\,\widehat{W}(F).[\xi \otimes \eta, \xi \otimes \eta] \geq c^+ \|\xi\|^2 \|\eta\|^2 \tag{6.0.2}$$

for all $F \in \mathbb{R}^{3\times3}$ and $\xi,\ \eta \in \mathbb{R}^3$.[32]

Moreover, Dacorogna gives existence theorems for quasi- and polyconvex functions.
With all this in mind, we notice that if $\widehat{W}$ is not rank-one convex or elliptic, it is also not quasiconvex and not polyconvex. Consequently, we know nothing about existence of a solution. So rank-one convexity is a necessary criterion in order to apply classical methodes for minimization of the energy functional in (2.2.15) or, like it is said in [16]:

> "[...] *it is just what is needed for a good existence and uniqueness theory for linear elastostatics and elastodynamics.*"

Thus, in the following we focus on (Legendre-Hadamard) ellipticity.

6.1 Ellipticity conditions for linear problems

Now we want to investigate if the energy function

$$\widehat{W}(F) = \mu \left[\frac{1}{2}\|F\|^2 - \log(\det(F))\right]$$

used in section 4.5.1 fulfills the condition of (Legendre-Hadamard) ellipticity. For this, we will refer to previous results from Sections 4.2 and 4.3.

[31] At this point we have to mention that the connection rank-one convex $\not\Rightarrow$ quasikonvex is only shown for $n \geq 3$, see [3].

[32] The operator $\otimes$ defines the tensor product: Let $\xi,\ \eta \in \mathbb{R}^3$, then $\xi \otimes \eta \in \mathbb{R}^{3\times3}$ and $\xi \otimes \eta = \xi \cdot \eta^T$. For the tensor product's properties see [14].

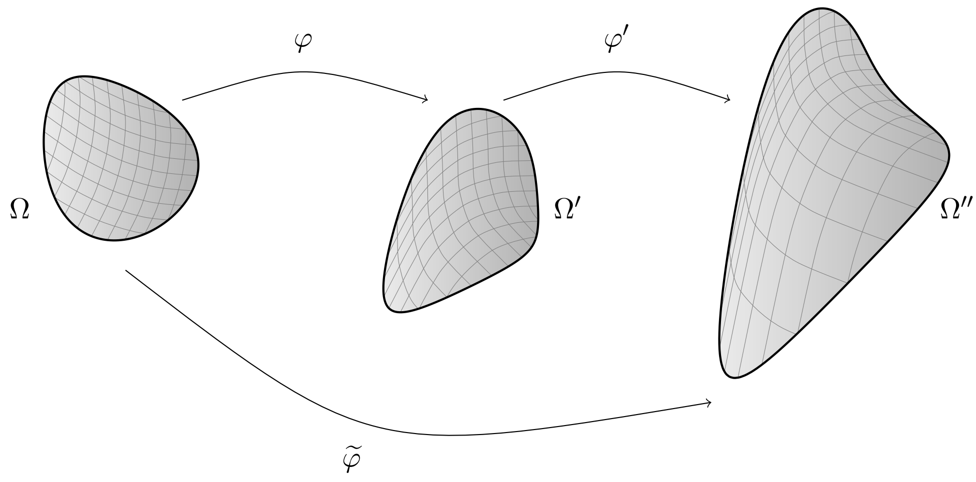

Figure 17: Reminder: Relationships between the deformations φ, φ', $\widetilde{\varphi}$ and the configurations Ω (reference configuration), Ω' (intermediate configuration), Ω'' (current configuration).

