A new paradigm: the linear isotropic Cosserat model with conformally invariant curvature energy.

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Abstract

This is an essay on a linear Cosserat model with weakest possible constitutive assumptions on the curvature energy still providing for existence, uniqueness and stability. The assumed curvature energy is the conformally invariant expression $\mu\, L_c^2\, \| \operatorname{dev} \operatorname{sym} \nabla \operatorname{axl} \overline{A}\|^2$, where $\operatorname{axl} \overline{A}$ is the axial vector of the skewsymmetric microrotation $\overline{A} \in \mathfrak{so}(3)$, dev is the orthogonal projection on the Lie-algebra $\mathfrak{sl}(3)$ of trace free matrices and sym is the orthogonal projection onto symmetric matrices. It is observed that unphysical singular stiffening for small samples is avoided in torsion and bending while size effects are still present. The number of Cosserat parameters is reduced from six to four: in addition to the (size-independent) classical linear elastic Lamé moduli μ and λ only one Cosserat coupling constant $\mu_c > 0$ and one length scale parameter $L_c > 0$ need to be determined. We investigate those deformations not leading to moment stresses for different curvature assumptions and we exhibit a novel invariance principle of linear, isotropic Cauchy elasticity which is extended to the Cosserat and couple-stress (Koiter-Mindlin) model with conformal curvature.

Key words: polar-materials, microstructure, conformal transformations, structured continua, solid mechanics, variational methods.
AMS 2000 subject classification: 74A35, 74A30, 74C05, 74C10

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1 Introduction

We investigate some of the salient novel features of a linear elastic Cosserat model with conformally invariant curvature energy. This work extends and precises previous work of the first author [22].

General continuum models involving **independent rotations** have been introduced by the Cosserat brothers [5] at the beginning of the last century. Their originally nonlinear, geometrically exact development has been largely forgotten for decades only to be rediscovered in a restricted linearized setting in the early sixties. Since then, the original Cosserat concept has been generalized in various directions, notably by Eringen and his coworkers who extended the Cosserat concept to include also microinertia effects and to rename it subsequently into **micropolar theory**. For an overview[1] of these so called **microcontinuum** theories we refer to [6, 4, 30]. The Cosserat model includes in a natural way **size effects**, i.e., small samples behave comparatively stiffer than large samples. In classical, size-independent models this would lead to an apparent increase of elastic moduli for smaller samples of the same material.

The micropolar theory is perhaps best viewed as a generalized continuum theory in which microstructure details are averaged out by a "characteristic internal length scale" L_c [2, 7]. This last parameter can be considered as the size of a representative volume element (RVE) in heterogeneous media and it is frequently used to model damage and fracture phenomenon in concrete [31]. A dislocated single crystal [24] is another example of a Cosserat continuum for which lattice curvature is due to geometrically necessary dislocations [29]. Extensions to plasticity have been considered in [26, 25, 10, 32].

The mathematical analysis establishing well-posedness for the infinitesimal strain, Cosserat elastic solid is presented e.g. in [11] and extended in [12] for so called linear microstretch models. This analysis has always been based on the **uniform positivity** of the free quadratic energy of the Cosserat solid. The first author has extended the existence results for both the Cosserat model and the more general micromorphic models to the geometrically exact, finite-strain case, see e.g. [27, 23]. More on the mathematical analysis for the nonlinear case can be found in [20, 34]

The important problem of the determination of Cosserat material parameters for continuous solids with random microstructure is still a major challenging problem both analytically [3] and practically. In the linear, isotropic case there are the classical linear elastic Lamé moduli μ and λ whose determination is simple and the possibility of four additional constants, one coupling constant $\mu_c \geq 0$ with dimension [MPa] and three curvature length scales. One of the major problems of the micropolar theory is therefore to determine these parameters in an experimental setting. Lakes [17, 1] proposed an experimental procedure to determine the four supplementary material moduli $(\mu_c, \alpha, \beta, \gamma)$ but the setup is difficult. Usually, a series of experiments with

[1] See http://www.mathematik.tu-darmstadt.de/fbereiche/analysis/pde/staff/neff/patrizio/Cosserat.html

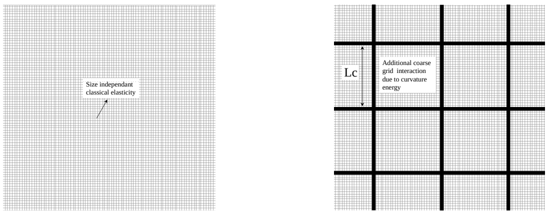

Figure 1: Left: Classical size independent linear elasticity indicated by the fine grid. Right: Additional interaction through Cosserat curvature energy, indicated by a coarse grid superposed on the fine grid, with spacing $L_c > 0$.

specimens of different tiny slenderness is performed in order to determine the additional four Cosserat parameters [17]. By using the traditional curvature energy complying with pointwise positive definiteness, one observes, however, an **unphysical unbounded singular stiffening** behavior for slender specimens [22] which makes it impossible to arrive at consistent values for the Cosserat parameters: the values for the parameters will depend strongly on the smallest investigated specimen size and for very small specimen the experimental values become dubious. Thus a size-independent determination of the material parameters (which must be the ultimate goal) is impossible. This inconsistency is in part responsible for the fact that 1. (linear, isotropic) Cosserat parameters for continuous solids have never gained general acceptance even in the "Cosserat community" and 2. that the linear elastic Cosserat model has never been really accepted by a majority of applied scientists as a useful model to describe size effects in continuous solids.[2]

As a possible answer to this problem we propose to use instead a **weaker curvature energy**[3] of the type

$$W_{\text{curv}}(\nabla \operatorname{axl} \overline{A}) = \mu \, L_c^2 \, \| \operatorname{dev} \operatorname{sym} \nabla \operatorname{axl} \overline{A} \|^2 \,, \tag{1.1}$$

which is **not pointwise positive**. This curvature expression is **conformally invariant** [8] and it reduces the number of additional Cosserat parameters to two: one coupling constant μ_c[MPa] and one internal length scale L_c[m]. Symmetry methods including the conformal group have been applied to extended continuum model, see e.g. [18, 19]. That the weak curvature energy used in (1.1) still gives rise to a well-posed boundary value problem has been recently demonstrated in [13].

Traditionally, a discussion of the Cosserat model starts with the statement that the material is made of particles having an extension and which may move and rotate. Then the question arises invariably whether one can really "see" these particles rotate and whether this rotation coincides with the Cosserat rotation. This has never been conclusively achieved albeit it is tempting to try to identify the rotations of grains in a granular material etc. with these rotations.

For our purposes, let us now shift attention from how the microstructure looks like (rigid = Cosserat, affine = micromorph etc.) to the additional interaction which the Cosserat model introduces. In fact, the effect of the curvature energy is to introduce, in addition to the always present arbitrary fine-scale, size-independent response of linear elasticity a certain additional "coarse grid"-interaction term with long range structure. The interaction strength of which is proportional to the internal length scale L_c, see Figure 1.

For the conformal curvature energy we will show that precisely conformal mappings are "coarse grid"- interaction free, while in the traditionally considered curvature cases, only homogeneous deformations are "coarse grid"-interaction free. What are infinitesimal conformal mappings anyway? Their jacobean consists additively of a dilation and a rotation in each point thus they are locally doing nothing else than dilating and rotating (see Figure 2). They leave the shape and angles of infinitesimal figures invariant. As such, they preserve the topologically structure of the coarse grid exactly. Since on the other extreme, plasticity is triggered by

[2]Note that Lakes himself arrived at consistent values [1] by sacrificing uniform positive definiteness for which he was wrongly criticized in the literature.

[3]In general, using a weaker curvature expression allows to determine larger values of the internal length scale L_c. A major problem in identifying the internal length scale for continuous solids using strong curvature is connected to the fact that the identified values for L_c are orders of magnitude smaller than would make sense from a homogenization point of view in which L_c is related to the size of a representative volume element.

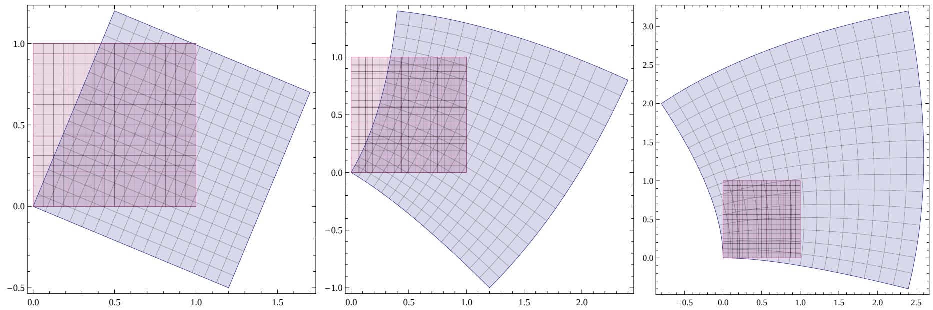

Figure 2: Infinitesimal conformal mappings which locally leave shapes invariant: a prototype elastic deformation. Shown is the coarse grid deformation.

changes in shape (von Mises flow rule) the conformal mappings are really the prototype linear elastic deformations. We remark immediately that conformal mappings are, in that picture, not entirely energy free: they only induce local linear elastic energy. In other words, the conformal mapping is moment free but inhomogeneous. Since the conformal mapping is inhomogeneous but nevertheless represents a certain long range order the constitutive hypothesis of zero "coarse grid"- interaction (conformal curvature) is not altogether unreasonable.

In order to exhibit the additional interaction term our method is to consider the limit case of $L_c \to \infty$ (which corresponds to the presence of only the coarse grid interaction structure) and to investigate what type of deformations do not induce coarse grid interaction. This is what we call subsequently the investigation of the curvature nullspace.

This contribution is now organized as follows. First, we present the linear elastic static isotropic Cosserat model in variational form and recall the necessary conditions for non-negativity of the energy. Then we present the strong form of the Cosserat balance equations together with some development of the scaling behaviour of a finite strain Cosserat model. Following is an investigation of the nullspaces of the curvature energy in the Cosserat model and in the indeterminate couple stress (Koiter-Mindlin) model together with an in depth analysis of the infinitesimal conformal transformations, their general form and related topics. Finally, we exhibit that for zero classical bulk modulus $K = \frac{3\lambda+2\mu}{3} = 0$, linear Cauchy elasticity is formally invariant under infinitesimal conformal transformations and we show that this feature holds true as well for the linear Cosserat model and the indeterminate couple stress model provided the conformal curvature energy is chosen. The conformal approach implies that the Cosserat moment stresses are symmetric and trace free. In the appendix we collect our notation, some relations for infinitesimal conformal mappings as well as a glance at finite conformal mappings.

In a companion paper [14] we already treat the FEM-simulation of our new model. It is our strong believe that the usually assumed pointwise positivity of the Cosserat curvature energy is responsible for the fact that material parameters for the Cosserat solid have not been successfully determined. Thus, relaxing the curvature energy might allow for a new chance of parameter determination, notably of the Cosserat couple modulus μ_c.

2 The linear elastic isotropic Cosserat model revisited

This section does not contain new results, rather it serves to accommodate the widespread notations used in Cosserat elasticity with our own use and to introduce the problem; it is not intended as an introduction to the Cosserat model.

