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THE COMPRESSIBILITY OF SOLIDS UNDER EXTREME PRESSURES

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INTRODUCTION

We shall attempt to show in the present paper that a careful and exact consideration of the energy principle suffices to account, to within the limits of experimental accuracy, for the observed compressibilities of solids even though the pressures to which the solids have been subjected are so great (e.g. 50,000 atmospheres) that the solid in question has been reduced to approximately one-half its original size. The problem of finite, i.e. non-infinitesimal, deformations has been studied over and over again, and we shall attempt to make clear at just what point our theory departs from these past (and present) studies and to what it owes, in our opinion, its success when subjected to the test of actual experiment.

THE DEFORMATION MATRICES M AND N

Let us denote by $a = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ a point of our solid which is deformed to the point $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$; we write these vectors a and x as matrices of one column rather than as matrices of one row, since we are thereby enabled to employ the powerful abbreviations of matrix algebra. The coordinates of the point x will be, in general, quite complicated functions of the point a but their differentials will be relatively simple functions (in fact, linear, homogeneous functions) of the differentials of the coordinates of the point a. This fact finds its expression in the formula

$$dx = P\,da$$

where P is the Jacobian matrix $\frac{\partial(x,y,z)}{\partial(a,b,c)}$. From this formula we read off at once the effect of the deformation upon the element of arc length. Denoting the transpose of a matrix (i.e. the matrix obtained by interchanging the rows and columns of the matrix) by an attached prime, we have

$$ds_0^2 = da'da \qquad ds^2 = dx'dx = da'P'P\,da$$

so that

$$ds^2 - ds_0^2 = da'(P'P - E)da = dx'(E - Q'Q)dx$$

where E denotes the unit matrix and Q denotes the reciprocal of the matrix P:

$$QP = E = PQ \qquad da = Q\,dx$$

We say, therefore, that the matrix $M = P'P$ suffices (insofar as measurements of lengths are concerned) to describe the deformation, this description being in terms of the initial position a; whilst the matrix $N = Q'Q$ describes the same deformation in terms of the final position x. It is important to observe that the reciprocal of M, i.e. QQ', differs from $N = Q'Q$ merely in the order in which the factors are written. Hence M^{-1} and N are similar to one another:

$$PM^{-1}P^{-1} = PQQ'Q = Q'Q = N$$

Let us denote the characteristic numbers of M and N by (m_1, m_2, m_3) and (n_1, n_2, n_3), respectively, and the invariants of M and N by (M_1, M_2, M_3) and (N_1, N_2, N_3), respectively. It follows at once from the relations

$$M_1 = m_1 + m_2 + m_3 \qquad M_2 = m_2 m_3 + m_3 m_1 + m_1 m_2 \qquad M_3 = m_1 m_2 m_3$$

$$N_1 = n_1 + n_2 + n_3 \qquad N_2 = n_2 n_3 + n_3 n_1 + n_1 n_2 \qquad N_3 = n_1 n_2 n_3$$

$$n_1 = m_1^{-1} \qquad n_2 = m_2^{-1} \qquad n_3 = m_3^{-1}$$

that

$$N_1 = M_2/M_3 \qquad N_2 = M_1/M_3 \qquad N_3 = 1/M_3$$

$$M_1 = N_2/N_3 \qquad M_2 = N_1/N_3 \qquad M_3 = 1/N_3$$

THE SYMMETRIZED SPACE-VARIATION MATRIX D

In order to express the principle of conservation of energy we adopt the dynamic viewpoint (as opposed to the static viewpoint which interests itself only in the initial and final positions of the body) in accordance with which x is a function not only of a but of an accessory parameter ϑ (which might well be, for example, the time). If we have any function f of x, we denote by δf the variation of f (i.e. the partial differential $\frac{\partial f}{\partial \vartheta} d\vartheta$, where the differentiation is performed under the assumption that a is kept constant), and observe that differentiations with respect to ϑ and a are commutative whilst those with respect to ϑ and x are not (because ϑ and a are independent variables whilst ϑ and x are not). Since $P = \partial x/\partial a$,

$$\delta P = \left(\delta \frac{\partial x}{\partial a}\right) = \left(\frac{\partial}{\partial a}\delta x\right) = \left(\frac{\partial}{\partial x}\delta x\right) P$$