We start with considerations on the intermediate configuration Ω'. First we have to figure out the weak form of the linearized equations of equilibrium

$$\begin{aligned} 0 = \operatorname{Div}_{x'} \Big[&\sigma\left(\nabla\varphi(\varphi^{-1}(x'))\right) \cdot \left[\left(1+\operatorname{tr}(H(x'))\right)\cdot \mathbb{1} - H(x')^T\right] \\ &+ D\sigma\left(\nabla\varphi(\varphi^{-1}(x'))\right). \left[H(x')\cdot\nabla\varphi(\varphi^{-1}(x'))\right]\Big], \end{aligned} \tag{6.1.1}$$

stated in (4.2.4). For $v' \in C_0^\infty(\Omega')$ with $F_0 = \nabla\varphi$ and $H = \nabla u$ we get

$$\begin{aligned} 0 &= -\int_{\Omega'} \left\langle \operatorname{Div}_{x'}\Big[\sigma(F_0)\cdot\left[(1+\operatorname{tr}(H))\cdot\mathbb{1} - H^T\right] + D\sigma(F_0).\left[HF_0\right]\Big], v'\right\rangle \,\mathrm{d}x' \\ &\overset{P.I.}{=} \int_{\Omega'} \left\langle \sigma(F_0)\cdot\left[(1+\operatorname{tr}(H))\cdot\mathbb{1} - H^T\right] + D\sigma(F_0).\left[HF_0\right], \nabla v'\right\rangle \,\mathrm{d}x' \\ &= \int_{\Omega'} \Bigg\langle \underbrace{\sigma(F_0)\cdot\Big[(1+\operatorname{tr}(\nabla u))\cdot\mathbb{1} - (\nabla u)^T\Big] + D\sigma(F_0).[\nabla u\cdot F_0]}_{=:\mathcal{C}(F_0).\nabla u}, \nabla v'\Bigg\rangle \, dx'. \end{aligned} \tag{6.1.2}$$

So we obtain a bilinear form $\mathcal{A} : C^\infty(\Omega')\times C^\infty(\Omega') \to \mathbb{R}$ for $u \in C^\infty(\Omega')$ and $v' \in C_0^\infty(\Omega')$ with

$$\mathcal{A}(u, v') = \int_{\Omega'} \left\langle \mathcal{C}(F_0).\nabla u, \nabla v'\right\rangle \, dx' = 0. \tag{6.1.3}$$

To determine whether $\widehat{W}$ satisfies the ellipticity condition

$$\langle \mathcal{C}(F_0).H, H\rangle \geq c\cdot \|H\|^2 \tag{6.1.4}$$

regarding the intermediate configuration for $\nabla u = H := \xi\otimes\eta$ with ξ, $\eta \in \mathbb{R}^3$ and a positive constant $c > 0$, we have to compute $\mathcal{C}(F_0).H$. For this we need the linearized equation of equilibrium in the intermediate configuration with respect to the energy function $\widehat{W}$. It is stated in (4.5.5) as

$$\begin{aligned} 0 &= \operatorname{Div}\left[\sigma\left(F_0\right) + \frac{\mu}{\det(F_0)}\left(HB_0 + B_0H^T\right) - \sigma\left(F_0\right)H^T\right] \\ &= \operatorname{Div}\left[\frac{\mu}{\det(F_0)}\left(HB_0 + B_0H^T\right) - \sigma\left(F_0\right)H^T\right], \end{aligned}$$

where the last step applies because we assume F_0 to be the gradient of an equilibrium solution, see Section 4.1 . Hence, with equation (4.5.2) we have to estimate

$$\begin{aligned} &\left\langle \frac{\mu}{\det(F_0)} \left(HB_0 + B_0H^T\right) - \sigma(F_0)H^T \, , \, H \right\rangle \\ &= \frac{\mu}{\det(F_0)} \left\langle HB_0 + B_0H^T - (B_0 - \mathbb{1})\, H^T, H\right\rangle \\ &= \frac{\mu}{\det(F_0)} \left\langle HB_0 + H^T, H\right\rangle \end{aligned} \tag{6.1.5}$$

with $H = \xi \otimes \eta$, ξ, $\eta \in \mathbb{R}^3$. Together with the requirement $\det(F_0) > 0$ and $\gamma_{\min}(B_0)$ the lowest eigenvalue of B_0, we get[33]