2.1 The linear elastic Cosserat model in variational form

For the **displacement** $u : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ and the **skew-symmetric infinitesimal microrotation** $\overline{A} : \Omega \subset \mathbb{R}^3 \mapsto \mathfrak{so}(3)$ we consider the **two-field** minimization problem

$$I(u, \overline{A}) = \int_\Omega W_{\mathrm{mp}}(\overline{\varepsilon}) + W_{\mathrm{curv}}(\nabla \operatorname{axl} \overline{A}) - \langle f, u \rangle \, \mathrm{dx} \mapsto \quad \min . \text{ w.r.t. } (u, \overline{A}), \tag{2.1}$$

under the following constitutive requirements and boundary conditions

$$
\begin{aligned}
\overline{\varepsilon} &= \nabla u - \overline{A}, \quad \textbf{first Cosserat stretch tensor}\\
u_{|\Gamma} &= u_{\mathrm{d}}\,, \quad \textbf{essential displacement boundary conditions}\\
W_{\mathrm{mp}}(\overline{\varepsilon}) &= \mu\, \|\operatorname{sym}\overline{\varepsilon}\|^2 + \mu_c\, \|\operatorname{skew}\overline{\varepsilon}\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[\operatorname{sym}\overline{\varepsilon}\right]^2 \qquad \textbf{strain energy}\\
\phi &:= \operatorname{axl}\overline{A} \in \mathbb{R}^3, \quad \overline{\mathfrak{k}} = \nabla\phi\,, \quad \|\operatorname{curl}\phi\|^2_{\mathbb{R}^3} = 4\|\operatorname{axl}\operatorname{skew}\nabla\phi\|^2_{\mathbb{R}^3} = 2\|\operatorname{skew}\nabla\phi\|^2_{\mathbb{M}^{3\times 3}}\,,\\
W_{\mathrm{curv}}(\nabla\phi) &= \frac{\gamma+\beta}{2}\|\operatorname{dev}\operatorname{sym}\nabla\phi\|^2 + \frac{\gamma-\beta}{2}\|\operatorname{skew}\nabla\phi\|^2 + \frac{3\alpha+(\beta+\gamma)}{6}\operatorname{tr}\left[\nabla\phi\right]^2. \qquad (2.2)
\end{aligned}
$$

Here, f are given volume forces while u_{d} are Dirichlet boundary conditions for the displacement at $\Gamma \subset \partial\Omega$. Surface tractions, volume couples and surface couples could be included in the standard way. The strain energy W_{mp} and the curvature energy W_{curv} are the most general isotropic quadratic forms in the **infinitesimal non-symmetric first Cosserat strain tensor** $\overline{\varepsilon} = \nabla u - \overline{A}$ and the **micropolar curvature tensor** $\overline{\mathfrak{k}} = \nabla \operatorname{axl}\overline{A} = \nabla\phi$ (curvature-twist tensor). The parameters μ, λ[MPa] are the classical size-independent Lamé moduli and α, β, γ are additional micropolar moduli with dimension $[\mathrm{Pa}\cdot\mathrm{m}^2] = [\mathrm{N}]$ of a force. The additional parameter $\mu_c \geq 0$[MPa] in the strain energy is the **Cosserat couple modulus** which ideally should be size-independent as well. For $\mu_c = 0$ the two fields of displacement and microrotations decouple and one is left formally with classical linear elasticity for the displacement u.

Remark 2.1 (Boundary conditions for the Cosserat model)
It is always possible to prescribe essential boundary values for the microrotations $\overline{A}$ but we abstain from such a prescription because the physical meaning of it is doubtful. Similarly, surface couples are not prescribed. Note that well-posedness of the Cosserat model is true for free-Neumann-type conditions on the microrotation anyway. Therefore, any artificial boundary requirement will heavily influence the solution.

2.2 Non-negativity of the energy

The condition for non-negativity of the energy are well known [22]. It must hold

$$
\begin{aligned}
\mu \geq 0\,, \quad \mu_c \geq 0\,, \quad 2\mu + 3\lambda \geq 0\,,\\
\gamma+\beta \geq 0\,, \quad \gamma-\beta \geq 0\,, \quad 3\alpha + (\beta+\gamma) \geq 0\,.
\end{aligned} \qquad (2.3)
$$

Certain of these inequalities need to be strict in order for the well-posedness of the model. However, the uniform pointwise positivity (strict inequalities everywhere) is not necessary [13], although it is assumed most often in treatments of linear Cosserat elasticity [30].

2.3 Bounded stiffness for small samples

For every physical material, it is essential that small samples still show bounded rigidity. However, this may or may not be true for Cosserat models, depending on the values of Cosserat parameters. Based on analytic solution formulas for simple three-dimensional Cosserat boundary value problems it has been shown in [22] that for **bounded stiffness for arbitrary slender cylindrical samples we must have**

1. in torsion of a slender cylinder: $\beta+\gamma = 0$ or $\Psi = \frac{\beta+\gamma}{\alpha+\beta+\gamma} = \frac{3}{2}$.
2. in bending of a slender cylinder: $(\beta+\gamma)\,(\gamma-\beta) = 0$.

The conformal curvature energy (1.1) satisfies both requirements through $\beta = \gamma$ and $\Psi = \frac{3}{2}$.[4] We note that bounded stiffness does not imply that there is no size effect. Rather, it bounds the size-effect away from unphysical limits.

[4]The additional conditions in [22] for bounded stiffness have been based on dimensionally reduced models and must therefore be taken with care.

2.4 The linear elastic Cosserat balance equations: strong form

The induced balance equations are

$$
\begin{aligned}
\operatorname{Div}\sigma &= f\,, \qquad \textbf{balance of linear momentum}\\
-\operatorname{Div} m &= 4\,\mu_c\cdot \operatorname{axl}\operatorname{skew}\overline{\varepsilon}\,, \quad \textbf{balance of angular momentum}\\
\sigma &= 2\mu\cdot \operatorname{sym}\overline{\varepsilon} + 2\mu_c\cdot \operatorname{skew}\overline{\varepsilon} + \lambda\cdot \operatorname{tr}[\overline{\varepsilon}]\cdot 1\!\!1\,,\\
m &= \gamma\,\nabla\phi + \beta\,\nabla\phi^T + \alpha\operatorname{tr}[\nabla\phi]\cdot 1\!\!1\,, \quad \phi = \operatorname{axl}\overline{A}\,, \quad u_{|_\Gamma} = u_{\mathrm{d}}\,.
\end{aligned}
\tag{2.4}
$$

Here, m is the (second order) **couple stress tensor** which is given as a linear function of the curvature $\nabla\phi = \nabla\operatorname{axl}\overline{A}$ and σ is the non-symmetric force stress tensor.

2.5 The investigated cases

We run the Cosserat model with basically three different sets of variables for the curvature energy which in each step relaxes the curvature energy. The cases are

1: **pointwise positive case:** $\frac{\mu L_c^2}{2}\|\nabla\phi\|^2$. This corresponds to $\alpha = 0, \quad \beta = 0, \quad \gamma = \mu\,L_c^2$. Eringen notes [6, p.151]: "often it is assumed that γ is the leading term and α,β are estimated to be small, non-negative quantities." In the linear setting this case can be arrived at by homogenization of materials with periodic microstructure like grid works and lattice beams, see again [6].

1.1: **deviatoric case:** $\frac{\mu L_c^2}{2}\|\operatorname{dev}\nabla\phi\|^2 = \frac{\mu L_c^2}{2}(\|\nabla\phi\|^2 - \frac{1}{3}\operatorname{tr}[\nabla\phi]^2)$. This corresponds to $\beta = 0$ and $\gamma = \frac{\mu\,L_c^2}{2}$ and $\alpha = -\frac{1}{3}\mu\,L_c^2$. This is the second case of Lakes [17]. Note that interpreting the coefficient α here as a "spring-constant" is impossible, since α takes negative values while the curvature energy is still positive semi-definite. The same remark applies, with appropriate changes, to case three.

2: **symmetric case:** $\frac{\mu L_c^2}{2}\|\operatorname{sym}\nabla\phi\|^2$. This corresponds to $\alpha = 0$, $\beta = \gamma$ and $\gamma = \frac{\mu\,L_c^2}{2}$. In [36, 35] it is proposed to use $\beta = \gamma$ based on non-standard curvature invariance principle. **It leads already to a symmetric couple stress tensor** m. The same requirement, based on another motivation has been arrived at in [33].

3: **conformal case:** $\frac{\mu L_c^2}{2}\|\operatorname{dev}\operatorname{sym}\nabla\phi\|^2 = \frac{\mu L_c^2}{2}(\|\operatorname{sym}\nabla\phi\|^2 - \frac{1}{3}\operatorname{tr}[\nabla\phi]^2)$. This corresponds to $\beta = \gamma$ and $\gamma = \frac{\mu\,L_c^2}{2}$ and $\alpha = -\frac{1}{3}\mu\,L_c^2$. This is the first case of Lakes and our conformal curvature. Here, the Cosserat couple stress tensor m is symmetric and trace free. For the indeterminate couple-stress problem (4.1) the last two cases coincide since the trace term is cancelled.

The reader should realize that all these cases are well-posed. The well-posedness of the last case is a new result, proved in [13], making use of a new coercive inequality for formally positive quadratic forms. The well-posedness in the second case is a consequence of Korn's second inequality applied to the curvature energy. The first case is representative of a pointwise positive curvature energy and therefore deserves no further comment. The first subcase can be subsumed in the third case.

Remark 2.2
In a plain-strain, two-dimensional setting the axis of rotations is constant and all these curvature cases coincide. This underlines the fact that the Cosserat model is essentially three-dimensional.

2.6 Scaling and geometry of microstructure

By a simple scaling argument one may see that very small samples of a material can be described by the Cosserat model with increased L_c. In this sense, $L_c \to \infty$ corresponds to arbitrary small samples. Let us present the scaling relations appearing in a finite-strain elastic Cosserat theory. We consider a finite-strain Cosserat model because the scaling relations are much more transparent then. Our goal is to relate the response of large and small samples of the same material and to asses the influence of the characteristic length L_c.

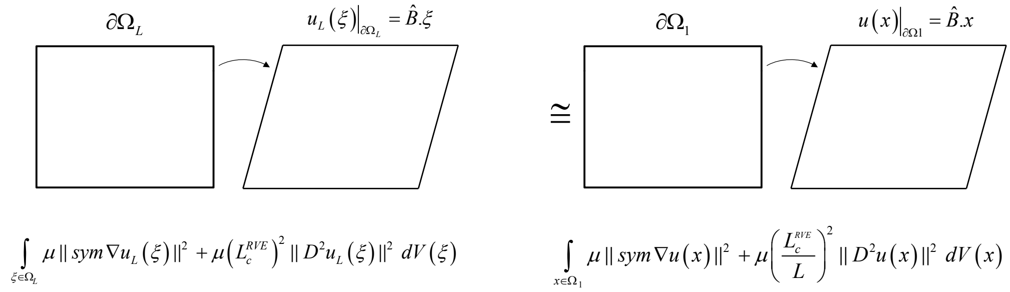

Figure 3: Scaling relations and homogeneous boundary conditions.

First, we define the **characteristic length** L_c^{RVE} as given material parameter, corresponding to the **smallest discernible distance** to be accounted for in the model. A simple consequence is that actual geometrical dimensions L of the bulk material must be larger than L_c^{RVE}, indeed for a continuum theory to apply at all L should be significantly larger than L_c^{RVE}. We may thus identify L_c^{RVE} with the size of a representative volume element RVE. The classical size-independent model ensues if L is arbitrary larger than L_c^{RVE} in which we have separation of scales.