Hence $\delta P' = P'\left(\frac{\partial}{\partial x}\delta x\right)'$, so that

$$\delta M = P'\delta P + \delta P' P = 2P'DP$$

where D is the symmetric matrix

$$D = \frac{1}{2}\left\{\left(\frac{\partial}{\partial x}\delta x\right) + \left(\frac{\partial}{\partial x}\delta x\right)'\right\}$$

It is in terms of this matrix, rather than in terms of δM, that the principle of conservation of energy finds, most conveniently, its exact (and not merely approximate expression. We term D the symmetrized space variation of the virtual displacement vector and note that

$$\delta M = 2P'DP \qquad 2D = Q'\delta M Q$$

THE VIRTUAL WORK OF ALL THE FORCES ACTING ON THE BODY

The forces acting on the elastic body are, in general, of two types:

(a) Body or mass forces. Examples of these are the weight of the body and, if the medium is not in equilibrium, the forces of inertia. We shall denote by F the vector which measures these body forces per unit mass; i.e. if dV is an element of volume of the medium, the resultant force on this element is $\rho F dV$ where ρ is the density of the medium.

(b) Surface forces. These are forces (such as a uniform hydrostatic pressure) applied to the surface of the elastic body. We shall denote by f the vector which measures these surface forces per unit area; i.e. if dS is an element of area of the surface, the resultant force on this element of area is $f dS$.

In a virtual displacement of the elastic body the virtual work of all the forces, both body and surface, which act upon it is

$$\int(\rho F\delta x)\,dV+\int(f\delta x)\,dS$$

If T is the stress matrix and ν is the unit normal drawn outwards from the surface we have $f=T\nu$, and our surface integral $\int(T\nu\,\delta x)\,dS$ may be replaced by the equivalent volume integral $\int \mathrm{div}(T\delta x)\,dV = \int\{(\mathrm{div}\,T\delta x)+[T\frac{\partial}{\partial x}\delta x]\}\,dV$.

Here $\mathrm{div}\,T$ is the vector whose first component is $\frac{\partial T_x^x}{\partial x}+\frac{\partial T_x^y}{\partial y}+\frac{\partial T_x^z}{\partial z}$, and $[T\frac{\partial}{\partial x}\delta x]$ denotes the trace, or sum of the diagonal elements, of the matrix $T\frac{\partial}{\partial x}\delta x$. Hence the virtual work of all the forces acting on the body appears as the volume integral

$$\int\{(\rho F+\mathrm{div}\,T)\,\delta x+[T\frac{\partial}{\partial x}\delta x]\}\,dV$$

We assume that this virtual work is zero if the body moves as a rigid whole (which is merely another way of saying that the equilibrium of an elastic body will not be disturbed if the body is imagined rigidified). On considering the virtual translations (for which the matrix $\frac{\partial}{\partial x}\delta x$ is zero) we obtain the basic equations of motion (it being remembered that F includes the forces of inertia):

$$\rho F+\mathrm{div}\,T=0$$

On considering the virtual rotations for which $\frac{\partial}{\partial x}\delta x$ is skew-symmetric (i.e. $(\frac{\partial}{\partial x}\delta x)+(\frac{\partial}{\partial x}\delta x)'=0$), we find that the stress matrix T must be symmetric (i.e. $T=T'$), and then the expression for the virtual work of all the forces acting on the medium takes the simple form:

$$\text{Virtual work} = \int[TD]\,dV \qquad 2D=\left(\frac{\partial}{\partial x}\delta x\right)+\left(\frac{\partial}{\partial x}\delta x\right)'$$

It is precisely here that we diverge from the classical or infinitesimal theory; in that theory it is assumed that the

$$\text{Virtual work} = \frac{1}{2}\int[T\delta M]\,dV$$

so that the approximation used in the infinitesimal theory consists in replacing $D=\frac{1}{2}Q'\delta MQ$ by $\frac{1}{2}\delta M$, i.e. in replacing Q, where it appears in the formula for D , by the unit matrix E .