$$\begin{aligned} &\frac{\mu}{\det(F_0)} \left\langle (\xi \otimes \eta)\, B_0 + (\xi \otimes \eta)^T \, , \, \xi \otimes \eta \right\rangle \\ &= \frac{\mu}{\det(F_0)} \left[\langle (\xi \otimes \eta)\, B_0 \, , \, \xi \otimes \eta\rangle + \left\langle (\xi \otimes \eta)^T \, , \, \xi \otimes \eta\right\rangle\right] \\ &= \frac{\mu}{\det(F_0)} \left[\|\xi\|^2 \langle B_0\, \eta \, , \, \eta\rangle + \langle \eta \, , \, \xi\rangle^2\right] \\ &=\geq \frac{\mu}{\det(F_0)} \left[\|\xi\|^2 \langle B_0\, \eta \, , \, \eta\rangle\right] \\ &\geq \frac{\mu}{\det(F_0)} \left[\|\xi\|^2 \gamma_{min}(B_0)\, \|\eta\|^2\right] \\ &= c^+ \|\xi\|^2 \, \|\eta\|^2 \end{aligned} \tag{6.1.6}$$

with a positive constant $c^+ = \frac{\mu}{\det(F_0)} \cdot \gamma_{\min}(B_0) \in \mathbb{R}$.
So the energy function $\widehat{W}$ satisfies the ellipticity condition with respect to the linearization on the intermediate configuration as well.
Now we try to do the same calculations on the linearized equation of equilibrium regarding the reference configuration

$$\begin{aligned} 0 = \mathrm{Div}_x \Big[&\left[\sigma\left(\nabla\varphi(x)\right) \cdot \left[(1 + \mathrm{tr}(H(\varphi(x)))) \cdot \mathbb{1} - H(\varphi(x))^T\right] \right.\\ &\left. + D\sigma\left(\nabla\varphi(x)\right).\left[H(\varphi(x)) \cdot \nabla\varphi(x)\right]\right] \cdot \mathrm{Cof}(\nabla\varphi(x))\Big], \end{aligned} \tag{6.1.7}$$

stated in (4.3.5). The weak form of this linearized equation for $v \in C_0^\infty(\Omega)$ with $F_0 = \nabla\varphi$ and $H = \nabla u$ is given by

$$\begin{aligned} 0 &= -\int_\Omega \left\langle \mathrm{Div}_x \left[\left[\sigma\left(F_0\right) \cdot \left[(1 + \mathrm{tr}(H)) \cdot \mathbb{1} - H^T\right] + D\sigma\left(F_0\right).\left[HF_0\right]\right] \cdot \mathrm{Cof}(F_0)\right], v \right\rangle \mathrm{d}x \\ &\overset{P.I.}{=} \int_\Omega \left\langle \left[\sigma\left(F_0\right) \cdot \left[(1 + \mathrm{tr}(H)) \cdot \mathbb{1} - H^T\right] + D\sigma\left(F_0\right).\left[HF_0\right]\right] \cdot \mathrm{Cof}(F_0), \nabla v \right\rangle \mathrm{d}x \\ &= \int_\Omega \left\langle \left[\sigma(F_0) \left[(1 + \mathrm{tr}(\nabla u)) \cdot \mathbb{1} - (\nabla u)^T\right] + D\sigma(F_0).[\nabla u \cdot F_0]\right] \cdot \mathrm{Cof}(F_0), \nabla v \right\rangle \mathrm{d}x. \end{aligned} \tag{6.1.8}$$

At this point, we observe that we cannot proceed as we did before, because u is a mapping on Ω' and v is a mapping on Ω. Instead, in order to establish a variational problem on Ω', consider the function $\widetilde{v} : \Omega' \to \mathbb{R}^3$ with $\widetilde{v}(\varphi(x)) := v(x)$ for any given $v \in C_0^\infty(\Omega)$. Then

$$\nabla v(x) = \nabla[\widetilde{v}(\varphi(x))] = \nabla\widetilde{v}(\varphi(x)) \cdot \nabla\varphi(x) = \nabla\widetilde{v}(\varphi(x))F_0 \, ,$$