Now let $\Omega_L = [0, L[\text{m}]] \times [0, L[\text{m}]] \times [0, L[\text{m}]]$ be the cube with (non-dimensional) edge length L, representing the bulk material. Consider a deformation $\varphi_L : \xi \in \Omega_L \mapsto \mathbb{R}^3$ and microrotation $\overline{R}_L(\xi) : \Omega_L \mapsto \text{SO}(3)$ as solution of the generic ($\mu_c = \mu$) minimization problem

$$\int_{\xi\in\Omega_L} \mu(\xi)\, \|\overline{R}_L^T(\xi) F_L(\xi) - 1\!\!1\|^2 + \widehat{\mu}\, (L_c^{\text{RVE}})^2\, \|\text{D}_\xi \overline{R}_L(\xi)\|^2 \, \text{d}\xi \mapsto \ \min.\,\text{w.r.t.}\ (\varphi_L, \overline{R}_L)\,, \tag{2.5}$$

$$\text{subject to homogeneous boundary conditions } \xi \in \partial\Omega_L : \quad \varphi_L(\xi) = (1\!\!1 + \widehat{B}).\xi\,, \quad \widehat{B} \in \mathfrak{gl}(3)\,.$$

This is the finite-strain problem which corresponds to the infinitesimal Cosserat model in variational form

$$\begin{aligned}\int_{\xi\in\Omega_L} & \mu(\xi)\, \|\,\text{sym}\, \nabla_\xi u_L(\xi)\|^2 + \mu(\xi)\, \|\,\text{skew}\, \nabla_\xi u_L(\xi) - \overline{A}_L(\xi)\|^2 \\ & + \widehat{\mu}\, (L_c^{\text{RVE}})^2\, \|\text{D}_\xi \overline{A}_L(\xi)\|^2 \, \text{d}\xi \mapsto \ \min.\,\text{w.r.t.}\ (u_L, \overline{A}_L)\,, \end{aligned} \tag{2.6}$$

$$\text{subject to homogeneous boundary conditions } \xi \in \partial\Omega_L : \quad u_L(\xi) = \widehat{B}.\xi\,, \quad \widehat{B} \in \mathfrak{gl}(3)\,.$$

The simple scaling transformation $\zeta : \mathbb{R}^3 \mapsto \mathbb{R}^3$, $\zeta(x) = L \cdot x$ maps the unit cube $\Omega_1 = [0, 1[\text{m}]] \times [0, 1[\text{m}]] \times [0, 1[\text{m}]]$ into Ω_L. Defining the related deformation $\varphi : x \in \Omega_1 \mapsto \mathbb{R}^3$ and microrotation $\overline{R}(x) : \Omega_1 \mapsto \text{SO}(3)$ as

$$\varphi(x) := \zeta^{-1}\left(\varphi_L(\zeta(x))\right), \qquad \overline{R}(x) := \overline{R}_L(\zeta(x))\,, \tag{2.7}$$

shows

$$\begin{aligned}\nabla_x \varphi(x) &= \frac{1}{L} \nabla_\xi \varphi_L(\zeta(x)) \nabla_x \zeta(x) = \nabla_\xi \varphi_L(\xi)\,, \\ \text{D}_\text{x} \overline{R}(x) &= \text{D}_\xi \overline{R}_L(\zeta(x)) \cdot \nabla_x \zeta(x) = \text{D}_\xi \overline{R}_L(\xi) \cdot L\,, \\ \varphi(x) &= \frac{1}{L} \varphi_L(L \cdot x) = (1\!\!1 + \widehat{B}).x\,, \quad x \in \partial\Omega_1\,. \end{aligned} \tag{2.8}$$

Hence, the minimization problem can be transformed to the unit cube[5]

$$\begin{aligned}&\int_{\xi\in\Omega_L} \mu(\xi)\, \|\overline{R}_L^T(\xi) \nabla_\xi \varphi_L(\xi) - 1\!\!1\|^2 + \widehat{\mu}\, L_c^2\, \|\text{D}_\xi \overline{R}_L(\xi)\|^2 \, \text{d}\xi \\ =&\int_{x\in\Omega_1} \mu(Lx)\, \|\overline{R}^T(x) \nabla_x \varphi(x) - 1\!\!1\|^2 \det[\nabla_x \zeta(x)] + \widehat{\mu}\, (L_c^{\text{RVE}})^2\, \|\frac{1}{L} \text{D}_\text{x} \overline{R}(x)\|^2 \det[\nabla_x \zeta(x)] \, \text{dx}\end{aligned}$$

[5] Homogeneous boundary conditions are invariant under the re-scaling, as is any one-homogeneous expression $\varphi_L(r\,\xi) = r\,\varphi_L(\xi)$ as e.g., $\varphi_L(\xi) = B.\xi + \frac{\xi\otimes\xi}{\|\xi\|}.b$.

$$=\int_{x\in\Omega_1} \mu(Lx)\,\|\overline{R}^T(x)\nabla_x\varphi(x)-1\!\!1\|^2\,L^3+\widehat{\mu}\,(L_c^{\mathrm{RVE}})^2\,L^{3-2}\,\|\mathrm{D_x}\overline{R}(x)\|^2\,\mathrm{dx}\,, \tag{2.9}$$

and dividing by L^3 we may consider at last the equivalent problem defined on the unit cube Ω_1:

$$\int_{x\in\Omega_1} \mu(Lx)\,\|\overline{R}^T(x)\nabla_x\varphi(x)-1\!\!1\|^2+\widehat{\mu}\,\frac{(L_c^{\mathrm{RVE}})^2}{L^2}\,\|\mathrm{D_x}\overline{R}(x)\|^2\,\mathrm{dx}\mapsto\ \min.\ \text{w.r.t. }(\varphi,\overline{R}).$$

still subject to homogeneous boundary conditions $x\in\partial\Omega_1:\quad \varphi(x)=(1\!\!1+\widehat{B}).x\,,\quad \widehat{B}\in\mathfrak{gl}(3)\,.$

Thus we are led to define a relative internal length $L_c:=\frac{(L_c^{\mathrm{RVE}})^2}{L^2}$, which is in fact that L_c which we use in this work most of the time. Comparison of different sample sizes is now afforded by transformation to the unit cube respectively, e.g., we compare two samples of the same material with bulk sizes $L_1>L_2$. Transformation to the unit cube shows that the response of sample Ω_{L_2} is stiffer than the response of sample Ω_{L_1}. It is plain to see that for L large compared to L_c^{RVE}, the influence of the rotations will be small and in the limit $\frac{L_c^{\mathrm{RVE}}}{L}\to 0$, classical, size-independent behaviour results. Otherwise, the larger $\frac{L_c^{\mathrm{RVE}}}{L}$, the more pronounced the Cosserat effects become and a small sample is relatively stiffer than a large one.

For a very small cube Ω_L with side length $L\ll 1$ we have $L_c=\frac{L_c^{\mathrm{RVE}}}{L}\gg 1$. Consider therefore (hypothetically) the limit $L_c\to\infty$. In a variational context the energy has to remain finite. In case $L_c=\infty$ it is understood that W_{curv} must vanish. Therefore, the precise form of the curvature energy determines, which deformation possibilities remain for the substructure itself. These deformation possibilities are given by the nullspace of the curvature contribution. The nullspace of the curvature determines therefore the "coarse grid"-interaction law. The hypothetical limit $L_c\to\infty$ therefore characterizes completely the interaction which is induced by the presence of a microstructure which induces a "coarse-grid" setting. Thus we investigate the null-space now.

3 Nullspace of the curvature energy

Since we are interested in the response of the Cosserat model primarily with respect to different curvature energies it is next expedient to investigate the nullspaces of the respective expressions. In the following, constant terms are denoted with a hat by $\widehat{W},\widehat{A}\in\mathfrak{so}(3),\widehat{b}\in\mathbb{R}^3,\widehat{p}\in\mathbb{R}$ etc.

3.1 The pointwise positive nullspace

The first case 1 is simple. In the following we abbreviate with $\phi:\mathbb{R}^3\mapsto\mathbb{R}^3$ the axial vector of the microrotation $\overline{A}\in\mathfrak{so}(3)$, i.e. $\phi=\mathrm{axl}\,\overline{A}$. Subsequently, when there is no danger of confusion, we use $\overline{A}$ also to denote an arbitrary skew-symmetric matrix.

The condition of zero curvature energy $\mu\,L_c^2\,\|\nabla\phi\|^2=0$ is simply $\nabla\phi=0$ and this implies

$$\phi(x_1,x_2,x_3):=\widehat{b}\,, \tag{3.1}$$

for some constant translational vector $\widehat{b}\in\mathbb{R}^3$. This is the three-dimensional space of translations. It implies strong stiffening behaviour as $L_c\to\infty$ which is also observed in our simulations in the companion paper [14] together with the solution for $L_c=\infty$.

Remark 3.1 (Boundary conditions)
If $\overline{A}=0$ (equivalently $\phi=0$) at $\Gamma\subset\partial\Omega$ for (3.1) then $\overline{A}\equiv 0\quad(\phi\equiv 0)$ in Ω. In fact, for smooth fields, it suffices to prescribe $\phi=0$ at an isolated point only.

3.2 The nullspace for dev alone

The first subcase 1.1 is also simple. The condition of zero curvature energy is $\operatorname{dev}\nabla\phi=0$ and this implies $\nabla\phi=p(x)\,1\!\!1$ for some scalar field $p:\Omega\subset\mathbb{R}^3\mapsto\mathbb{R}$. Taking the Curl on both sides of the last equation yields

$$0=\mathrm{Curl}[p(x)\,1\!\!1]=\begin{pmatrix}0 & p_{x_3} & -p_{x_2}\\ -p_{x_3} & 0 & p_{x_1}\\ p_{x_2} & -p_{x_1} & 0\end{pmatrix}\in\mathfrak{so}(3)\,. \tag{3.2}$$

Thus $\nabla p(x) = 0$ and we have after integration

$$\phi(x_1, x_2, x_3) := \widehat{p}\, x + \widehat{b}\,, \tag{3.3}$$

for some constant translational vector $\widehat{b} \in \mathbb{R}^3$ and a constant number $\widehat{p} \in \mathbb{R}$. This is a four-dimensional space. The decisive new feature as compared to (3.1) is that now a linear variation of microrotations does not necessarily lead to curvature energy or moment stresses. This will also obtain in the next case.

Remark 3.2 (Boundary conditions)
If $\overline{A} = 0 \quad (\phi = 0)$ at $\Gamma \subset \partial\Omega$ for (3.3) then $\overline{A} \equiv 0 \quad (\phi \equiv 0)$ in Ω. In fact, for smooth fields, it suffices to prescribe $\phi = 0$ on a one-dimensional curve.

3.3 The symmetric nullspace

In the second symmetric curvature situation, case 2, we obtain from the zero curvature requirement that $\operatorname{sym} \nabla\phi = 0$ which locally means

$$\nabla\phi(x_1, x_2, x_3) = \overline{A}(x_1, x_2, x_2) \in \mathfrak{so}(3) \;\Rightarrow\; 0 = \operatorname{Curl} \overline{A}(x_1, x_2, x_2) \;\Rightarrow\; \overline{A}(x) = \widehat{A} = \text{const.}\,, \tag{3.4}$$

on using formula $(3.6)_4$. This implies that

$$\phi(x_1, x_2, x_3) := \widehat{A}.x + \widehat{b}\,, \tag{3.5}$$

where $\widehat{A} \in \mathfrak{so}(3)$ and $\widehat{b} \in \mathbb{R}^3$ are some constant skew-symmetric matrix and constant translation, respectively. This is the well known six-dimensional space of infinitesimal rigid movements. Let us collect some useful formulas for this case (three space dimensions):

$$\begin{aligned}
-\operatorname{Curl} \overline{A} &= [\nabla \operatorname{axl} \overline{A}]^T - \operatorname{tr}\left[\nabla \operatorname{axl} \overline{A}\right] 1\!\!1\,, \quad \operatorname{tr}\left[\operatorname{Curl} \overline{A}\right] = 2 \operatorname{tr}\left[\nabla \operatorname{axl} \overline{A}\right],\\
\|\operatorname{Curl} \overline{A}\|^2 &= \|\nabla \operatorname{axl}(\overline{A})\|^2 + \operatorname{tr}\left[\nabla \operatorname{axl} \overline{A}\right]^2 \geq \|\nabla \operatorname{axl}(\overline{A})\|^2\,,\\
-\operatorname{sym} \operatorname{Curl} \overline{A} &= \operatorname{sym}[\nabla \operatorname{axl} \overline{A}] - \operatorname{tr}\left[\nabla \operatorname{axl} \overline{A}\right] 1\!\!1\,,\\
\|\operatorname{sym} \operatorname{Curl} \overline{A}\|^2 &= \|\operatorname{sym} \nabla \operatorname{axl}(\overline{A})\|^2 + \operatorname{tr}\left[\nabla \operatorname{axl} \overline{A}\right]^2.
\end{aligned} \tag{3.6}$$

The last equality suggest that the parameter values $\beta = \gamma = \alpha$ could also be an interesting constitutive choice. Inequality $(3.6)_2$ admits a (surprising) generalization to exact rotations [28]. Considering the deviator, we observe, moreover

$$\begin{aligned}
\|\operatorname{dev} \operatorname{sym} \operatorname{Curl} \overline{A}\|^2 &= \|\operatorname{sym} \operatorname{Curl} \overline{A}\|^2 - \frac{1}{3}\operatorname{tr}\left[\operatorname{Curl} \overline{A}\right]^2\\
&= \|\operatorname{sym} \nabla \operatorname{axl}(\overline{A})\|^2 + \operatorname{tr}\left[\nabla \operatorname{axl} \overline{A}\right]^2 - \frac{1}{3}\operatorname{tr}\left[\operatorname{Curl} \overline{A}\right]^2\\
&= \|\operatorname{sym} \nabla \operatorname{axl}(\overline{A})\|^2 + \operatorname{tr}\left[\nabla \operatorname{axl} \overline{A}\right]^2 - \frac{4}{3}\operatorname{tr}\left[\nabla \operatorname{axl} \overline{A}\right]^2\\
&= \|\operatorname{sym} \nabla \operatorname{axl}(\overline{A})\|^2 - \frac{1}{3}\operatorname{tr}\left[\nabla \operatorname{axl} \overline{A}\right]^2 = \|\operatorname{dev} \operatorname{sym} \nabla \operatorname{axl} \overline{A}\|^2\,.
\end{aligned} \tag{3.7}$$

Remark 3.3 (Boundary conditions)
If $\overline{A} = 0 \quad (\phi = 0)$ at $\Gamma \subset \partial\Omega$ for (3.5) then $\overline{A} \equiv 0 \quad (\phi \equiv 0)$ in Ω. This result follows as in linear elasticity for the displacement. Here it is decisive that Γ is a two-dimensional surface, otherwise the infinitesimal rotations are not fixed.

