The Γ-limit of a finite-strain Cosserat model for asymptotically thin domains and a consequence for the Cosserat couple modulus μ_c

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We study the behaviour of a geometrically exact 3D Cosserat continuum model for an asymptotically flat domain. Despite the inherent nonlinearity, the Γ-limit of a corresponding canonically rescaled problem on a domain with constant thickness can be explicitly calculated. This "membrane" limit exhibits no bending contributions scaling with h^3 (similar to classical approaches) but features a transverse shear resistance scaling with h for strictly positive Cosserat couple modulus $\mu_c > 0$. This result is physically inacceptable for a zero-thickness "membrane" limit model. Therefore it is suggested that the physically consistent value of the Cosserat couple modulus μ_c is zero. In this case, however, the Γ-limit looses coercivity for the midsurface deformation in $H^{1,2}(\omega, \mathbb{R}^3)$. For numerical purposes then, a transverse shear resistance can be reintroduced, establishing coercivity.

1 The finite-strain 3D-Cosserat model in variational form

We consider a fully frame-indifferent finite-strain Cosserat [2] formulation on an asymptotically thin domain $\Omega_h = \omega \times [-\frac{h}{2}, \frac{h}{2}]$, where $h > 0$ is the characteristic thickness and $\omega \subset \mathbb{R}^2$ is the referential midsurface. The **two-field** Cosserat problem will be introduced in a variational setting. The task is to find a pair $(\varphi, \overline{R}) : \Omega_h \subset \mathbb{R}^3 \mapsto \mathbb{R}^3 \times \mathrm{SO}(3,\mathbb{R})$ of **deformation** φ and **independent microrotation** $\overline{R}$ minimizing the energy functional I,

$$
\begin{aligned}
I(\varphi, \overline{R}) &= \int_{\Omega_h} W(\overline{U}) + \mu\, L_c^p \|\mathrm{D_x} \overline{R}\|^p \,\mathrm{dV} \mapsto \min . \text{ w.r.t. } (\varphi, \overline{R}), \quad \varphi_{|_{\Gamma_h}} = g_{\mathrm{d}}, \quad \overline{R}_{|_{\Gamma_h}} \quad \text{free}\,, \\
W(\overline{U}) &= \mu \,\|\,\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \frac{\lambda}{2} \,\mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2 + \mu_c \,\|\,\mathrm{skew}(\overline{U} - 1\!\!1)\|^2 \,, \qquad (1.1)\\
\overline{U} &= \overline{R}^T \nabla\varphi, \quad \text{non-symmetric Cosserat stretch tensor}\,, \\
\mathrm{D_x} \overline{R} &:= (\nabla(\overline{R}.e_1) | \nabla(\overline{R}.e_2) | \nabla(\overline{R}.e_3))\,, \quad \Gamma_h = \gamma_0 \times [-\frac{h}{2}, \frac{h}{2}]\,,
\end{aligned}
$$

with Dirichlet boundary condition of place for the deformation φ on a part of the lateral boundary Γ_h with $\gamma_0 : \mathbb{R} \mapsto \partial\omega \subset \mathbb{R}^2$ and everywhere Neumann conditions on the Cosserat rotations $\overline{R}$. The parameters $\mu, \lambda > 0$ are the classical Lamé constants of isotropic elasticity, the additional parameter $\mu_c \geq 0$ is called the **Cosserat couple modulus**, whose **value is controversial**. The parameter $L_c > 0$ (with dimension length) introduces an **internal length** which is **characteristic** for the material, e.g. related to the grain size in a polycrystal. The internal length $L_c > 0$ is responsible for **size effects** in the sense that smaller samples are relatively stiffer than larger samples.

In this setting, the variational problem (1.1) admits minimizers for any given thickness $h > 0$ and for all $\infty \geq \mu_c \geq 0$ ($\mu_c = \infty$ formally implies a symmetry constraint). For more information and mathematical existence results concerning this Cosserat bulk model we refer to [7, 6, 4, 9]. In the following, we are interested in characterizing the behaviour of minimizers to (1.1) as $h \to 0$.

2 The rescaled Cosserat model

In order to do so, it is customary to consider a corresponding **rescaled problem**, i.e. transforming the problem (1.1) on a domain with constant thickness. This is achieved by letting $\Omega_1 = \omega \times [-\frac{1}{2}, \frac{1}{2}]$ and defining the rescaled deformations and rotations by $\varphi^\sharp(x,y,z) := \varphi(x,y,h\,z)$, $\quad \overline{R}^\sharp(x,y,z) := \overline{R}(x,y,h\,z)$. The rescaled variational problem reads then

$$
\begin{aligned}
I^\sharp(\varphi^\sharp, \overline{R}^\sharp) &= h \int_{\Omega_1} W(\overline{U}_h^\sharp) + \mu\, L_c^p \|\mathrm{D_x}^h \overline{R}^\sharp\|^p \,\mathrm{dV} \mapsto \min . \text{ w.r.t. } (\varphi^\sharp, \overline{R}^\sharp), \quad \varphi^\sharp_{|_{\Gamma^1}} = g^\sharp_{\mathrm{d}}, \quad \overline{R}^\sharp_{|_{\Gamma^1}} \quad \text{free}\,, \\
\overline{U}_h^\sharp &:= \overline{R}^{\sharp,T}\, \nabla^h \varphi^\sharp, \quad \nabla^h \varphi^\sharp := (\partial_x \varphi^\sharp | \partial_y \varphi^\sharp | \frac{1}{h} \partial_z \varphi^\sharp) \quad (= \nabla\varphi)\,, \qquad (2.1)\\
\mathrm{D_x}^h \overline{R}^\sharp &:= (\nabla^h(\overline{R}^\sharp.e_1) | \nabla^h(\overline{R}^\sharp.e_2) | \nabla^h(\overline{R}^\sharp.e_3))\,, \quad \Gamma_1 = \gamma_0 \times [-\frac{1}{2}, \frac{1}{2}]
\end{aligned}
$$

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and we consider the **sequence of variational problems** $I_h^\sharp(\varphi^\sharp,\overline{R}^\sharp) := \frac{1}{h}\, I^\sharp(\varphi^\sharp,\overline{R}^\sharp)$.