THE ENERGY OF DEFORMATION OF THE BODY

We assume that the work done by the forces acting on the body is stored up (as elastic energy), and we denote by φ the density of elastic energy per unit mass; thus the elastic energy is furnished by the volume integral $\int\rho\varphi\,dV$ In a virtual displacement the variation of ρdV is zero (by the principle of mass conservation), and so the variation of the elastic energy is $\int\rho\delta\varphi\,dV$. On equating this to the virtual work we obtain the basic relation

$$\rho\delta\varphi=[TD]=\frac{1}{2}[TQ'\delta MQ]=\frac{1}{2}[QTQ'\delta M]$$

since the trace of the product of two matrices is independent of the order in which they are taken. On the assumption that φ is completely determined by the matrix M we have $\delta\varphi=[\frac{\partial\varphi}{\partial M}\delta M]$ where $\frac{\partial\varphi}{\partial M}$ is the matrix of derivatives of φ with respect to the elements of M (it being understood that in forming these partial derivatives no account is taken of the fact that M is symmetric, and that φ is so expressed as a function of M that $\frac{\partial\varphi}{\partial M}$ is symmetric). On taking into account the fact that δM is an arbitrary (symmetric) matrix we obtain the basic equation

$$\rho\frac{\partial\varphi}{\partial M}=\frac{1}{2}QTQ' \qquad T=2\rho P\frac{\partial\varphi}{\partial M}P'$$

THE FUNDAMENTAL EQUATION CONNECTING STRESS AND STRAIN IN AN ISOTROPIC BODY

We now come to the essential remark which we regard as distinctive of the theory of this paper. The basic equation just written,

$$T=2\rho P\frac{\partial\varphi}{\partial M}P'$$

may be put in a practically usable form when the elastic body is isotropic; the importance of this remark is at once evident when we point out that the matrix P is completely unknown. The elastic body is said to be isotropic if φ is unaffected when the mass element ρdV is subjected to an arbitrary rotation before the deformation takes place; in mathematical terms φ must be unaffected when a is replaced by Oa and x by Ox where O is an arbitrary proper orthogonal matrix. The effect of this replacement is to replace P by OPO' and hence $M=P'P$ by OMO'. Thus, for an isotropic body, φ is such a special function of M that it is completely determined by the characteristic numbers (m_1, m_2, m_3) of M (or, what is the same thing, by the invariants (M_1, M_2, M_3) of M; for O may be so chosen, since M is symmetric, that OMO' is a diagonal matrix with m_1, m_2, and m_3 as its diagonal elements. This implies, as we have seen above, that φ is a function of the invariants N_1, N_2, N_3 of N, and it is only when we regard φ as a function of N, rather than as a function of M, that our fundamental equation (which connects the stress T with the strain) takes a usable form. To obtain this form we observe that the invariants M_1, M_2, M_3 of M are, respectively, the sums of the one-rowed, two-rowed, and three-rowed principal (i.e. diagonal) minors of M. Hence

$$\frac{\partial M_1}{\partial M} = E \qquad \frac{\partial M_2}{\partial M} = M_1E - M \qquad \frac{\partial M_3}{\partial M} = M_3M^{-1}$$

and, similarly,

$$\frac{\partial N_1}{\partial N} = E \qquad \frac{\partial N_2}{\partial N} = N_1E - N \qquad \frac{\partial N_3}{\partial N} = N_3N^{-1}$$

Since $M_1 = \frac{N_2}{N_3}$, $M_2 = \frac{N_1}{N_3}$, $M_3 = \frac{1}{N_3}$, and since N satisfies its own characteristic equation (i.e. since $N^3 - N_1N^2 + N_2N - N_3E = 0$), it follows that

$$\frac{\partial M_1}{\partial N} = -N^{-2} \qquad \frac{\partial M_2}{\partial N} = \frac{1}{N_3}(E - N_1N^{-1}) \qquad \frac{\partial M_3}{\partial N} = -\frac{1}{N_3}N^{-1}$$

Since $PP' = N^{-1}$, $PMP' = PP'PP' = N^{-2}$, so that

$$P\frac{\partial M_1}{\partial M}P' = N^{-1} = -N\frac{\partial M_1}{\partial N}$$

$$P\frac{\partial M_2}{\partial M}P' = M_1N^{-1} - N^{-2} = \frac{1}{N_3}(N_2N^{-1} - N_3N^{-2}) = -\frac{1}{N_3}(N - N_1E) = -N\frac{\partial M_2}{\partial N}$$

$$P\frac{\partial M_3}{\partial N}P' = M_3PQQ'P' = M_3E = \frac{1}{N_3}E = -N\frac{\partial M_3}{\partial N}$$