3.4 The conformal nullspace

In the last case 3 we obtain for vectorfields $\phi : \mathbb{R}^3 \mapsto \mathbb{R}^3$ the condition $\operatorname{dev} \operatorname{sym} \nabla\phi = 0$. One can show that the nullspace in dimension $n \geq 3$ has dimension $(n+1)\,(n+2)/2$.[6] To see this

[6] In dimension $n = 2$ the kernel is infinite-dimensional. Consider

$$\nabla\phi(x) = \begin{pmatrix} \phi_{1,x_1} & \phi_{1,x_2} \\ \phi_{2,x_1} & \phi_{2,x_2} \end{pmatrix} = \begin{pmatrix} \widehat{p} & \widehat{a} \\ -\widehat{a} & \widehat{p} \end{pmatrix} \Rightarrow \operatorname{dev}_2 \operatorname{sym} \nabla\phi = 0\,. \tag{3.8}$$

Thus ϕ_1, ϕ_2 satisfy the Cauchy-Riemann equations and all harmonic functions are in the kernel.

for dimension $n=3$, consider that $\operatorname{dev}\operatorname{sym}\nabla\phi=0$ implies $\nabla\phi=p(x)\,1\!\!1+\overline{A}(x)$ for a scalar field $p:\Omega\mapsto\mathbb{R}$ and a skew-symmetric field $\overline{A}:\Omega\mapsto\mathfrak{so}(3)$. Taking the Curl yields

$$
\begin{aligned}
\nabla\phi=p(x)\,1\!\!1+\overline{A}(x)\;&\Rightarrow\;0=\underbrace{\operatorname{Curl}[p(x)\,1\!\!1]}_{\in\mathfrak{so}(3),\,(3.2)}+\operatorname{Curl}\overline{A}(x)\;\Rightarrow\\
0=\operatorname{sym}\operatorname{Curl}\overline{A}(x)\;&\underbrace{\Rightarrow}_{(3.6)_4}\;\operatorname{sym}\nabla\operatorname{axl}(\overline{A}(x))=0\;\Rightarrow\\
\nabla\operatorname{axl}(\overline{A}(x))=\overline{W}(x)\in\mathfrak{so}(3)\;&\Rightarrow\;\operatorname{axl}(\overline{A}(x))=\widehat{W}.x+\widehat{\eta}\;\Rightarrow \qquad (3.9)\\
-\operatorname{Curl}\overline{A}=[\nabla\operatorname{axl}\overline{A}]^T-\operatorname{tr}\left[\nabla\operatorname{axl}\overline{A}\right]1\!\!1&=\widehat{W}^T-0\;\Rightarrow\;\operatorname{Curl}[p(x)\,1\!\!1]=\widehat{W}^T=-\widehat{W}\;\Rightarrow\\
\operatorname{Curl}[p(x)\,1\!\!1]=\begin{pmatrix}0&p_{x_3}&-p_{x_2}\\-p_{x_3}&0&p_{x_1}\\p_{x_2}&-p_{x_1}&0\end{pmatrix}&=-\widehat{W}\;\Rightarrow\;\nabla p=\begin{pmatrix}p_{x_1}\\p_{x_2}\\p_{x_3}\end{pmatrix}=\begin{pmatrix}-\widehat{W}_{23}\\\widehat{W}_{13}\\-\widehat{W}_{12}\end{pmatrix}=\operatorname{axl}(\widehat{W})\,.
\end{aligned}
$$

Integration yields $p(x)=\langle\operatorname{axl}(\widehat{W}),x\rangle+\widehat{p}$, which implies

$$
\begin{aligned}
\nabla\phi&=p(x)\,1\!\!1+\overline{A}(x)=[\langle\operatorname{axl}(\widehat{W}),x\rangle+\widehat{p}]\,1\!\!1+\operatorname{anti}(\widehat{W}.x+\widehat{\eta})\\
&=[\langle\operatorname{axl}(\widehat{W}),x\rangle+\widehat{p}]\,1\!\!1+\operatorname{anti}(\widehat{W}.x)+\widehat{A}=\operatorname{anti}(\widehat{W}.x)+\langle\operatorname{axl}(\widehat{W}),x\rangle\,1\!\!1+[\widehat{p}\,1\!\!1+\widehat{A}]\;\Rightarrow\\
\phi&=\frac{1}{2}\left(2\langle\operatorname{axl}(\widehat{W}),x\rangle\,x-\operatorname{axl}(\widehat{W})\,\|x\|^2\right)+[\widehat{p}\,1\!\!1+\widehat{A}].x+\widehat{b}\,. \qquad (3.10)
\end{aligned}
$$

We have thus shown that for $n=3$ the kernel is ten-dimensional.[7] It consists of all **infinitesimal conformal transformations (ICT)** having the form (abbreviate $\widehat{k}=\frac{1}{2}\operatorname{axl}(\widehat{W})$)

$$
\begin{aligned}
\phi_C(x_1,x_2,x_3):&=\sum_{i=1}^{3}\widehat{k}_i\;Q_i(x,x)+\widehat{M}.x+\widehat{b}=2\langle\widehat{k},x\rangle-\widehat{k}\,\|x\|^2+\widehat{M}.x+\widehat{b}\,,\quad \widehat{M}=\widehat{p}\,1\!\!1+\widehat{A}\,,\\
\nabla\phi_C(x_1,x_2,x_3)&=2\left[\operatorname{anti}(\widehat{W}.x)+\langle\operatorname{axl}(\widehat{W}),x\rangle\,1\!\!1\right]+\widehat{M}\,, \qquad (3.11)\\
D^2\phi_C(x).h&=2\left[\operatorname{anti}(\widehat{W}.h)+\langle\operatorname{axl}(\widehat{W}),h\rangle\,1\!\!1\right]\in\mathbb{R}\,1\!\!1\oplus\mathfrak{so}(3)\,,
\end{aligned}
$$

where $\widehat{p},\widehat{k}_1,\widehat{k}_2,\widehat{k}_3\in\mathbb{R}$ and $\widehat{A}\in\mathfrak{so}(3)$ and $\widehat{b}$ are constant numbers, constant skew-symmetric matrix and constant translation, respectively. Here, $Q_i:\mathbb{R}^3\times\mathbb{R}^3\mapsto\mathbb{R}^3$ are three **infinitesimal special conformal transformations (ISCT)** (which we have shown to be second order polynomials):

$$
\begin{aligned}
Q_1(x,x):=2\,x_1\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}-\begin{pmatrix}\|x\|^2\\0\\0\end{pmatrix}&=\begin{pmatrix}x_1^2-(x_2^2+x_3^2)\\2\,x_1x_2\\2\,x_1x_3\end{pmatrix},\quad \nabla Q_1(x,x)=2\begin{pmatrix}x_1&-x_2&-x_3\\x_2&x_1&0\\x_3&0&x_1\end{pmatrix},\\
Q_2(x,x):=2\,x_2\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}-\begin{pmatrix}0\\\|x\|^2\\0\end{pmatrix}&=\begin{pmatrix}2\,x_1\,x_2\\x_2^2-(x_1^2+x_3^2)\\2\,x_2x_3\end{pmatrix},\quad \nabla Q_2(x,x)=2\begin{pmatrix}x_2&x_1&0\\-x_1&x_2&-x_3\\0&x_3&x_2\end{pmatrix},\\
Q_3(x,x):=2\,x_3\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}-\begin{pmatrix}0\\0\\\|x\|^2\end{pmatrix}&=\begin{pmatrix}2\,x_3\,x_1\\2\,x_3\,x_2\\x_3^2-(x_1^2+x_2^2)\end{pmatrix},\quad \nabla Q_3(x,x)=2\begin{pmatrix}x_3&0&x_1\\0&x_3&x_2\\-x_1&-x_2&x_3\end{pmatrix}
\end{aligned}
\qquad (3.12)
$$

It is easy to check that $\nabla Q_i=p(x)\,1\!\!1+\overline{A}(x)$ for $\overline{A}(x)\in\mathfrak{so}(3)$ and $p(x)\in\mathbb{R}$. A short form representation is given by

$$
Q_i(x,x)=2\langle x,e_i\rangle\,x-\|x\|^2\,e_i\,. \qquad (3.13)
$$

3.5 3D-ICT and boundary conditions

Let us show that infinitesimal conformal transformations (ICT), while offering richer possibilities than rigid movements, are still uniquely determined when set to zero on a two-dimensional smooth surface Γ. More precisely, we show

[7] It is plain to see that ϕ_C forms a ten-dimensional linear space which can be endowed with the structure of a Lie-algebra by using as Lie-bracket the usual commutator bracket for vectorfields.

Lemma 3.4
If $\overline{A} = 0 \quad (\phi = 0)$ *at* $\Gamma \subset \partial\Omega$ *for* (3.11) *then* $\overline{A} \equiv 0 \quad (\phi \equiv 0)$ *in* Ω.

Proof. We choose curves $\gamma_i : \mathbb{R} \mapsto \Gamma \subset \mathbb{R}^3$ which lie on the surface Γ. Along these curves it holds by assumption that

$$\phi(\gamma(t)) = 0 \Rightarrow 0 = \frac{\mathrm{d}}{\mathrm{dt}}\phi(\gamma(t)) = \nabla\phi(\gamma(t)).\gamma'(t)\,. \tag{3.14}$$

Since we can always choose curves which pass through a given point $\gamma(t_0) = x_0 \in \Gamma$ and since Γ is a smooth two-dimensional surface, there exist always two-linear independent directions τ_1, τ_2 such that

$$\nabla\phi(x_0).\tau_i = 0\,, \quad \tau_i = \gamma_i\prime(t_0)\,,\ i = 1, 2, \quad \gamma(t_0) = x_0\,. \tag{3.15}$$

Therefore, we conclude that if $\phi = 0$ on Γ then the rank of $\nabla\phi$ is maximally one on Γ. On the other hand, ϕ being infinitesimal conformal, we have

$$\begin{aligned}\nabla\phi(x_0) &= \mathrm{anti}(\widehat{W}.x_0) + \langle \mathrm{axl}(\widehat{W}), x_0\rangle\, 1\!\!1 + [\widehat{p}\, 1\!\!1 + \widehat{A}]\\ &= \mathrm{anti}(\widehat{W}.x_0) + \widehat{A} + [\langle \mathrm{axl}(\widehat{W}), x_0\rangle + \widehat{p}]\, 1\!\!1\,.\end{aligned} \tag{3.16}$$

Let us check the rank of this expression on Γ. Since it has the form $\nabla\phi = \mathfrak{so}(3) + \mathbb{R}\cdot 1\!\!1$ we only have to show that there exists an $x_0 \in \Gamma$ at which not both summands vanish simultaneously. This suffices since, if either of them is nonzero, then the rank is at least two: if only the skew symmetric part vanishes then the rank is three, if only the dilation (spherical) part vanishes, then the rank is two.

