Existence of minimizers for a geometrically exact Cosserat solid.

Patrizio Neff*

Fachbereich Mathematik, TU Darmstadt, Schlossgartenstrasse 7, 64289 Darmstadt, Germany

We study a geometrically exact Cosserat continuum model. The model is investigated in variational form as a two-field minimization problem for the deformation φ and the independent microrotation $\overline{R}$. The elastic energy is assumed to depend quadratically on the micropolar stretch tensor $\overline{U}$ and super-quadratically on the curvature $\mathfrak{K}$. Depending on the values of constitutive parameters, existence of minimizers in Sobolev-spaces can be shown.

1 The finite-strain Cosserat model in variational form

Extended continuum models with additional degrees of freedom of rotational type have been introduced in [2]. Modern accounts of the theory can be found in [3, 4, 1]. Recently, extended continuum models incorporating length scale effects have gained renewed attention. These models are used e.g to describe the length scale dependence of plastic yielding.

The mathematical analysis of the corresponding infinitesimal formulations is well-established. We are concerned with a frame-indifferent finite-strain formulation. In [6] a finite-strain, fully frame-indifferent, three-dimensional Cosserat micropolar model is introduced. The **two-field** problem has been posed in a variational setting. The task is to find a pair $(\varphi, \overline{R}) : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3 \times \mathrm{SO}(3, \mathbb{R})$ of deformation φ and **independent microrotation** $\overline{R}$ minimizing the energy functional I,

$$
\begin{aligned}
I(\varphi, \overline{R}) := \int_\Omega W_{\mathrm{mp}}(\overline{R}^T \nabla\varphi) + W_{\mathrm{curv}}(\overline{R}^T \mathrm{D_x}\overline{R}) - \Pi_f(\varphi) - \Pi_M(\overline{R})\,\mathrm{dV} \\
- \int_{\Gamma_S} \Pi_N(\varphi)\,\mathrm{dS} - \int_{\Gamma_C} \Pi_{M_c}(\overline{R})\,\mathrm{dS} \mapsto \min .\ \text{w.r.t. } (\varphi, \overline{R})\,,
\end{aligned}
\tag{1.1}
$$

with the Dirichlet boundary condition of place for the deformation φ on Γ: $\varphi_{|_\Gamma} = g_{\mathrm{d}}$ and various possible **alternative** boundary conditions for the microrotations $\overline{R}$ on Γ,

$$
\overline{R}_{|_\Gamma} = \begin{cases} \overline{R}_{\mathrm{d}}\,, & \text{the case of } \textbf{rigid} \text{ prescription}\,, \\ \mathrm{polar}(\nabla\varphi)\,, & \text{the case of } \textbf{consistent coupling}\,, \\ \text{no condition for } \overline{R} \text{ on } \Gamma, & \textbf{induced Neumann-type} \text{ relations for } \overline{R} \text{ on } \Gamma\,. \end{cases}
\tag{1.2}
$$

The constitutive assumptions on the densities are

$$
\begin{aligned}
W_{\mathrm{mp}}(\overline{U}) &= \mu\,\|\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c\,\|\mathrm{skew}(\overline{U})\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2,\ \overline{U} = \overline{R}^T F\,,\ F = \nabla\varphi\,, \\
W_{\mathrm{curv}}(\mathfrak{K}) &= \mu \frac{L_c^{1+p}}{12}\,(1 + \alpha_4\, L_c^q\,\|\mathfrak{K}\|^q)\,\left(\alpha_5\,\|\mathrm{sym}\,\mathfrak{K}\|^2 + \alpha_6\,\|\mathrm{skew}\,\mathfrak{K}\|^2 + \alpha_7\,\mathrm{tr}\,[\mathfrak{K}]^2\right)^{\frac{1+p}{2}}\,, \\
\mathfrak{K} &= \overline{R}^T \mathrm{D_x}\overline{R} := \left(\overline{R}^T\nabla(\overline{R}.e_1), \overline{R}^T\nabla(\overline{R}.e_2), \overline{R}^T\nabla(\overline{R}.e_3)\right)\,, \text{ the third order } \textbf{curvature tensor}\,.
\end{aligned}
\tag{1.3}
$$

The total elastically stored energy $W = W_{\mathrm{mp}} + W_{\mathrm{curv}}$ is quadratic in the **micropolar stretch tensor** $\overline{U}$ (first Cosserat deformation tensor) and possibly super-quadratic in the curvature $\mathfrak{K}$. The strain energy W_{mp} depends on the deformation gradient $F = \nabla\varphi$ and the microrotations $\overline{R} \in \mathrm{SO}(3, \mathbb{R})$, which do not necessarily coincide with the **continuum rotations** $R = \mathrm{polar}(F)$. The curvature energy W_{curv} depends moreover on the space derivatives $\mathrm{D_x}\overline{R}$ through $\mathfrak{K}$ describing the self-interaction of the microstructure. In general, the micropolar stretch tensor $\overline{U}$ is **not symmetric** and does not coincide with the **symmetric continuum stretch** tensor $U = R^T F = \sqrt{F^T F}$. By abuse of notation we set $\|\mathrm{sym}\,\mathfrak{K}\|^2 := \sum_{i=1}^3 \|\mathrm{sym}\,\mathfrak{K}^i\|^2$ for third order tensors $\mathfrak{K}$.