Hence, since φ is a function of M_1, M_2, M_3,

$$P\frac{\partial \varphi}{\partial M}P' = -N\frac{\partial \varphi}{\partial N}$$

so that

$$T = -2\rho N \frac{\partial \varphi}{\partial N}$$

This is the fundamental equation of the theory. It is usual to write $N = E - 2\epsilon$ and to term ϵ the strain or deformation matrix. In this notation our basic equation takes the form

$$T = \rho(E - 2\epsilon)\frac{\partial \varphi}{\partial \epsilon}$$

The approximation furnished by the classical theory is obtained by setting $\rho = \rho_0$ (i.e. by ignoring the change in density of the body) and by replacing $E - 2\epsilon$ by E. When we recall that the high pressures realizable at present suffice to double ρ, it is clear how poor the approximation given by the classical theory must be. The general statement of Hooke's Law, as given by the classical theory, is

$$T = \rho_0 \frac{\partial \varphi}{\partial \epsilon}$$

i.e. the stress is the gradient of the elastic energy (per unit volume rather than per unit mass) relative to the strain. This must, in the exact theory, be replaced by our equation

$$T = \rho(E - 2\epsilon)\frac{\partial \varphi}{\partial \epsilon} = -2\rho N \frac{\partial \varphi}{\partial N}$$

It is interesting to observe that if we introduce the matrix R defined by

$$N = e^{-2R}$$

we have

$$T = \rho \frac{\partial \varphi}{\partial R}$$

In other words, if allowance is made for the change in density, the statement of Hooke's Law which is given in the classical theory becomes an exact statement (and not merely an approximation) if we replace the strain matrix ϵ by R where

$$R = -\frac{1}{2}\log(E - 2\epsilon) = \epsilon + \epsilon^2 + \frac{4\epsilon^3}{3} + \cdots$$

In this connection a remark concerning Biot's[1] theory of finite strain may be made. The essential feature of this theory is the introduction of the positive symmetric matrix J whose square is M:

$$J^2 = M = P'P$$

It is, however, not convenient to write the exact equations in terms of J; we must, rather, introduce the positive square root K of N^{-1}:

$$K^2 = N^{-1} = PP'$$

Then $\frac{\partial\varphi}{\partial K} = -2K^{-3}\frac{\partial\varphi}{\partial N}$ so that $K\frac{\partial\varphi}{\partial K} = -2N\frac{\partial\varphi}{\partial N}$. Hence

$$T = \rho K\frac{\partial\varphi}{\partial K}$$

or, equivalently,

$$\rho_0\frac{\partial\varphi}{\partial K} = \frac{\rho_0}{\rho}K^{-1}T$$

Since $\rho_0/\rho = \det P$ we have $\rho_0/\rho = \det K$, since $(\det K)^2 = (\det K^2) = \det PP' = (\det P)^2$. Hence

$$\rho_0\frac{\partial\varphi}{\partial K} = (\det K)K^{-1}T$$

If we write $K = E + \epsilon$ and use the approximations $\det(E+\epsilon) = 1 + [\epsilon]$ $K^{-1} = E - \epsilon$, we obtain the (approximative) result

$$\rho_0\frac{\partial\varphi}{\partial K} = (1+[\epsilon])T - \epsilon T$$

given by Biot. The identification of J and K made by Biot is legitimate as an approximation, since P is of the form $P = E + \alpha$ where α is infinitesimal, so that

$$P'P = E + \alpha + \alpha' + \alpha'\alpha \qquad PP' = E + \alpha + \alpha' + \alpha\alpha'$$

Hence if we neglect the difference between $\alpha'\alpha$ and $\alpha\alpha'$, $P'P = PP'$, forcing $J = K$. It is worthy of note that the exact equation

$$T = \rho K\frac{\partial\varphi}{\partial K} = \frac{\rho_0}{\det K}K\frac{\partial\varphi}{\partial K}$$

has such a relatively simple form.