Individually, if both vanish, we have

$$\langle \mathrm{axl}(\widehat{W}), x_0\rangle = -\widehat{p}\,, \quad 0 = \mathrm{anti}(\widehat{W}.x_0) + \widehat{A} \Rightarrow \widehat{W}.x_0 = -\,\mathrm{axl}(\widehat{A})\,. \tag{3.17}$$

In matrix form

$$\begin{pmatrix} \mathrm{axl}(\widehat{W})^T \\ \widehat{W} \end{pmatrix} x_0 = \begin{pmatrix} -\widehat{p} \\ -\,\mathrm{axl}(\widehat{A}) \end{pmatrix}_{\mathbb{R}^4}\,. \tag{3.18}$$

The first case is $\widehat{W} = 0$. Then $\widehat{p}$ and $\widehat{A}$ are both zero, which implies $\phi \equiv 0$. In the second case assume now that $\widehat{W} \neq 0$. A simple calculation shows that

$$\mathrm{rank}\begin{pmatrix} \mathrm{axl}(\widehat{W})^T \\ \widehat{W} \end{pmatrix}_{3\times 4} = 3 \quad \Leftrightarrow \quad \widehat{W} \neq 0\,. \tag{3.19}$$

Thus the solution of (3.18) is given by

$$x_0 = x_{\mathrm{inhom}} + s\cdot x_{\mathrm{hom}}\,, \quad s \in \mathbb{R}\,, \tag{3.20}$$

which parameterizes a straight line, but Γ is two-dimensional; the contradiction. ■

3.6 The geometry of the nullspaces: visualization

In order to get a feeling for the transformations which lie in the nullspace we show now how the two-dimensional first quadrant is mapped by infinitesimal conformal transformations (ICT). The basis for the visualization is a two-dimensional, projected version of (3.11). The only relevant components in the (x_1, x_2)-plane are therefore given by[8]

$$Q_1(x,x) = \begin{pmatrix} x_1^2 - x_2^2 \\ 2x_1\, x_2 \end{pmatrix}, \quad Q_2(x,x) = \begin{pmatrix} 2x_1\, x_2 \\ x_2^2 - x_1^2 \end{pmatrix}, \quad \widehat{M} = \begin{pmatrix} \widehat{p} & \widehat{a} \\ -\widehat{a} & \widehat{p} \end{pmatrix}, \quad \widehat{b} = \begin{pmatrix} \widehat{b}_1 \\ \widehat{b}_2 \end{pmatrix}.$$

We plot subsequently the transformation of the first quadrant $[0,1]\times[0,1]$ under the mapping

$$x \mapsto x + \phi_C^{2\mathrm{D}}(x) = x + \widehat{k}_1\, Q_1(x,x) + \widehat{k}_2\, Q_2(x,x) + \widehat{M}.x + \widehat{b}\,, \tag{3.21}$$

[8] For the visualization we drop the out of plane component in Q_3.

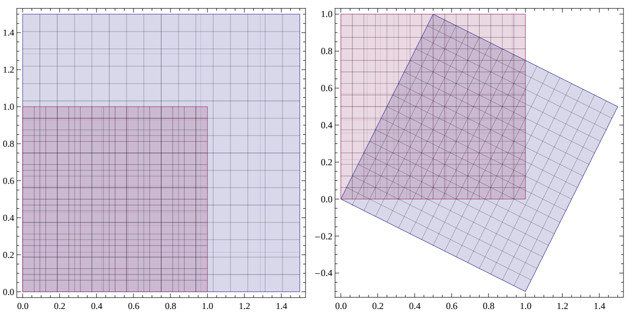

Figure 4: **Left: Mappings in the nullspace for** dev **alone**. In this case, apart for the ubiquitous constant translation vector $\widehat{b} \in \mathbb{R}^3$, we have $\widehat{p} \neq 0$ but $\widehat{a} = 0, \widehat{k}_1, \widehat{k}_2 = 0$. The first quadrant is **homogeneously scaled** with $\widehat{p}$. Here $\widehat{p} = 0.5$. **Right: Mappings in the symmetric nullspace**. Here, the transformation possibilities are encoded by $\widehat{a} \neq 0$ but $\widehat{p} = 0, \widehat{k}_1, \widehat{k}_2 = 0$. The first quadrant is **homogeneously rotated** with infinitesimal rotation angle $\widehat{a} = 0.5$. Note that the infinitesimal rotation also leads to a homogeneous increase in volume which is an artifact of the linear model. In any of these cases, the deformation is homogeneous only!

for numbers $\widehat{k}_1, \widehat{k}_2, \widehat{p}, \widehat{a}, \widehat{b}_1, \widehat{b}_2$. The parameter $\widehat{a}$ is the infinitesimal rotation angle, $\widehat{b}$ is a simple translation and will therefore be neglected, $\widehat{p}$ is the infinitesimal change in length and $\widehat{k}_1, \widehat{k}_2$ parametrize the two-dimensional inhomogeneous infinitesimal special conformal transformations (ISCT).

In Figure 4 and Figure 5 we show the encoded deformation possibilities. The mappings in the nullspace for pointwise positive curvature can only shift the first quadrant by the constant vector $\widehat{b} \in \mathbb{R}^3$ and are therefore not visualized. Note that this is not the deformation of the substructure itself, since the transformation corresponds to the axial vector of the infinitesimal rotation of the substructure, i.e, $\overline{A}(x) = \operatorname{anti}(\phi(x))$, but we use ϕ to show the transformation.

4 Nullspaces for the indeterminate couple stress problem

The indeterminate couple stress problem [21, 15] is characterized by the identification $\frac{1}{2} \operatorname{curl} u = \operatorname{axl} \overline{A} = \phi$ which can be formally obtained from the genuine Cosserat model by setting $\mu_c = \infty$. Since here the infinitesimal microrotations $\overline{A}$ cease to be an independent field the model has the advantage of conceptional simplicity and improved physical transparency.[9] We can completely characterize what type of displacement u does not induce curvature energy for the different curvature cases. This allows to us to understand what kind of "torsional spring analogy" may be implied by the respective curvatures. Let us recall this Koiter-Mindlin model, for simplicity without external loads.

4.1 The indeterminate couple stress model

For the **displacement** $u : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ we consider the **one-field** minimization problem

$$I(u) = \int_\Omega W_{\mathrm{mp}}(\nabla u) + W_{\mathrm{curv}}(\nabla \operatorname{curl} u)\, \mathrm{dV} \mapsto \quad \min.\ \text{w.r.t. } u,$$

under the constitutive requirements and boundary conditions

$$\begin{aligned} W_{\mathrm{mp}}(\overline{\varepsilon}) &= \mu\, \|\operatorname{sym} \nabla u\|^2 + \frac{\lambda}{2} \operatorname{tr}[\operatorname{sym} \nabla u]^2, \quad u_{|_\Gamma} = u_{\mathrm{d}}, \\ W_{\mathrm{curv}}(\nabla \operatorname{curl} u) &= \frac{\gamma + \beta}{8}\, \|\operatorname{sym} \nabla \operatorname{curl} u\|^2 + \frac{\gamma - \beta}{8}\, \|\operatorname{skew} \nabla \operatorname{curl} u\|^2. \end{aligned} \tag{4.1}$$

In this limit model, the curvature parameter α, related to the spherical part of the (higher order) couple stress tensor m remains **indeterminate**, since $\operatorname{tr}[\nabla \phi] = \operatorname{Div} \operatorname{axl} \overline{A} = \operatorname{Div} \frac{1}{2} \operatorname{curl} u = 0$. A motivation for this model in a finite-strain, multiplicative elasto-plastic context has been given

[9] At the prize of being a fourth order boundary value problem.

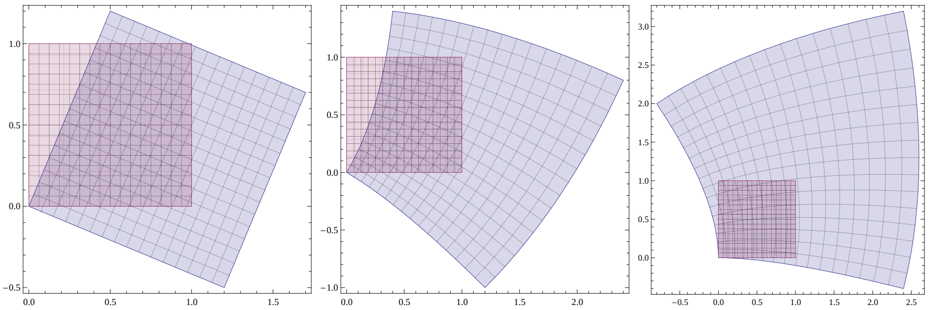

Figure 5: **Mappings in the conformal nullspace**. Finally, in the conformal case we can vary $\widehat{a}, \widehat{p}, \widehat{k}_1, \widehat{k}_2$. We plot the transformation of the first quadrant that does not induce curvature energy. If $\widehat{k}_1^2 + \widehat{k}_2^2 > 0$, then the first quadrant is **inhomogeneously** transformed. Clock wise: homogeneous rotation and scaling with $\widehat{p} = 0.2, \widehat{a} = 0.5$, inhomogeneous mapping $\widehat{a} = 0, \widehat{p} = 0.6, \widehat{k}_1 = 0.2, \widehat{k}_2 = 0.4$ and $\widehat{a} = 0, \widehat{p} = 0.6, \widehat{k}_1 = 0.8, \widehat{k}_2 = 0.4$. The effect of the curvature energy is to introduce, in addition to the always present arbitrary fine-scale, size-independent response of linear elasticity a certain additional "coarse grid"-interaction term with long range structure. The interaction strength of which is proportional to L_c. For the conformal curvature energy, the conformal mappings are therefore "coarse grid" interaction free, while in the previous cases, only homogeneous deformations are "coarse grid" interaction free. Remark that the above mappings are not entirely energy free: they only induce local linear elastic energy. In other words, the conformal mapping is moment free but inhomogeneous. Since the conformal mapping is inhomogeneous but nevertheless represents a certain long range order the constitutive hypothesis of zero "coarse grid" interaction is not altogether unreasonable.

recently in [9]. Following [15], it is practically always assumed that $-1 < \eta := \frac{\beta}{\gamma} < 1$ in order to guarantee uniform positive definiteness [3]. For the conformal case, we use, on the contrary $\frac{\beta}{\gamma} = 1$, which makes the couple stress tensor symmetric and trace free. The curvature free displacements u are, by definition, those displacements that "survive" in the limit of internal length scale $L_c \to \infty$ (i.e. $\gamma + \beta \to \infty$, $\gamma - \beta \to \infty$).

4.2 Curvature free displacement for pointwise positive curvature

This is the case where, formally, $\gamma + \beta, \gamma - \beta > 0$. Here, from $\|\nabla \operatorname{curl} u\| = 0$ it must hold for a given constant vector $\widehat{b} \in \mathbb{R}^3$, see (3.1)

$$\frac{1}{2}\operatorname{curl} u = \operatorname{axl} \overline{A}(x) = \widehat{b} \;\Rightarrow\; \operatorname{curl} u = 2\,\widehat{b} \;\Rightarrow\; u(x) = \nabla\zeta(x) + \underbrace{\operatorname{anti}(\widehat{b}).x + \widehat{\xi}}_{\text{infinitesimal rigid movement}}\,, \tag{4.2}$$

We find the solution in the form $u = u_{\text{hom}} + u_{\text{spec}}$. The homogeneous solution $\operatorname{curl} u_{\text{hom}} = 0$ is $u_{\text{hom}} = \nabla\zeta + \xi$ where $\zeta : \mathbb{R}^3 \mapsto \mathbb{R}$ is a scalar potential and $\widehat{\xi} \in \mathbb{R}^3$ is another constant vector. One special solution of $\operatorname{curl} u = 2\,\widehat{b}$ is given by $u_{\text{spec}} = \operatorname{anti}(\widehat{b}).x$ since $\operatorname{curl} u = 2\operatorname{axl}(\operatorname{skew}\nabla u)$. Altogether, the displacement gradient follows as

$$\nabla u(x) = D^2\zeta(x) + \operatorname{anti}(\widehat{b})\,. \tag{4.3}$$

Hence, in the elastic energy only the symmetric part appears with energy

$$\mu\,\|\operatorname{dev}\operatorname{sym}\nabla u(x)\|^2 + \frac{K}{2}\operatorname{tr}\left[\nabla u(x)\right]^2 = \mu\,\|\operatorname{dev} D^2\zeta(x)\|^2 + \frac{K}{2}\operatorname{tr}\left[D^2\zeta(x)\right]^2. \tag{4.4}$$

Thus only the irrotational part contributes to the elastic energy and for $L_c \to \infty$ and $-1 < \eta < 1$ the limit variational problem reduces to a second order energy on a scalar potential ζ.