3 The Γ-limit Cosserat "membrane" model

We define the metric space $X = L^r(\Omega_1,\mathbb{R}^3)\times L^p(\Omega_1,\mathrm{SO}(3,\mathbb{R}))$, $r = p' = \frac{2p}{p-2}$, $p > 3$ and note the compact embeddings $H^{1,2}(\Omega_1,\mathbb{R}^3)\subset L^r(\Omega_1,\mathbb{R}^3)$, $W^{1,p}(\Omega_1,\mathrm{SO}(3,\mathbb{R}))\subset L^p(\Omega_1,\mathrm{SO}(3,\mathbb{R}))$. The following result has been obtained in [8]. The Γ-limit [3, 1] to the sequence $I_h^\sharp(\varphi^\sharp,\overline{R}^\sharp) : X \mapsto \mathbb{R}^+$ is given by the variational problem (after de-scaling) for the **midsurface deformation** $m : \omega\subset\mathbb{R}^2\mapsto\mathbb{R}^3$ and the **independent microrotation** of the plate $\overline{R} : \omega\subset\mathbb{R}^2\mapsto \mathrm{SO}(3,\mathbb{R})$:

$$
\begin{aligned}
I_0(m,\overline{R}) &= \int_\omega h\, W^{\mathrm{hom}}(\nabla m,\overline{R}) + h\,\mu {L_c}^p\,\|\mathfrak{K}_s\|^p \,\mathrm{d}\omega \mapsto \min .\ \text{w.r.t. } (m,\overline{R})\,,\\
m_{|\gamma_0} &= g_{\mathrm{d}}(x,y,0) \quad \text{simply supported}\,,\quad \overline{R}_{|\gamma_0} \quad \text{free}\,,\\
W^{\mathrm{hom}}(\nabla m,\overline{R}) &= \mu\, \underbrace{\|\,\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T\nabla m - 1\!\!1_2)\|^2}_{\text{"intrinsic" shear-stretch energy}} + \mu_c\, \underbrace{\|\,\mathrm{skew}((\overline{R}_1|\overline{R}_2)^T\nabla m - 1\!\!1_2)\|^2}_{\text{"intrinsic" first order drill energy}} \qquad (3.1)\\
&\quad + \underbrace{2\mu\,\frac{\mu_c}{\mu+\mu_c}\left(\langle \overline{R}_3, m_x\rangle^2 + \langle \overline{R}_3, m_y\rangle^2\right)}_{\text{homogenized transverse shear energy}} + \frac{\mu\lambda}{2\mu+\lambda}\, \underbrace{\mathrm{tr}\left[\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T\nabla m - 1\!\!1_2)\right]^2}_{\text{homogenized elongational stretch energy}},\\
\mathfrak{K}_s &= \left((\nabla(\overline{R}.e_1)|0),(\nabla(\overline{R}.e_2)|0),(\nabla(\overline{R}.e_3)|0)\right) \quad \text{reduced third order curvature tensor}\,,
\end{aligned}
$$

where we set $\overline{R}_i = \overline{R}.e_i$. Note that $\frac{2\mu\,\mu_c}{\mu+\mu_c} = \mathcal{H}(\mu,\mu_c)$, $\frac{\mu\lambda}{2\mu+\lambda} = 1/2\,\mathcal{H}(\mu,\lambda/2)$, where $\mathcal{H}$ denotes the **harmonic mean**. This variational limit formulation looses coercivity for the midsurface deformation $m\in H^{1,2}(\omega,\mathbb{R}^3)$ if $\mu_c = 0$. However, this loss of coercivity is not related to the missing drill-energy contribution but only due to the missing transverse shear term in that case. The proof of this Γ-limit result is first obtained for $\mu_c > 0$ (in which case equicoercivity for the sequence $I_h^\sharp$ over X greatly facilitates the task) and thereafter it is shown, that the result remains true also for $\mu_c = 0$ where, however, one is faced with an unusual loss of equicoercivity of this sequence. For dimensionally reduced Cosserat models based on a formal ansatz we refer to [5] and rerefences therein.

4 A surprising consequence for the Cosserat couple modulus μ_c

The Γ-limit describes rigourously the limit of zero-thickness, hence a two-dimensional object. Such a "membrane"-model should neither have bending-resistance (scaling with h^3) nor transverse shear resistance, since both effects can only be explained by some remaining small (but finite) thickness. The Γ-limit does not have a bending resistance. The resistance τ against transverse shearing is, however, proportional to $\tau \sim 2\mu\,\frac{\mu_c}{\mu+\mu_c}\left(\langle\overline{R}_3,m_x\rangle + \langle\overline{R}_3,m_y\rangle\right)$. This strongly suggests that $\mu_c \equiv 0$ is the physically consistent value, thus providing us with an answer to the controversy about the value of μ_c. From a practical point of view, for the computation of thin structures with a remaining finite thickness $h > 0$, one should use the Cosserat Γ-limit model (3.1) with $\mu_c = 0$ but augment the stretch energy expression W^{hom} exclusively with some transverse shear contribution. This will restore coercivity for $m\in H^{1,2}(\omega,\mathbb{R}^3)$ and lead to stable computations.

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