ASSUMPTION CONCERNING THE ELASTIC ENERGY

No further progress can be made until something further is known as to the structure of $\rho_0\varphi$ as a function of the invariants N_1, N_2, N_3 of N. For this we shall have to await further progress in quantum theory, but it is surprising that the naive hypotheses of the classical theory suffice to explain the experimental results already secured in the case where the applied force is a uniform hydrostatic pressure. In order to keep our notation as close as possible to that of the classical theory we set

$$N = E - 2\epsilon$$

so that our fundamental equation takes the form

$$T = \rho(E - 2\epsilon)\frac{\partial\varphi}{\partial\epsilon}$$

On denoting the invariants of the deformation matrix ϵ by I_1, I_2, I_3 we observe that these are infinitesimals of the first, second, and third orders, respectively, if ϵ is an infinitesimal matrix. We regard $\rho_0\varphi$ as expanded in a power series in I_1, I_2, I_3 and keep only terms of order not higher than the second. If the unstrained position is one of zero stress, $\rho_0\varphi$ cannot contain terms of the first order, and since $\rho_0\varphi$ is indeterminate to the extent of an additive constant, it may be assumed to be given by the formula

$$\varphi = \frac{\lambda + 2\mu}{2} I_1^2 - 2\mu I_2$$

and since

$$\frac{\rho}{\rho_0} = \det Q = (\det N)^{\frac{1}{2}}$$

our equation for T becomes

$$T = \{\det(E - 2\epsilon)\}^{\frac{1}{2}}(E - 2\epsilon)(\lambda I_1 E + 2\mu\epsilon)$$

UNIFORM HYDROSTATIC PRESSURE

Let ϵ be a scalar matrix, so that T is also a scalar matrix, and write

$$\epsilon = -fE \qquad T = -pE$$

so that p is the uniform hydrostatic pressure to which the body is subjected. Our formula becomes

$$p = (1+2f)^{\frac{5}{2}}(3\lambda + 2\mu)f$$

so that it involves but one empirical constant $3\lambda + 2\mu$. Despite this lack of flexibility it suffices to account, to within the limits of experimental accuracy, for the compressibility of a whole series of solids under pressures ranging from 0 to 50,000 atm. For details of this comparison of theory with experiment we refer to papers by Birch[2], Bridgman[3], and the writer[4]. The quantity f is determined by the formula

$$\frac{V_0}{V} = \frac{\rho}{\rho_0} = (1+2f)^{\frac{3}{2}} \qquad f = \frac{1}{2}\left[\left(\frac{V_0}{V}\right)^{\frac{2}{3}} - 1\right]$$

where V_0 is the initial volume of the solid and V is the volume of the solid when it is subjected to the pressure p.

In order to facilitate the application of our formula we have prepared the following table furnishing f and $\log\{(1+2f)^{5/2}\, 2f\}$ in terms of $\Delta V = (V_0 - V)/V_0$ for values of ΔV from 0.01 to 0.51 at intervals of 0.01.

TABLE I

ΔV	f	$\log\{(1+2f)^{\frac{5}{2}}2f\}$	ΔV	f	$\log\{(1+2f)^{\frac{5}{2}}2f\}$	ΔV	f	$\log\{(1+2f)^{\frac{5}{2}}2f\}$
0.01	0.00335	7.83332	0.18	0.07075	9.29446	0.35	0.16635	9.83387
.02	.00680	8.14816	.19	.07540	9.33090	.36	.17325	9.86272
.03	.01025	8.33380	.20	.08020	9.36670	.37	.18035	9.89157
.04	.01380	8.47046	.21	.08510	9.40156	.38	.18765	9.92040
.05	.01740	8.57870	.22	.09010	9.43555	.39	.19515	9.94918
.06	.02185	8.68693	.23	.09515	9.46864	.40	.20285	9.97796
.07	.02480	8.74800	.24	.10040	9.50146	.41	.21080	.00682
.08	.02860	8.81775	.25	.10570	9.53330	.42	.21890	.03556
.09	.03245	8.88052	.26	.11115	9.56484	.43	.22730	.06451
.10	.03640	8.93841	.27	.11675	9.59609	.44	.23595	.09353
.11	.04040	8.99176	.28	.12240	9.62661	.45	.24485	.12265
.12	.04445	9.04142	.29	.12825	9.65699	.46	.25400	.15188
.13	.04865	9.08891	.30	.13420	9.68698	.47	.26345	.18125
.14	.05290	9.13369	.31	.14035	9.71684	.48	.27320	.21083
.15	.05720	9.17603	.32	.14660	9.74636	.49	.28330	.24066
.16	.06165	9.21716	.33	.15300	9.77560	.50	.29370	.27065
.17	.06615	9.25646	.34	.15960	9.80484	.51	.30445	.30087