4.3 Curvature free displacement for dev-curvature

Here, we consider the first subcase 1.1. Looking at (3.3) and using the identification $\frac{1}{2}\operatorname{curl} u = \operatorname{axl}(\overline{A}) = \phi$ we obtain for a given constant vector $\widehat{b} \in \mathbb{R}^3$ and a given constant number $\widehat{p}$

$$\operatorname{curl} u = \widehat{p}\,x + \widehat{b} \;\Rightarrow\; u(x) = \nabla\zeta(x) + \operatorname{anti}(\widehat{b}).x + \widehat{\xi}\,, \tag{4.5}$$

where $\zeta : \mathbb{R}^3 \mapsto \mathbb{R}$ is a scalar potential and $\widehat{\xi} \in \mathbb{R}^3$ is another constant vector. This case coincides with the previous one! To see this, consider

$$\operatorname{curl} u = \widehat{p}\, x + \widehat{b} \;\Rightarrow\; 0 = \operatorname{Div}\operatorname{curl} u(x) = \operatorname{Div}[\widehat{p}\, x] = \operatorname{tr}\left[\nabla[\widehat{p}\, x]\right] = \operatorname{tr}\left[\widehat{p}\,\mathbb{1}\right] = 3\,\widehat{p}\,. \tag{4.6}$$

Thus, $\widehat{p}$ must be zero and we are back in the previous case.

4.4 Curvature free displacement for symmetric curvature

Here, for a given constant vector $\widehat{b} \in \mathbb{R}^3$ and a given constant skew-symmetric matrix $\overline{A} \in \mathfrak{so}(3)$ we must have (see (3.5))

$$\operatorname{curl} u = \widehat{A}.x + \widehat{b} \;\Rightarrow\; u(x) = \nabla\zeta(x) + P_2(x) + \operatorname{anti}(\widehat{b}).x + \widehat{\xi}\,, \tag{4.7}$$

where $\zeta : \mathbb{R}^3 \mapsto \mathbb{R}$ is a scalar potential, $\widehat{\xi} \in \mathbb{R}^3$ is another constant vector and $P_2 : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is a homogeneous polynomial of second order such that

$$\operatorname{curl} P_2(x) = \widehat{A}.x\,. \tag{4.8}$$

A simple calculation confirms the (for us at first surprising) result that for some constant vector $\widehat{\eta} \in \mathbb{R}^3$, depending on the entries of $\widehat{A} \in \mathfrak{so}(3)$, the polynomial P_2 can be chosen as

$$P_2(x) = 2\langle \eta, x\rangle\, x - \eta\, \|x\|^2\,. \tag{4.9}$$

To see this, we compute the total differential of P_2 for $h \in \mathbb{R}^3$

$$\begin{aligned}
DP_2(x).h &= 2\,\langle \eta, h\rangle\, x + 2\,\langle \eta, x\rangle\, h - 2\,\eta\,\langle x, h\rangle = 2\,\left(x \otimes \eta + \langle \eta, x\rangle\,\mathbb{1} - \eta \otimes x\right).h\\
&= 2\,\left(2\,\operatorname{skew}(x \otimes \eta) + \langle \eta, x\rangle\,\mathbb{1}\right).h \;\Rightarrow\\
\nabla P_2(x) &= 2\,\left(2\,\operatorname{skew}(x \otimes \eta) + \langle \eta, x\rangle\,\mathbb{1}\right)\\
\operatorname{curl} P_2(x) &:= 2\,\operatorname{axl}(\operatorname{skew}\nabla P_2) = 8\,\operatorname{axl}(\operatorname{skew}(x \otimes \eta)) = 4\,\widehat{\eta} \times x = 4\,\operatorname{anti}(\widehat{\eta}).x\,,
\end{aligned} \tag{4.10}$$

where we have used, in this order of appearance, that $\operatorname{curl} u = 2\,\operatorname{axl}(\operatorname{skew}\nabla u)$ and $\operatorname{axl}(\operatorname{skew}(a \otimes b)) = -\frac{1}{2}\,a \times b$ and $\operatorname{axl}(A) \times x = A.x$. Therefore, choosing $\widehat{\eta} = \frac{1}{4}\operatorname{axl}(\widehat{A})$ shows the claim. The polynomial P_2 is nothing else than the infinitesimal special conformal transformation (ISCT).[10] Regarding (4.7) we may always subsume the scalar potential to be given in the form $\zeta + \frac{\widehat{p}}{2}\,\|x\|^2$ by misuse of notation for ζ. Thus, the curvature free displacements in the indeterminate couple stress theory with symmetric curvature (symmetric moment stresses) are of the form

$$u(x) = \nabla\zeta(x) + \underbrace{\frac{1}{2}\left(2\langle \operatorname{axl}(\widehat{W}), x\rangle\, x - \operatorname{axl}(\widehat{W})\,\|x\|^2\right) + [\widehat{p}\,\mathbb{1} + \widehat{A}].x + \widehat{b}}_{\text{infinitesimal conformal mapping}}\,, \tag{4.11}$$

with arbitrary constant terms $\widehat{W}, \widehat{A} \in \mathfrak{so}(3)$, $\widehat{b} \in \mathbb{R}^3$ and $\widehat{p} \in \mathbb{R}$. The corresponding displacement gradient is given by

$$\nabla u(x) = \underbrace{D^2\zeta(x)}_{\in \operatorname{Sym}(3):\ \text{irrotational}} + \underbrace{[\langle \operatorname{axl}(\widehat{W}), x\rangle + \widehat{p}]\,\mathbb{1} + \operatorname{anti}(\widehat{W}.x) + \widehat{A}}_{\text{conformal derivative}}\,. \tag{4.12}$$

Hence, in the elastic energy of the formal limit problem $L_c = \infty$ only the symmetric part appears with energy

$$\begin{aligned}
&\mu\,\|D^2\zeta(x) + \operatorname{sym}\nabla P_2(x)\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[D^2\zeta(x) + \nabla P_2(x)\right]^2\\
&= \mu\,\|\operatorname{dev}\operatorname{sym}(D^2\zeta(x) + \nabla P_2(x))\|^2 + \frac{2\mu + 3\lambda}{6}\operatorname{tr}\left[D^2\zeta(x) + \nabla P_2(x)\right]^2\\
&= \mu\,\|\operatorname{dev} D^2\zeta(x)\|^2 + \frac{K}{2}\operatorname{tr}\left[D^2\zeta(x) + [\langle \operatorname{axl}(\widehat{W}), x\rangle + \widehat{p}]\,\mathbb{1}\right]^2.
\end{aligned} \tag{4.13}$$

Assuming the formal limit case of zero bulk-modulus $K = 0$, the elastic energy consists only of $\mu\,\|\operatorname{dev} D^2\zeta\|^2$.

[10] Thus, the infinitesimal special conformal transformations ϕ can be equivalently characterized through the condition $\operatorname{curl}\phi = \widehat{A}.x$ for arbitrary $\widehat{A} \in \mathfrak{so}(3)$!

4.5 Curvature free displacement for conformal curvature

Here, we must have $\operatorname{curl} u = \phi$, where $\phi : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is just an infinitesimal conformal mapping (ICT), which is, because of (3.11) given as

$$\begin{aligned}\phi(x) &= \frac{1}{2}\left(2\langle \operatorname{axl}(\widehat{W}), x\rangle\, x - \operatorname{axl}(\widehat{W})\, \|x\|^2\right) + [\widehat{p}\, 1\!\!1 + \widehat{A}].x + \widehat{b}\,,\\ \nabla\phi(x) &= [\langle \operatorname{axl}(\widehat{W}), x\rangle + \widehat{p}]\, 1\!\!1 + \operatorname{anti}(\widehat{W}.x) + \widehat{A}\,,\end{aligned} \tag{4.14}$$

where $\widehat{W}, \widehat{A} \in \mathfrak{so}(3)$ and $\widehat{b} \in \mathbb{R}^3$ and $\widehat{p}$ are given constants. Consider

$$\begin{aligned}0 = \operatorname{Div}\operatorname{curl} u = \operatorname{Div}\phi = \operatorname{tr}[\nabla\phi] = 3\,[\langle \operatorname{axl}(\widehat{W}), x\rangle + \widehat{p}] \;\Rightarrow\\ \forall\, x \in \mathbb{R}^3: \quad -\widehat{p} = \langle \operatorname{axl}(\widehat{W}), x\rangle\,.\end{aligned} \tag{4.15}$$

The last equation determines x to lie on a plane with normal $\operatorname{axl}(\widehat{W})$ but $x \in \mathbb{R}^3$ is arbitrary. Since $\widehat{p}$ and $\widehat{W}$ are both constant, they must therefore vanish. Hence, $\operatorname{curl} u = \widehat{A}.x + \widehat{b}$. Thus, the conformal case is indistinguishable from the symmetric case as far as the formal limit $L_c = \infty$ is concerned in the indeterminate couple stress problem.

4.6 Curl-operator and infinitesimal conformal functions

Let us note a remarkable property concerning the curl-operator and infinitesimal conformal functions (ICT). We have

Lemma 4.1 (Infinitesimal conformal functions are closed under curl**)**
Let $\phi \in ICT$ be given. Then $\operatorname{curl}\phi = \widehat{A}.x + \widehat{b} \in ICT$ for some constant skew-symmetric matrix $\widehat{A} \in \mathfrak{so}(3)$ and some constant vector $\widehat{b}$.

Proof. We have derived a complete characterization of infinitesimal conformal functions ICT given in (3.11). With constant terms $\widehat{W}, \widehat{A} \in \mathfrak{so}(3), \widehat{p} \in \mathbb{R}, \widehat{b} \in \mathbb{R}^3$ they have the form

$$\begin{aligned}\phi(x) &= \frac{1}{2}\left(2\langle \operatorname{axl}(\widehat{W}), x\rangle\, x - \operatorname{axl}(\widehat{W})\, \|x\|^2\right) + [\widehat{p}\, 1\!\!1 + \widehat{A}].x + \widehat{b}\,,\\ \nabla\phi(x) &= [\langle \operatorname{axl}(\widehat{W}), x\rangle + \widehat{p}]\, 1\!\!1 + \operatorname{anti}(\widehat{W}.x) + \widehat{A}\,.\end{aligned} \tag{4.16}$$

Thus, applying the curl-operator, we obtain

$$\operatorname{curl}\phi = 2\ \operatorname{axl}(\operatorname{skew}(\nabla\phi)) = 2\ \operatorname{axl}(\operatorname{anti}(\widehat{W}.x) + \widehat{A})) = 2\,[\widehat{W}.x + \operatorname{axl}(\widehat{A})]\,, \tag{4.17}$$

so that $\operatorname{curl}\phi$ is in fact an infinitesimal rigid movement since $\widehat{W} \in \mathfrak{so}(3)$ and $\operatorname{axl}(\widehat{A}) \in \mathbb{R}^3$. ■

5 Conformal invariance for zero bulk modulus

The infinitesimal conformal invariance of the curvature energy does, however, not imply that the fully Cosserat coupled problem has this invariance in general. However, infinitesimal conformal invariance is true in one special formal case: the case of zero bulk modulus. Of course, standard engineering materials have positive bulk modulus $K > 0$, which is also necessary for the well-posedness. Here, we set formally $K = 0$ but we remark that composite man made materials may have small K or even $K = 0$ [16].[11] Let us consider linear elasticity as a starting point.