[33] For B_0 and $(\xi \otimes \xi) \in \mathbb{R}^{3\times 3}$ the following equation holds:

$$\langle B_0\xi, \xi\rangle \geq \gamma_{\min}(B_0) \cdot \|\xi\|^2,$$

see [14, cor. 12].

thus we obtain the bilinear form

$$(u,\widetilde{v}) \mapsto \int_\Omega \left\langle \left[\sigma(F_0)\left[(1+\operatorname{tr}(\nabla u))\cdot \mathbb{1} - (\nabla u)^T\right] + D\sigma(F_0).[\nabla u\cdot F_0]\right]\cdot \operatorname{Cof}(F_0), \nabla\widetilde{v}\cdot F_0\right\rangle \,\mathrm{d}x$$

and thus, for $H = \nabla u = \nabla\widetilde{v}$, we obtain the quadratic form given by the expression

$$\int_\Omega \left\langle \left[\sigma(F_0)\left[(1+\operatorname{tr}(H))\cdot \mathbb{1} - H^T\right] + D\sigma(F_0).[HF_0]\right]\cdot \operatorname{Cof}(F_0), HF_0\right\rangle \,\mathrm{d}x\,. \tag{6.1.9}$$

Therefore, in order to determine whether $\widehat{W}$ satisfies the ellipticity condition regarding the reference configuration, we have to find a lower bound for (6.1.9). For this, we need the linearized equation of equilibrium in the reference configuration regarding the energy function $\widehat{W}$, which is given by

$$\begin{aligned} 0 &= \operatorname{Div}\left[\left(\sigma(F_0) + \frac{\mu}{\det(F_0)}\left(HB_0 + B_0H^T\right) - \sigma(F_0)\,H^T\right)\cdot \operatorname{Cof}(F_0)\right] \\ &= \operatorname{Div}\left[\left(\frac{\mu}{\det(F_0)}\left(HB_0 + B_0H^T\right) - \sigma(F_0)\,H^T\right)\cdot \operatorname{Cof}(F_0)\right]. \end{aligned} \tag{6.1.10}$$

Hence, we have to find an estimate for the inner product

$$\begin{aligned} &\left\langle \left[\frac{\mu}{\det(F_0)}\left(HB_0 + B_0H^T\right) - \sigma(F_0)H^T\right]\cdot \operatorname{Cof}(F_0)\,,\, HF_0\right\rangle \\ &= \frac{\mu}{\det(F_0)}\left\langle \left[HB_0 + B_0H^T - (B_0 - \mathbb{1})\,H^T\right]\cdot \operatorname{Cof}(F_0), HF_0\right\rangle \\ &= \frac{\mu}{\det(F_0)}\left\langle \left[HB_0 + H^T\right]\cdot \det(F_0)\cdot F_0^{-T}, HF_0\right\rangle \\ &= \mu\left\langle HF_0F_0^TF_0^{-T} + H^TF_0^{-T}, HF_0\right\rangle \\ &= \mu\left\langle HF_0 + H^TF_0^{-T}, HF_0\right\rangle \end{aligned} \tag{6.1.11}$$

for $HF_0 \coloneqq \xi\otimes\eta$ with $\xi,\,\eta\in\mathbb{R}^3$. Due to $\widetilde{H} \coloneqq HF_0 \iff H = \widetilde{H}F_0^{-1}$, we get

$$\begin{aligned} \mu\left\langle HF_0 + H^TF_0^{-T}, HF_0\right\rangle &= \mu\left[\langle \xi\otimes\eta, \xi\otimes\eta\rangle + \langle H^T, H\rangle\right] \\ &= \mu\left[\langle \xi\otimes\eta, \xi\otimes\eta\rangle + \left\langle F_0^{-T}\,(\xi\otimes\eta)^T, (\xi\otimes\eta)\,F_0^{-1}\right\rangle\right] \\ &= \mu\cdot\|\xi\|^2\cdot\|\eta\|^2 + \mu\cdot\left\langle \left(F_0^{-T}\eta\right)\otimes\xi, \xi\otimes\left(F_0^{-T}\eta\right)\right\rangle \\ &= \mu\cdot\|\xi\|^2\cdot\|\eta\|^2 + \mu\cdot\left\langle F_0^{-T}\eta, \xi\right\rangle^2 \\ &\geq \mu\cdot\|\xi\|^2\cdot\|\eta\|^2 \\ &= \widetilde{c}^+\cdot\|\xi\|^2\cdot\|\eta\|^2 \end{aligned} \tag{6.1.12}$$

with a positive constant $\widetilde{c}^+ = \mu\in\mathbb{R}$.