Here $\Omega \subset \mathbb{R}^3$ is a domain with boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ is that part of the boundary, where Dirichlet conditions $g_{\mathrm{d}}, \overline{R}_{\mathrm{d}}$ for deformations and microrotations or coupling conditions for microrotations, are prescribed. $\Gamma_S \subset \partial\Omega$ is a part of the boundary, where traction boundary conditions in the form of the potential of applied surface forces Π_N are given with $\Gamma \cap \Gamma_S = \emptyset$. In addition, $\Gamma_C \subset \partial\Omega$ is the part of the boundary where the potential of external surface couples Π_{M_c} are applied with

* Corresponding author: e-mail: neff@mathematik.tu-darmstadt.de, Phone: +49 6151 16 3495, Fax: +49 6151 16 4011

$\Gamma \cap \Gamma_C = \emptyset$. On the free boundary $\partial\Omega \setminus \{\Gamma \cup \Gamma_S \cup \Gamma_C\}$ corresponding natural boundary conditions for $(\varphi, \overline{R})$ apply. The potential of the external applied volume force is Π_f and Π_M takes on the role of the potential of applied external volume couples. For simplicity we assume

$$\Pi_f(\varphi) = \langle f, \varphi \rangle , \quad \Pi_M(\overline{R}) = \langle M, \overline{R} \rangle , \quad \Pi_N(\varphi) = \langle N, \varphi \rangle , \quad \Pi_{M_c}(\overline{R}) = \langle M_c, \overline{R} \rangle , \tag{1.4}$$

for the potentials of applied loads with given functions $f \in L^2(\Omega, \mathbb{R}^3)$, $M \in L^2(\Omega, \mathbb{M}^{3\times 3})$, $N \in L^2(\Gamma_S, \mathbb{R}^3)$, $M_c \in L^2(\Gamma_C, \mathbb{M}^{3\times 3})$.

The parameters μ, $\lambda > 0$ are the Lamé constants of classical isotropic elasticity, the additional parameter $\mu_c \geq 0$ is called the **Cosserat couple modulus**. For $\mu_c > 0$ the elastic strain energy density $W_{\mathrm{mp}}(\overline{U})$ is **uniformly convex** in $\overline{U}$. In contrast, for $\mu_c = 0$ the strain energy density is **merely convex** w.r.t. $\overline{U}$.

The parameter $L_c > 0$ (with dimension length) introduces an **internal length** which is **characteristic** for the material, e.g. related to the grain size in a polycrystal. The internal length $L_c > 0$ is responsible for **size effects** in the sense that smaller samples are relatively stiffer than larger samples. We assume throughout that $\alpha_5 > 0, \alpha_6 > 0, \alpha_7 \geq 0$. This implies the **coercivity of curvature**

$$W_{\mathrm{curv}}(\mathfrak{K}) \geq c^+ \, ||\mathfrak{K}||^{1+p} , \tag{1.5}$$

which is a basic ingredient of the mathematical analysis.

The non-standard boundary condition of **consistent coupling** ensures that no unwanted non-classical, polar effects may occur at the Dirichlet boundary Γ. It implies for the micropolar stretch that $\overline{U}_{|\Gamma} \in \mathrm{Sym}$ and for the second Piola-Kirchhoff stress tensor $S_2 := F^{-1} D_F W_{\mathrm{mp}}(\overline{U}) \in \mathrm{Sym}$ on Γ. A linearization of this Cosserat bulk model with $\mu_c = 0$ for small displacement and small microrotations completely decouples the two fields of deformation and microrotations and leads to the classical linear elasticity problem for the deformation. Thinking in the context of an infinitesimal-displacement Cosserat theory one might have thought that $\mu_c > 0$ is strictly necessary also for a "true" finite-strain Cosserat theory. This is not the case. The Cosserat couple modulus μ_c has a decisive influence on the mechanical response of the Cosserat solid: For positive couple modulus $\mu_c > 0$ the response in (inhomogeneous) torsion is markedly stiffer than would be expected from measurements in (homogeneous) tension only.

Let us present the existence results for the Cosserat model. We state only the obtained results for the case without external loads. It is shown in [6, 7]:

Theorem 1.1 (Existence for finite-strain elastic Cosserat model with $\mu_c > 0$) *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\Omega, \mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p}(\Omega, \mathrm{SO}(3,\mathbb{R}))$. Then (1.1) with $\mu_c > 0$, $\alpha_4 \geq 0$, $p \geq 1$, $q \geq 0$ and either free or rigid prescription for $\overline{R}$ on Γ admits at least one minimizing solution pair $(\varphi, \overline{R}) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,1+p}(\Omega, \mathrm{SO}(3,\mathbb{R}))$.* □

Using the extended Korn's inequality [6, 8] the following has also been shown in [6, 7]:

Theorem 1.2 (Existence for finite-strain elastic Cosserat model with $\mu_c = 0$) *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\Omega, \mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p+q}(\Omega, \mathrm{SO}(3,\mathbb{R}))$. Then (1.1) with $\mu_c = 0$, $\alpha_4 > 0$, $p \geq 1$, $q > 1$ and either free or rigid prescription for $\overline{R}$ on Γ admits at least one minimizing solution pair $(\varphi, \overline{R}) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega, \mathrm{SO}(3,\mathbb{R}))$.* □

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