5.1 Conformal invariance in Cauchy elasticity

Considering the free energy of linear, isotropic Cauchy elasticity in the form

$$\int_\Omega \mu(x)\, \|\operatorname{dev}\operatorname{sym}\nabla u\|^2 + \frac{K(x)}{2}\operatorname{tr}[\nabla u]^2\,\mathrm{dx}\,, \tag{5.1}$$

[11] http://silver.neep.wisc.edu/ lakes/Poisson.html

we observe that in the formal limit of zero bulk modulus $K = \frac{2\mu+3\lambda}{3} = 0$ the energy is invariant under the transformation $u \mapsto u+\phi$, whenever ϕ is an infinitesimal conformal mapping, because

$$\operatorname{dev}\operatorname{sym}\nabla(u+\phi) = \operatorname{dev}\operatorname{sym}\nabla u + \operatorname{dev}\operatorname{sym}\nabla\phi = \operatorname{dev}\operatorname{sym}\nabla u\,. \tag{5.2}$$

Since the first, deviatoric term measures only change in shape it does not see those transformations, which, infinitesimally, do not change shape - precisely the infinitesimal conformal mappings $\phi \in ICT$. Thus, for zero bulk modulus $K = 0$, displacements u which are infinitesimal conformal mappings, see Figure 5, do not contribute to the elastic energy at all on the linear elastic "macroscopic level". A similar conclusion has been reached in [19] for incompressible isotropic linear elasticity with zero pressure.

5.2 Conformal invariance in Cosserat elasticity

We consider the free energy of linear, isotropic Cosserat elasticity for zero bulk modulus $K = 0$ in the form

$$\int_\Omega \mu\,\|\operatorname{dev}\operatorname{sym}\nabla u\|^2 + \frac{\mu_c}{2}\,\|\operatorname{curl} u - 2\operatorname{axl}\overline{A}\|^2 + \mu\,L_c^2\,\|\operatorname{dev}\operatorname{sym}\nabla\operatorname{axl}\overline{A}\|^2\,\mathrm{dx}\,. \tag{5.3}$$

With our preparation we see now immediately, that this energy is invariant under the transformation of displacement and microrotations through

$$(u,\,\operatorname{axl}\overline{A}) \mapsto (u+\phi,\,\operatorname{axl}\overline{A} + \frac{1}{2}\operatorname{curl}\phi)\,, \tag{5.4}$$

for all $\phi \in ICT$. The invariance of the first term is clear as in linear elasticity. For the third term use Lemma 4.1 to note that $\frac{1}{2}\operatorname{curl}\phi \in ICT$. For the second coupling term observe that

$$\operatorname{curl}(u+\phi) - 2[\operatorname{axl}\overline{A} + \frac{1}{2}\operatorname{curl}\phi] = \operatorname{curl} u - 2\operatorname{axl}\overline{A}\,. \tag{5.5}$$

5.3 Conformal invariance in indeterminate couple stress theory

We consider at last the free energy of the linear, isotropic indeterminate couple stress theory for zero bulk modulus $K = 0$ in the form

$$\int_\Omega \mu\,\|\operatorname{dev}\operatorname{sym}\nabla u\|^2 + \frac{\mu\,L_c^2}{4}\,\|\operatorname{dev}\operatorname{sym}\nabla\operatorname{curl} u\|^2\,\mathrm{dx}\,. \tag{5.6}$$

As before this energy is invariant under the transformation

$$u \mapsto u+\phi\,, \tag{5.7}$$

exactly as linear elasticity is for zero bulk modulus.

Surprisingly, therefore, conformal invariance for zero bulk modulus can also be obtained for the indeterminate couple stress model and the genuine Cosserat model provided we choose the conformal curvature expression. The line of argument is therefore not, why a model should have conformal invariance, but to realize that linear elasticity has it on the outset for a certain parameter range, and, therefore, the hypothesis is not altogether unreasonable that the extended continuum models should have it as well for the same parameter range! We summarize these findings in the novel

Postulate I: If an isotropic linear elastic solid (whether it be linear Cauchy elastic or a more general linear extended continuum model) with positive bulk modulus is (infinitesimally) conformally deformed then the elastic energy must consist only of a purely volumetric term.

In the companion paper [14] we use conformal invariance to obtain inhomogeneous analytical solutions for boundary value problems in Cosserat elasticity.

5.4 Universal potential solutions for variable bulk modulus

Consider linear, isotropic Cauchy elasticity with constant shear modulus μ and variable bulk modulus $K(x)$:

$$\int_\Omega \mu \, \| \operatorname{dev}\operatorname{sym}\nabla u\|^2 + \frac{K(x)}{2}\operatorname{tr}[\nabla u]^2 \, \mathrm{dx} \ \mapsto \min . \, u$$

$$u_{|_\Gamma}(x) = \nabla \zeta(x) \,, \quad \zeta : \mathbb{R}^3 \mapsto \mathbb{R} \,, \quad \Delta\zeta = 0 \,, \tag{5.8}$$

where ζ is a given harmonic function. This problem has a unique solution irrespective of the variation of the bulk modulus $K(x)$ and the solution is $u(x) \equiv \nabla\zeta(x)$. We see this from

$$\begin{aligned}
\sigma &= 2\,\mu\, \operatorname{dev}\operatorname{sym}\nabla u + K(x)\operatorname{tr}[\nabla u]\mathbb{1} = 2\,\mu\,(\operatorname{sym}\nabla u - \frac{1}{3}\operatorname{tr}[\nabla u]\mathbb{1}) + K(x)\operatorname{tr}[\nabla u]\mathbb{1} \\
&= 2\,\mu\, \operatorname{sym}\nabla u + (K(x) - \frac{2\mu}{3})\operatorname{tr}[\nabla u]\mathbb{1} = \mu\,(\nabla u + \nabla u^T) + (K(x) - \frac{2\mu}{3})\operatorname{Div} u\,\mathbb{1}\,, \\
\operatorname{Div}\sigma &= \mu\,\Delta u + \mu\,\nabla \operatorname{Div} u + \operatorname{Div}(K(x) - \frac{2\mu}{3})\operatorname{Div} u\,\mathbb{1}) \\
&= \mu\,\Delta u + \mu\,\nabla \operatorname{Div} u + \nabla(K(x) - \frac{2\mu}{3})\operatorname{Div} u) \\
&= \mu\,\Delta u + \nabla\left((\mu + (K(x) - \frac{2\mu}{3}))\operatorname{Div} u\right) = \mu\,\Delta u + \nabla\left((K(x) + \frac{\mu}{3})\operatorname{Div} u\right) .
\end{aligned} \tag{5.9}$$

For $u = \nabla\zeta$ we have $\operatorname{Div} u = \Delta\zeta = 0$. Moreover,

$$\Delta u = \begin{pmatrix} \Delta u^1 \\ \Delta u^2 \\ \Delta u^3 \end{pmatrix} = \begin{pmatrix} \Delta(\nabla\zeta)^1 \\ \Delta(\nabla\zeta)^2 \\ \Delta(\nabla\zeta)^3 \end{pmatrix} = \begin{pmatrix} \Delta\zeta_x \\ \Delta\zeta_y \\ \Delta\zeta_z \end{pmatrix} = \begin{pmatrix} (\Delta\zeta)_x \\ (\Delta\zeta)_y \\ (\Delta\zeta)_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} . \tag{5.10}$$

Thus $\operatorname{Div}\sigma = 0$ and the boundary conditions are trivially satisfied. It is clear that the same holds true for the general Cosserat and the indeterminate couple stress problem since by the appearance of $\operatorname{curl} u$ in both models, the term $\nabla\zeta$ will be annihilated.

We turn this result as well into a novel requirement

Postulate II: If an isotropic linear elastic solid (whether it be linear Cauchy elastic or a more general linear extended continuum model) is subject to harmonic gradient Dirichlet boundary conditions $u_{|_\Gamma}(x) = \nabla\zeta(x)\,,\ \zeta : \mathbb{R}^3 \mapsto \mathbb{R}\,,\ \Delta\zeta = 0$, then the unique solution must be given by $u(x) \equiv \nabla\zeta(x)$.

Remark 5.1
It is tempting to assume that Postulates I and II together would exclude any higher order derivative dependence other than that on $\nabla \operatorname{curl} u$ (or on $\nabla \operatorname{axl}\overline{A}$ in the Cosserat model) in a higher gradient model. But this is open.

6 Conclusion and open problems

The reduction in Cosserat parameters from six to four was first necessitated by the newly observed physical principle of bounded stiffness for very small samples. Here we related this reduction to the conformal invariance of linear Cauchy elasticity for vanishing bulk modulus. We investigated the curvature null-spaces and showed for both Cosserat and indeterminate couple stress problem what kind of (quite inhomogeneous) mappings do not contribute to the curvature energy. This led us to require two new Postulates which can be applied to narrow down the multitude of constitutive choices for extended continuum models.

Certainly the linear elastic models have a restricted range of applications. Thus it is pressing to come up with a geometrically exact extension of the conformal curvature expression. Formally,

$$\| \operatorname{dev}\operatorname{sym}\overline{R}^T \operatorname{Curl}\overline{R}\|^2_{\mathbb{M}^{3\times 3}} \tag{6.1}$$

is linearization equivalent to the conformal expression $\| \operatorname{dev} \operatorname{sym} \nabla \operatorname{axl} \overline{A} \|^2$. But there are many other expressions like (6.1) having the same linearization. Here, a deeper differential geometric insight is called for, perhaps in combination with the group of special conformal transformations. Note finally, that a geometrically exact model based on (6.1) would not be coercive when simultaneously putting $\mu_c = 0$. since from (6.1) it is not clear how to obtain $\overline{R} \in W^{1,2}(\Omega, \mathrm{SO}(3))$.

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Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 second order tensors, written with capital letters and Sym denotes symmetric second orders tensors. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times 3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times 3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3\times 3}$. The identity tensor on $\mathbb{M}^{3\times 3}$ will be denoted by $1\!\!1$, so that $\operatorname{tr}[X] = \langle X, 1\!\!1\rangle$. We set $\operatorname{sym}(X) = \frac{1}{2}(X^T + X)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \operatorname{sym}(X) + \operatorname{skew}(X)$. For $X \in \mathbb{M}^{3\times 3}$ we set for the deviatoric part $\operatorname{dev} X = X - \frac{1}{3}\operatorname{tr}[X]\,1\!\!1 \in \mathfrak{sl}(3)$ where $\mathfrak{sl}(3)$ is the Lie-algebra of traceless matrices. The set $\operatorname{Sym}(n)$ denotes all symmetric $n \times n$-matrices. The Lie-algebra of $\mathrm{SO}(3) := \{X \in \mathrm{GL}(3)\ | X^T X = 1\!\!1,\ \det[X] = 1\}$ is given by the set $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times 3}\ | X^T = -X\}$ of all skew symmetric tensors. The canonical identification of $\mathfrak{so}(3)$ and $\mathbb{R}^3$ is denoted by $\operatorname{axl}\overline{A} \in \mathbb{R}^3$ for $\overline{A} \in \mathfrak{so}(3)$. The Curl operator on the three by three matrices acts row-wise, i.e.

$$\operatorname{Curl}\begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33}\end{pmatrix} = \begin{pmatrix}\operatorname{curl}(X_{11}, X_{12}, X_{13})^T \\ \operatorname{curl}(X_{21}, X_{22}, X_{23})^T \\ \operatorname{curl}(X_{31}, X_{32}, X_{33})^T\end{pmatrix}. \tag{6.1}$$

Moreover, we have

$$\forall\, A \in \mathrm{C}^1(\mathbb{R}^3, \mathfrak{so}(3)): \quad \operatorname{Div} A(x) = -\operatorname{curl}\operatorname{axl}(A(x)). \tag{6.2}$$

Note that $(\operatorname{axl}\overline{A}) \times \xi = \overline{A}.\xi$ for all $\xi \in \mathbb{R}^3$, such that

$$\begin{gathered}\operatorname{axl}\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0\end{pmatrix} := \begin{pmatrix}-\gamma \\ \beta \\ -\alpha\end{pmatrix}, \quad \overline{A}_{ij} = \sum_{k=1}^{3} -\varepsilon_{ijk}\cdot \operatorname{axl}\overline{A}_k, \\ \|\overline{A}\|^2_{\mathbb{M}^{3\times 3}} = 2\,\|\operatorname{axl}\overline{A}\|^2_{\mathbb{R}^3}, \qquad \langle \overline{A}, \overline{B}\rangle_{\mathbb{M}^{3\times 3}} = 2\langle \operatorname{axl}\overline{A}, \operatorname{axl}\overline{B}\rangle_{\mathbb{R}^3},\end{gathered} \tag{6.3}$$

where ε_{ijk} is the totally antisymmetric permutation tensor. Here, $\overline{A}.\xi$ denotes the application of the matrix $\overline{A}$ to the vector ξ and $a \times b$ is the usual cross-product. Moreover, the inverse of axl is denoted by anti and defined by

$$\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0\end{pmatrix} := \operatorname{anti}\begin{pmatrix}-\gamma \\ \beta \\ -\alpha\end{pmatrix}, \quad \operatorname{axl}(\operatorname{skew}(a \otimes b)) = -\frac{1}{2}\, a \times b, \tag{6.4}$$

and

$$2\operatorname{skew}(b \otimes a) = \operatorname{anti}(a \times b) = \operatorname{anti}(\operatorname{anti}(a).b). \tag{6.5}$$

Moreover,

$$\operatorname{curl} u = 2\operatorname{axl}(\operatorname{skew}\nabla u). \tag{6.6}$$

By abuse of notation we denote the differential $D\varphi$ of the deformation $\varphi : \mathbb{R}^3 \mapsto \mathbb{R}^3$ by $\nabla\varphi$. This implies a transposition in certain comparisons with other literature since here $(\nabla\varphi)_{ij} = \partial_j \varphi^i$ is understood.