Now we want to compare this result with the classical concept of ellipticity. First we have to linearize the *classical* Piola-Kirchhoff stress tensor. Then we take a look at its equation of equilibrium and especially to the weak form of the equation like we have done before.
For the equation of equilibrium concerning the linearization, we find

$$\begin{aligned} \operatorname{Div}\left[S_1(F_0 + \widetilde{\widetilde{H}})\right] = 0 &\iff \operatorname{Div}\left[S_1(F_0) + D_{F_0}S_1(F_0).\left[\widetilde{H}\right] + O\left(\|\widetilde{H}\|^2\right)\right] = 0 \\ &\implies \operatorname{Div}\left[D_{F_0}S_1(F_0).[\widetilde{H}]\right] = 0. \end{aligned} \tag{6.1.13}$$

Thus with partial integration and $v' \in C_0^\infty(\Omega')$ we get

$$0 = -\int_\Omega \left\langle \operatorname{Div}\left[D_{F_0}S_1(F_0).[\widetilde{H}]\right], v\right\rangle \,\mathrm{d}x \overset{P.I.}{=} \int_\Omega \left\langle D_{F_0}S_1(F_0).[\widetilde{H}], \nabla v\right\rangle \,\mathrm{d}x. \tag{6.1.14}$$

Now we have to calculate the first Piola-Kirchhoff stress tensor with respect to the given energy function $\widehat{W}$:

$$S_1(F_0) = D_{F_0}\widehat{W}(F_0) = \mu\cdot F_0 - \mu\cdot\frac{1}{\det(F_0)}\cdot\operatorname{Cof}(F_0) = \mu\left[F_0 - F_0^{-T}\right]. \tag{6.1.15}$$

Hence

$$\begin{aligned} S_1(F_0+\widetilde{H}) &= \mu\left[F_0+\widetilde{H} - \left(F_0+\widetilde{H}\right)^{-T}\right] \\ &= \mu\cdot F_0 + \mu\cdot\widetilde{H} - \mu\cdot F_0^{-T} + \mu\cdot F_0^{-T}\widetilde{H}^T F_0^{-T} + O\left(\|\widetilde{H}\|^2\right) \\ &= S_1(F_0) + \mu\left[\widetilde{H} + F_0^{-T}\widetilde{H}^T F_0^{-T}\right] + O\left(\|\widetilde{H}\|^2\right). \end{aligned} \tag{6.1.16}$$

Consequently,

$$D_{F_0}S_1(F_0).[\widetilde{H}] = \mu\left[\widetilde{H} + F_0^{-T}\widetilde{H}^T F_0^{-T}\right]. \tag{6.1.17}$$

Therefore, we have to estimate

$$\left\langle D_{F_0}S_1(F_0).[\widetilde{H}], \widetilde{H}\right\rangle \tag{6.1.18}$$

for $\widetilde{H} := \xi\otimes\eta$. At this point we notice that we simply have to repeat the same estimates as in (6.1.12), since in direction $\widetilde{H} = HF_0$ we get

$$\begin{aligned} \left\langle D_{F_0}S_1(F_0).[\widetilde{H}], \widetilde{H}\right\rangle &= \mu\left\langle \widetilde{H} + F_0^{-T}\widetilde{H}^T F_0^{-T}, \widetilde{H}\right\rangle = \mu\left\langle HF_0 + F_0^{-T}F_0^T H^T F_0^{-T}, HF_0\right\rangle \\ &= \mu\left\langle HF_0 + H^T F_0^{-T}, HF_0\right\rangle. \end{aligned}$$

That is just what we expected from equation (4.3.8). Thus, together with (6.1.12) we obtain

$$\langle D_F S_1(F).[\xi\otimes\eta], \xi\otimes\eta\rangle \geq \widetilde{c}^+\cdot\|\xi\|^2\cdot\|\eta\|^2$$

with a positive constant $\widetilde{c}^+ = \mu\in\mathbb{R}$, as well.