As an example of the use of this table, let us consider the data on the compression of NaCl at room temperature over a range of 5000 to 50,000 atm. In order to give our formula the severest possible test, we adjust our single constant so that our formula gives the correct value 0.0192 for ΔV at the lowest pressure (5000 atm.) for which ΔV is recorded; i.e. we use our formula to extrapolate over the range 5000 to 50,000 atm., taking from experiment only the measurement at 5000 atm. To make our adjustment from the table we interpolate between 0.01 and 0.02 and find for $\log\{(1+2f)^{5/2}2f\}$ the value 8.12297. On subtracting this from $\log 5000$ we obtain 5.57600 as the logarithm of the constant $(3\lambda+2\mu)/2$. Then, to obtain from the table the value of ΔV for 10,000 atm. we subtract 5.57600 from $\log 10{,}000$, obtaining 8.42400, so that ΔV lies between 0.03 and 0.04; by interpolation we find $\Delta V = 0.0366$. The experimentally measured value is 0.0365. The following gives the result of the comparison of theory with experiment over the entire range:

p	ΔV (calculated)	ΔV (observed)
10,000	0.0366	0.0365
15,000	.0520	.0523
20,000	.0662	.0664
25,000	.0806	.0798
30,000	.0935	.0919
35,000	.1056	.1029
40,000	.1159	.1130
45,000	.1275	.1223
50,000	.1376	.1309

The agreement between theory and experiment must be regarded as extraordinarily good. At the highest pressure recorded (50,000 atm.) the calculated value is in excess of the observed value by 5 percent.

In concluding this section we refer to the fact that Bridgman gives (Ref. 3, p. 46) data on the compressibility of rubber for which our formula fails to account. We think that this failure may be attributed to the porous nature of the material. For any such porous or spongy material the compressibility will be much greater at the lower pressures. The deviations from the results given by our formula could be partly accounted for if we knew what percentage of the initial volume was air; or, vice versa, the observed deviations could be used to yield information on this point. From the results given by Bridgman it is indicated by the following (rough) argument that about 7 percent of the initial volume of the rubber specimen was air. If we denote the volumes of air and rubber by subscripts 1 and 2, respectively, it follows from the relation $\Delta V = 1 - V/V_0$ that

$$(V_0)_1 \Delta V_1 + (V_0)_2 \Delta V_2 = V_0 \Delta V$$

If we regard $(V/V_0)_1$ as negligible at pressures of 5000 atm. and up, we have $\Delta V_1 = 1$, so that

$$\frac{(V_0)_2}{V_0} \Delta V_2 = \Delta V - \frac{(V_0)_1}{V_0}$$

If we write $(V_0)_1/V_0 = 7$ percent, $(V_0)_2/V_0 = 93$ percent, and the values for ΔV are to be obtained from the values of ΔV given by Bridgman by subtracting 0.07 and multiplying the result by 100/93. The values so obtained follow the pattern of the data on metals and can be approximated fairly closely by our formula. When we pass to the data at -78.8°, the estimate of 7 percent air (by volume) would have to be reduced to about 2 percent in order that the theory should, similarly, account for the experimental results.