7 Appendix

7.1 Infinitesimal conformal mappings (ICT) at a glance

Here we gather some useful formulas for infinitesimal conformal mappings. (**Needs to be checked**)

$$
\begin{aligned}
\phi_C(x) &= \frac{1}{2}\left(2\langle \operatorname{axl}(\widehat{W}), x\rangle\, x - \operatorname{axl}(\widehat{W})\,\|x\|^2\right) + [\widehat{p}\,1\!\!1 + \widehat{A}].x + \widehat{b}\,,\\
\nabla\phi_C(x) &= [\langle \operatorname{axl}(\widehat{W}), x\rangle + \widehat{p}]\,1\!\!1 + \operatorname{anti}(\widehat{W}.x) + \widehat{A}\,,\\
\operatorname{tr}[\nabla\phi_C(x)] &= 3\,\left[\langle \operatorname{axl}(\widehat{W}), x\rangle + \widehat{p}\right]\,,\\
\operatorname{skew}\nabla\phi_C(x) &= \operatorname{anti}(\widehat{W}.x) + \widehat{A}\,,\\
\operatorname{sym}\nabla\phi_C(x) &= [\langle \operatorname{axl}(\widehat{W}), x\rangle + \widehat{p}]\,1\!\!1\,,\\
\operatorname{dev}\operatorname{sym}\nabla\phi_C(x) &= 0\,,\\
\operatorname{Div}\phi_C(x) &= \operatorname{tr}[\nabla\phi_C] = 3\,\left[\langle \operatorname{axl}(\widehat{W}), x\rangle + \widehat{p}\right]\,,\\
\nabla\operatorname{Div}\phi_C(x) &= 3\,\operatorname{axl}(\widehat{W})\,,\\
\operatorname{curl}\phi_C(x) &= 2\,[\widehat{W}.x + \operatorname{axl}(\widehat{A})] = \widehat{\widehat{A}}.x + \widehat{\widehat{b}}\,,\\
\nabla\operatorname{curl}\phi_C(x) &= 2\,\widehat{W}\,,\\
\operatorname{curl}(\operatorname{curl}\phi_C(x)) &= 2\operatorname{axl}(\operatorname{skew}\nabla\operatorname{curl}\phi_C(x)) = 4\,\operatorname{axl}(\widehat{W})\,,\\
\Delta\phi_C(x) &= \operatorname{Div}\nabla\phi_C(x) = \nabla\operatorname{Div}\phi_C(x) - \operatorname{curl}\operatorname{curl}\phi_C(x) = -\operatorname{axl}(\widehat{W})\,,\\
D^2\phi_C(x).h &= \operatorname{anti}(\widehat{W}.h) + \langle \operatorname{axl}(\widehat{W}, h\rangle 1\!\!1 \qquad \in \mathbb{R}\,1\!\!1 \oplus \mathfrak{so}(3)\,.
\end{aligned} \tag{7.1}
$$

For **infinitesimal special conformal functions (ISCT)** we have thus

$$
\begin{aligned}
\phi_C^{ISCT}(x) &= \frac{1}{2}\left(2\langle \operatorname{axl}(\widehat{W}), x\rangle\, x - \operatorname{axl}(\widehat{W})\,\|x\|^2\right)\,,\\
\operatorname{curl}\phi_C^{ISCT}(x) &= 0 \;\Rightarrow\; \phi_C^{ISCT}(x) = 0\,,\\
\operatorname{Div}\phi_C^{ISCT}(x) &= 0 \;\Rightarrow\; \phi_C^{ISCT}(x) = 0\,.
\end{aligned} \tag{7.2}
$$

7.2 Conformal transformations

A conformal transformation (CT) is a continuous invertible mapping preserving the form of infinitesimal figures. Any conformal map on a portion of Euclidean space of dimension greater than 2 can always be composed from three types of transformation: a homothetic transformation (uniform dilation), an isometry (rigid rotation and translation), and a **special conformal transformation (SCT)**, where a **"special conformal transformation" is the composition of a reflection and an inversion on a sphere**. Thus, the group of conformal transformations in spaces of dimension greater than 2 are much more restricted than the planar case, where the Riemann mapping theorem provides a large group of conformal transformations and where indeed all holomorphic functions are conformal.

The conformal property may be described in terms of the Jacobean derivative matrix of a coordinate transformation. If the Jacobean matrix of the transformation is everywhere a scalar times a rotation matrix, then the transformation is conformal. Thus, the deformation gradient of a conformal mapping satisfies $\nabla\varphi \in \mathbb{R}^+\,\mathrm{SO}(3)$. This implies that infinitesimal shapes of bodies (our unit square for example) are preserved. What is not preserved, is the size of the body. For more on conformal field theory we refer to [8].[12]

7.3 Finite special conformal transformations (FSCT)

The **inversion on a sphere** of a point $x \in \mathbb{R}^3$ with respect to a sphere with center $\eta \in \mathbb{R}^3$ and radius $k > 0$ is given by

$$
\begin{aligned}
\operatorname{inv}_\eta(x) :&= \eta + \frac{k^2\,(x-\eta)}{\|x-\eta\|^2} = \frac{1}{\|x-\eta\|^2}\,\left(\eta\,\|x-\eta\|^2 + k^2\,(x-\eta)\right)\,,\\
D[\operatorname{inv}_\eta(x)].h &= k^2\,\|x-\eta\|^{-2}\left(1\!\!1 - 2\frac{(x-\eta)\otimes(x-\eta)}{\|x-\eta\|^2}\right)\,h \;\Rightarrow\\
\nabla\operatorname{inv}_\eta(x) &= k^2\,\|x-\eta\|^{-2}\underbrace{\left(1\!\!1 - 2\frac{(x-\eta)\otimes(x-\eta)}{\|x-\eta\|^2}\right)}_{\in\mathrm{O}(3)\,,\quad \det[\cdot]=-1} \in \mathbb{R}^+\;\mathrm{O}(3)\,.
\end{aligned} \tag{7.3}
$$

This is an **anti-conformal map**, i.e., it preserves angles but the orientation is reversed. Therefore, it needs to be composed with an orientation reversing map like a reflection at a hyperplane to give rise to a conformal map. The **reflection at a plane through the origin** with unit-normal $\vec{n}$ is given by

$$
\begin{aligned}
\operatorname{reflect}(x) :&= x - 2\,\langle x, \vec{n}\rangle\,\vec{n} = Q.x\,,\\
Q &= 1\!\!1 - 2\,\vec{n}\otimes\vec{n}\,,\quad Q^TQ = 1\!\!1\,,\quad \det[Q] = -1\,.
\end{aligned} \tag{7.4}
$$

[12] In two-dimensions, every Mobius transformation is a conformal map. The group of Mobius-transformations in dimension two has dimension three. For Mobius-Transformations check: http://www.youtube.com/watch?v=JX3VmDgiFnY.

Composing the inversion with this reflection yields

$$
\begin{aligned}
\widehat{\Phi}(x) = \text{reflect}(\text{inv}_\eta(x)) &= Q.\left(\eta + \frac{k^2\,(x-\eta)}{\|x-\eta\|^2}\right)\\
&= \eta + \frac{k^2\,(x-\eta)}{\|x-\eta\|^2} - 2\langle \eta + \frac{k^2\,(x-\eta)}{\|x-\eta\|^2}, \vec{n}\rangle\,\vec{n}\,, \qquad (7.5)\\
\nabla[\widehat{\Phi}(x)] &= k^2\,\|x-\eta\|^{-2}\,Q\,\underbrace{\left(1\!\!1 - 2\frac{(x-\eta)\otimes(x-\eta)}{\|x-\eta\|^2}\right)}_{\in \mathrm{O}(3)\,,\quad \det[\cdot]=-1} \in \mathbb{R}^+\,\mathrm{SO}(3)\,,
\end{aligned}
$$

which shows that this composition is a conformal map.

In order to see the relation between the infinitesimal special conformal functions (3.13) and the finite conformal functions (7.5) we consider the conformal map in (7.5) with $\eta = 0$, $k^2 = 1$, $\vec{n} = e_i$ and expand at e_i with respect to $\delta x \in \mathbb{R}^3$. This yields

$$
\begin{aligned}
\widehat{\Phi}(e_1 + \delta x) &= \frac{e_i + \delta x}{\|e_i + \delta x\|^2} - 2\langle \frac{e_i + \delta x}{\|e_i + \delta x\|^2}, e_i\rangle\, e_i\\
&= e_i + \delta x - 2\langle \delta x, e_i\rangle e_i - 2\langle \delta x, e_i\rangle \delta x + 4\langle \delta x, e_i\rangle^2 e_i - \|\delta x\|^2 e_i + \dots \qquad (7.6)\\
&\quad - 2\langle e_i + \delta x - 2\langle \delta x, e_i\rangle e_i - 2\langle \delta x, e_i\rangle \delta x + 4\langle \delta x, e_i\rangle^2 e_i - \|\delta x\|^2 e_i + \dots, e_i\rangle e_i\\
&= e_i + \delta x - 2\langle \delta x, e_i\rangle e_i - 2\langle \delta x, e_i\rangle \delta x + 4\langle \delta x, e_i\rangle^2 e_i - \|\delta x\|^2 e_i + \dots\\
&\quad - 2\left(1 + \langle \delta x, e_i\rangle - 2\langle \delta x, e_i\rangle - 2\langle \delta x, e_i\rangle^2 + 4\langle \delta x, e_i\rangle^2 - \|\delta x\|^2 + \dots\right) e_i\\
&= -e_i + \delta x - 2\langle \delta x, e_i\rangle \delta x + 4\langle \delta x, e_i\rangle^2 e_i - \|\delta x\|^2 e_i\\
&\qquad + 4\langle \delta x, e_i\rangle^2 e_i - 8\langle \delta x, e_i\rangle^2 e_i + 2\|\delta x\|^2 e_i + \dots\\
&= -e_i + \delta x - 2\langle \delta x, e_i\rangle \delta x + \|\delta x\|^2 e_i + \dots = -e_i + \delta x - Q_i(\delta x, \delta x) + \dots\\
\widehat{\Phi}(e_1 + \delta x) &= \widehat{\Phi}(e_1) + D\widehat{\Phi}(e_1).\delta x + \frac{1}{2} D^2\widehat{\Phi}(e_1).(\delta x, \delta x) + \dots,
\end{aligned}
$$

where Q_i is given in (3.13).