Remark 6.1.1. If we assume the deformation to be purely volumetric, i.e. $B_0 = \lambda\cdot\mathbb{1}\in\operatorname{Sym}^+(3)$ with $\lambda\in\mathbb{R}_+$, the constant c^+ in (6.1.6) reduces to

$$c^+ = \frac{\mu}{\det(F_0)}\cdot\gamma_{\min}(B_0) = \frac{\mu\lambda}{\sqrt{\det(B_0)}} = \frac{\mu\lambda}{\sqrt{\lambda^3}} = \frac{\mu}{\sqrt{\lambda}}. \tag{6.1.19}$$

Furthermore, in the two dimensional case we get

$$c^+ = \sqrt{\frac{\gamma_{\min}(B_0)}{\gamma_{\max}(B_0)}}\cdot\mu = \sqrt{\frac{\lambda}{\lambda}}\cdot\mu = \mu. \tag{6.1.20}$$

So in both (6.1.6) and (6.1.19) we obtain the same positive constant $\widetilde{c}^+ = c^+ = \mu$.

7 Outlook

We demonstrated that Liu did not find a new stability criterion regarding linearization in [10], so it does not matter whether we use the linearized equation of equilibrium in the reference or in the intermediate configuration. And indeed, all linearized equations in whatever configuration are equivalent. Moreover, he only considers the 2-dimensional case for application, and even there still rules out cases like uniform compression and the identity [2]. As mentioned before, the identity in particular is an important case for linearization. Moreover, Liu does not even clarify that he ignores theses cases.

It is also important to note that Liu does not emphasize clearly that he means different constants when he discusses the equivalence of coercivity and positive semidefiniteness in Section 4.5.1 in the given context, see the Appendix. Furthermore, for his pointwise considerations and with respect to his exclusion of compression and identity he does not need Korn's inequality, but as we have shown we cannot generally neglect it.

In summery, there are many additional things which should be investigated in the future. What about the 3-dimensional case? Do Liu's considerations also apply there? And what would happen if we do not look at a volumetric inflation with $F = \lambda \cdot \mathbb{1}$? Are Liu's Theorem 4.5.2 and his conclusions still valid? Finally, are we able to apply his considerations to other energy functions?

A Appendix

A.1 Coercivity and positive semidefiniteness

In [2], Liu defines the coercivity of $\mathcal{A}$ for an arbitrary constant $\beta > 0$ with $\mathcal{A}(x; \nabla u, \nabla u) \geq \beta \cdot \|\nabla u\|^2$ and asserts that this in turn is equivalent to the positive semidefiniteness of $A - \beta\,\mathbb{1}$ for all $x \in \Omega$. At this point, it is not clear whether he means the same constant $\beta > 0$ or another arbitrary constant $\widetilde{\beta} > 0$. Assuming he means the same constant β, his assertion does not hold in general, as we will show in the following.

Let $\gamma_1 = \gamma_2 = 2$, so that $B_0 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. In addition, choose $\mu = \sqrt{2}$ and $X = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Then $\nabla u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus, we get $\|X\|^2 = 1$ and $\|\nabla u\|^2 = 2$.

Before we continue with the counterexample, we take a closer look at the constant β. We know that β is bounded below by zero ($\beta > 0$). Now we would also like to indicate an upper bound such that $A - \beta\,\mathbb{1}$ is semipositive definite.