HOLLOW CIRCULAR CYLINDER UNDER INTERNAL PRESSURE

We consider a hollow circular cylinder of internal and external radii r_1 and r_2, respectively, and so long that the end effects may be neglected. The internal surface $r = r_1$ is subjected to a uniform hydrostatic pressure p, whilst the external surface $r = r_2$ is free from pressure. We seek the displacement u (necessarily radial, by symmetry) of any point of the cylinder. On denoting the initial and final values of r by a and r, respectively, we have

$$r = a + u$$

where u is a function of r alone. Since $da = (1 - u')dr$, we have

$$\frac{1}{2}\{(ds)^2 - (ds_0)^2\} = \{u' - \frac{1}{2}(u')^2\}(dr)^2 + \frac{1}{2}(r^2 - a^2)(d\theta)^2$$

$$= \{u' - \frac{1}{2}(u')^2\}(dr)^2 + \{\frac{u}{r} - \frac{1}{2}\left(\frac{u}{r}\right)^2\}(r\,d\theta)^2$$

so that the strain matrix (which may be here regarded as a two-rowed matrix) is diagonal, its diagonal elements being

$$\epsilon_r^r = u' - \frac{1}{2}(u')^2 \qquad \epsilon_\theta^\theta = \frac{u}{r} - \frac{1}{2}\left(\frac{u}{r}\right)^2$$

Hence the matrix N is diagonal, its diagonal elements being

$$N_r^r = (1-u')^2 \qquad N_\theta^\theta = \left(1 - \frac{u}{r}\right)^2$$

It follows that

$$\frac{\rho}{\rho_0} = (1-u')\left(1 - \frac{u}{r}\right)$$

and that

$$T_r^r = (1-u')^3\left(1 - \frac{u}{r}\right)\left[(\lambda+2\mu)\left\{u' - \frac{1}{2}(u')^2\right\} + \lambda\left\{\frac{u}{r} - \frac{1}{2}\left(\frac{u}{r}\right)^2\right\}\right]$$

$$T_\theta^\theta = (1-u')\left(1 - \frac{u}{r}\right)^3\left[\lambda\left\{u' - \frac{1}{2}(u')^2\right\} + (\lambda+2\mu)\left\{\frac{u}{r} - \frac{1}{2}\left(\frac{u}{r}\right)^2\right\}\right]$$

In the plane-polar coordinates here in use the equation $div\, T = 0$ takes the form*

$$\frac{d}{dr}(rT_r^r) = T_\theta^\theta$$

Writing, for the sake of brevity, $\lambda + 2\mu = 2\alpha$ and $\lambda = 2\beta$, we have

$$r\,T_r^r = (1-u')^3\left(1 - \frac{u}{r}\right)\left\{2\alpha(ru') + 2\beta u - \alpha(ru'^2) - \beta\frac{u^2}{r}\right\}$$

so that

$$\frac{d}{dr}(rT_r^r) = (1-u')^3\left(1 - \frac{u}{r}\right)\left\{2\alpha(ru')' + 2\beta u' - \alpha(ru'^2)' - \beta\left(\frac{u^2}{r}\right)'\right\}$$

$$-(1-u')^2\left\{(1-u')\left(\frac{u}{r}\right)' + 3u''\left(1 - \frac{u}{r}\right)\right\}\left\{2\alpha(ru') + 2\beta u - \alpha(ru'^2) - \beta\frac{u^2}{r}\right\}$$

On equating this expression to the one given for T_ϑ^ϑ, we obtain the differential equation which u must satisfy. We obtain the equation of the classical theory by treating u/r and u' as infinitesimals and by retaining only terms of the lowest (i.e. the first) order. This equation is

$$2\alpha(ru')' + 2\beta u' = 2\beta u' + 2\alpha\frac{u}{r}$$

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*Cf., for instance, Ames and Murnaghan[5], p. 59, formula (16.6).

i.e.

$$(ru')' = \frac{u}{r}$$

so that $u = Ar + B/r$, where A and B are constants of integration which must be determined so that $T_r^r = -p$ when $r = r_1$ and $T_r^r = 0$ when $r = r_2$. Since

$$T_r^r = 2\left(\alpha u' + \beta \frac{u}{r}\right) = 2(\alpha+\beta)A - 2(\alpha-\beta)\frac{B}{r^2}$$

we find

$$A = \frac{pr_1^2}{2(\alpha+\beta)(r_2^2-r_1^2)} = \frac{pr_1^2}{2(\lambda+\mu)(r_2^2-r_1^2)} \qquad B = \frac{pr_1^2r_2^2}{2(\alpha-\beta)(r_2^2-r_1^2)} = \frac{pr_1^2r_2^2}{2\mu(r_2^2-r_1^2)}$$

We observe that the infinitesimal u/r is the sum of A and B/r^2 and that $A : B/r_2^2 = \mu : (\lambda+\mu)$.