For $\gamma_1 = \gamma_2 = 2$ we calculate

$$
\begin{aligned}
A - \beta\,\mathbb{1} &= \begin{pmatrix} 3\,s_1 - \beta & 0 & 0 & 0 \\ 0 & 2\,s_1 - \beta & 0 & 0 \\ 0 & 0 & 8\,s_1 - 2\,s_1^2 - \beta & 0 \\ 0 & 0 & 0 & 2\,s_1^2 - \beta \end{pmatrix} \\
&= \begin{pmatrix} \frac{3}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{3}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 4\sqrt{2} - 1 - \beta & 0 \\ 0 & 0 & 0 & 1 - \beta \end{pmatrix}.
\end{aligned} \tag{A.1.1}
$$

Now the eigenvalues have to be bigger or equal than zero. Hence, the following inequalities have to apply:

$$
\begin{array}{llllllll}
\text{I} & \frac{3}{\sqrt{2}} - \beta & \geq & 0 & \Longleftrightarrow & \beta & \leq & \frac{3}{\sqrt{2}} \\
\text{II} & 4\sqrt{2} - 1 - \beta & \geq & 0 & \Longleftrightarrow & \beta & \leq & 4\sqrt{2} - 1 \\
\text{III} & 1 - \beta & \geq & 0 & \Longleftrightarrow & \beta & \leq & 1.
\end{array} \tag{A.1.2}
$$

Thus β is bounded above and we can choose $\beta := 1 > 0$. So for $\beta = 1$, $A - \beta\,\mathbb{1}$ is positive semidefinite, i.e. $X^T\,(A - \beta\,\mathbb{1})\,X \geq 0$. Hence,

$$
\mathcal{A}(x; \nabla u, \nabla u) = X^T A\, X = 1 < 2 = 1 \cdot \|\nabla u\|^2 = \beta \cdot \|\nabla u\|^2,
$$

which contradicts the claim that $\mathcal{A}(x; \nabla u, \nabla u) \geq \beta \cdot \|\nabla u\|^2$ for the same constant $\beta > 0$.

A.2 Caley-Hamilton

The Caley-Hamilton theorem tells us that each square matrix is zero of its characteristic polynom [18]. So for $A := \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathbb{R}^{3\times 3}$ and $\lambda \in \mathbb{R}$, we get

$$\begin{aligned}
0 = \det(A - \lambda \mathbb{1}) &= \det \begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{pmatrix} \\
&= (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}(a_{22}-\lambda)a_{13} - a_{32}a_{23}(a_{11}-\lambda) \\
&\quad - (a_{33}-\lambda)a_{21}a_{12} \\
&= (a_{11}a_{22} - a_{11}\lambda - a_{22}\lambda + \lambda^2)(a_{33}-\lambda) + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} + a_{31}a_{13}\lambda \\
&\quad - a_{32}a_{23}a_{11} + a_{32}a_{23}\lambda - a_{33}a_{21}a_{12} + a_{21}a_{12}\lambda \\
&= a_{11}a_{22}a_{33} - a_{11}a_{22}\lambda - a_{11}a_{33}\lambda + a_{11}\lambda^2 - a_{22}a_{33}\lambda + a_{22}\lambda^2 + a_{33}\lambda^2 - \lambda^3 + a_{12}a_{23}a_{31} \\
&\quad + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} + a_{31}a_{13}\lambda - a_{32}a_{23}a_{11} + a_{32}a_{23}\lambda - a_{33}a_{21}a_{12} + a_{21}a_{12}\lambda \\
&= -\lambda^3 + (a_{11}+a_{22}+a_{33})\lambda^2 - (a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{31}a_{13} - a_{32}a_{23} - a_{21}a_{12})\lambda \\
&\quad + a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \\
&= -\lambda^3 + \mathrm{tr}(A)\lambda^2 - \mathrm{tr}(\mathrm{Cof}(A))\lambda + \det(A)\cdot \mathbb{1} \\
&= -\lambda^3 + \mathrm{tr}(A)\lambda^2 - \mathrm{tr}(\mathrm{Adj}(A))\lambda + \det(A)\cdot \mathbb{1}.
\end{aligned}$$

Thus

$$A^3 - \mathrm{tr}(A)\cdot A^2 + \mathrm{tr}(\mathrm{Adj}(A))\cdot A - \det(A)\cdot \mathbb{1} = 0.$$

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