Returning now to our exact equation we retain terms of the second order. We find

$$\left(1-3u'-\frac{u}{r}\right)\left\{2\alpha(ru')' + 2\beta u' - \alpha(ru'^2)' - \beta\left(\frac{u^2}{r}\right)'\right\} - \left\{\left(\frac{u}{r}\right)' + 3u''\right\}\left\{2\alpha(ru') + 2\beta u\right\}$$

$$= \left(1-u'-3\frac{u}{r}\right)\left\{2\alpha\left(\frac{u}{r}\right) + 2\beta u' - \alpha\left(\frac{u}{r}\right)^2 - \beta(u')^2\right\}$$

or, equivalently,

$$2\alpha\left\{(ru')' - \frac{u}{r}\right\} - \left(3u' + \frac{u}{r}\right)\left\{2\alpha(ru')' + 2\beta u'\right\} - \left\{\left(\frac{u}{r}\right)' + 3u''\right\}\left\{2\alpha(ru') + 2\beta u\right\} - \alpha(ru'^2)'$$

$$- \beta\left(\frac{u^2}{r}\right)' + \left(u' + 3\frac{u}{r}\right)\left\{2\alpha\frac{u}{r} + 2\beta u'\right\} + \alpha\left(\frac{u}{r}\right)^2 + \beta u'^2 = 0$$

To obtain the second approximation we substitute the first approximation $u = Ar + B/r$ in the second order terms. After some reduction we find

$$\alpha\left\{(ru')' - \frac{u}{r}\right\} = -\frac{8B^2}{r^4}(\alpha-\beta) - \frac{2B^2}{r^4}(\alpha+\beta)$$

or, equivalently,

$$(ru')' - \frac{u}{r} = -\frac{2B^2}{r^4}\left(5 - 3\frac{\beta}{\alpha}\right) = -\frac{4(\lambda+5\mu)}{\lambda+2\mu}\frac{B^2}{r^4}$$

and it is at once seen that

$$u = -\frac{(\lambda+5\mu)}{2(\lambda+2\mu)}\frac{B^2}{r^3}$$

is a particular solution of this linearized equation. Hence the second approximation to u is

$$u = Ar + \frac{B}{r} - \frac{(\lambda+5\mu)}{2(\lambda+2\mu)}\frac{B^2}{r^3}$$

In order to obtain a concrete idea of the direction and magnitude of the correction to the classical theory which is furnished by our theory, let us consider an iron tube which is subjected to so great an internal pressure that the internal radius would undergo (according to the classical theory) an increase of 20 percent. In other words, u/a_2 is, to a first approximation, 20 percent so that u/r_1 is, to a first approximation 16 percent. This is composed of the two parts A and B/r_1^2 ; an average value of $\mu:\lambda$ for iron is 0.75, so that $A : B/r_2^2 = 3:7$. Hence $A : B/r_1^2 = 3r_1^2 : 7r_2^2$. Putting this equal to 1/3, roughly (so that $r_2^2 : r_1^2 = 9:7$), the term B/r_1^2 takes care of 75 percent of the relative increase of the internal radius, so that $B/r_1^2 = 12$ percent. Thus our theory predicts a relative increase of the internal radius which is less than that predicted by the classical theory by approximately 9 percent (12 percent of 75 percent). The pressure p_0 in question is approximately $16\mu/300$; μ is approximately 8.10^5 atm., so that p_0 is around 40,000 to 45,000 atm.

CONCLUSION

We have presented in the present paper arguments in support of the thesis that the simple principle of conservation of energy is adequate to explain compressibility phenomena and this without altering the expression for the elastic energy which is familiar in the classical theory. We have pointed out that the main defect of the classical theory has been its neglect of the variation in density of the compressed body. From the mathematical point of view our theory is immensely more difficult than the classical theory for problems other than those dealing with hydrostatic pressure, for the stress is not a linear function of the strain and the principle of superposition is not valid. However, we may express the hope that some of the simpler problems may yet be solved.

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