The Hencky strain energy $\|\log U\|^2$ measures the geodesic distance of the deformation gradient to $\mathrm{SO}(n)$ in the canonical left-invariant Riemannian metric on $\mathrm{GL}(n)$

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Offen im Denken

Strain measures in linear and nonlinear elasticity

We consider the deformation of an elastic body:

- $\Omega \subset \mathbb{R}^3$, Ω bounded domain, the reference configuration,
- $\varphi : \Omega \to \mathbb{R}^3$ is the deformation mapping,
- $\varphi(x)$ is the deformed position of the material point $x \in \Omega$.

Definition

- $F = \nabla\varphi$ *(the deformation gradient)*
- $U = \sqrt{F^T F}$ *(the right Biot-stretch tensor)*
- $C = F^T F = U^2$ *(the right Cauchy-Green deformation tensor)*
- $V = \sqrt{F F^T}$ *(the left Biot stretch tensor)*
- $B = F F^T = V^2$ *(the Finger tensor)*

Definition (Strain)

Strain is a measure of the deformation with respect to a chosen reference configuration that vanishes if and only if φ is a rigid movement of Ω in space.

Material and spatial strain measures

Lagrangian strain measures:

- $E_r(U) = \frac{1}{2\,r}(U^{2r} - 1\!\!1)$ Seth-Hill family
- $E_1(U) = \frac{1}{2}(U^2 - 1\!\!1) = \frac{1}{2}(C - 1\!\!1)$ Green-Lagrange strain
- $E_{1/2}(U) = U - 1\!\!1$ Biot strain
- $E_{-1}(U) = \frac{1}{2}(1\!\!1 - C^{-1})$
- $E_0(U) = \log U$ **Hencky strain**

Eulerian strain measures:

- $\widehat{E}_r(V) = \frac{1}{2\,r}(V^{2r} - 1\!\!1)$
- $\widehat{E}_1(V) = \frac{1}{2}(V^2 - 1\!\!1) = \frac{1}{2}(B - 1\!\!1)$
- $\widehat{E}_{1/2}(V) = V - 1\!\!1$
- $\widehat{E}_{-1}(V) = \frac{1}{2}(1\!\!1 - B^{-1})$ Almansi strain
- $\widehat{E}_0(V) = \log V$

Material and spatial strain measures

Lagrangian symmetrized strain measures:

- $\widetilde{E}_r \;=\frac{1}{2}[E_r + E_{-r}]$
- $\widetilde{E}_{1/2} = \frac{1}{2}[E_{1/2} + E_{-\frac{1}{2}}] = \frac{1}{2}(U - U^{-1})$ Bažant approximative Hencky strain
- $\widetilde{E}_0 \;= \log U = \lim\limits_{r\to 0} \widetilde{E}_r$

Eulerian symmetrized strain measures:

- $\widetilde{\widehat{E}}_r \;= \frac{1}{2}[\widehat{E}_r + \widehat{E}_{-r}]$
- $\widetilde{\widehat{E}}_{1/2} = \frac{1}{2}[\widehat{E}_{1/2} + \widehat{E}_{-\frac{1}{2}}] = \frac{1}{2}(V - V^{-1})$
- $\widetilde{\widehat{E}}_0 \;= \log V = \lim\limits_{r\to 0} \widetilde{\widehat{E}}_r$

Material and spatial strain measures in terms of stretches λ

Strain may be represented through a scale function on the principal stretches λ_i:

$$U = \sum_{i=1}^{3} \lambda_i \;\; n_i \otimes n_i \,, \qquad V = \sum_{i=1}^{3} \lambda_i \;\; \tilde{n}_i \otimes \tilde{n}_i \,,$$

$$E(U) = \sum_{i=1}^{3} e(\lambda_i) \;\; n_i \otimes n_i \,, \qquad E(V) = \sum_{i=1}^{3} e(\lambda_i) \;\; \tilde{n}_i \otimes \tilde{n}_i \,.$$

Strain measures in terms of the principal stretches λ_i:

- $e_r(\lambda) = \frac{1}{2\,r}(\lambda^{2\,r} - 1)$ — Seth-Hill family
- $e_1(\lambda) = \frac{1}{2}(\lambda^2 - 1)$ — Green-Lagrange strain
- $e_{1/2}(\lambda) = \lambda - 1$ — Engineering strain
- $e_{-1}(\lambda) = \frac{1}{2}(1 - \frac{1}{\lambda^2})$ — Almansi strain
- $e_0(\lambda) = \ln \lambda$ — **Hencky strain**
- $\tilde{e}_{1/2}(\lambda) = \frac{1}{2}(\lambda - \frac{1}{\lambda})$ — Bažant strain

Some reasonable requirements on $e : \mathbb{R}^+ \to \mathbb{R}$:

- ✓ e monotonically increasing, smooth
- ✓ $e(1) = 0$
- ✓ $e'(1) = 1$ (linearizations all coincide)
- ✓ $e(\lambda) \to +\infty \quad \text{as } \lambda \to +\infty$ (not fulfilled by Almansi strain)
- ✓ $e(\lambda) \to -\infty \quad \text{as } \lambda \to 0^+$ (not fulfilled by Biot/Green strain, ...)
- ✓ $e(\lambda^{-1}) = -e(\lambda)$ (fulfilled by Hencky and Bažant strain family)
- ✓ $e(\lambda^\alpha) = \alpha\, e(\lambda)\,, \quad \alpha \in \mathbb{R}$ (fulfilled only by Hencky strain)

Interesting properties of the Hencky strain tensor:

- ✓ Incompressibility condition takes on the simple form $\operatorname{tr}(\log U) \equiv 0$.
- ✓ Additive volumetric-isochoric split:

$$\log U = \log \Big[\underbrace{\frac{1}{(\det U)^{1/3}}\, U}_{\text{isochoric}} \cdot \underbrace{(\det U)^{1/3}\, \mathbb{1}}_{\text{volumetric}} \Big] = \operatorname{dev} \log U + \tfrac{1}{3} \operatorname{tr} \log U$$

- ✓ Simple lift of geometrically linear plasticity theory to geometrically nonlinear plasticity in terms of Hencky strain $\log U$
- ✓ No polar decomposition is needed to compute $\log U$ $(= \frac{1}{2} \log C)$.
- ✓ Uniaxial Hencky strains form a group - strains can be added:

$$\varepsilon_{\log}^{n,n+1} := \int_{L_n}^{L_{n+1}} \frac{1}{L}\, \mathrm{dL} = \ln(L_{n+1}) - \ln(L_n) = \ln\left(\frac{L_{n+1}}{L_n}\right)$$

$$\varepsilon_{\log}^{3,1} = \ln\left(\frac{L_3}{L_1}\right) = \ln\left(\frac{L_3}{L_2}\,\frac{L_2}{L_1}\right) = \ln\left(\frac{L_3}{L_2}\right) + \ln\left(\frac{L_2}{L_1}\right) = \varepsilon_{\log}^{3,2} + \varepsilon_{\log}^{2,1}$$

What's in a strain?

Is there any fundamental property that singles out the Hencky strain tensor $\log U$?

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No.

What's in a strain? Much ado about nothing.

Is there any fundamental property that singles out the Hencky strain tensor $\log U$?

No.

All strain measures are created equal

The choice of a strain measure is immaterial: any strain measure can be used to obtain any stress-strain response (any elastic energy)!

Decisive is the used strain energy $W(F)$!

Thus the Hencky strain has no intrinsic advantage over other strain measures!

> *"[...] while logarithmic measures of strain are a favorite in one-dimensional or semi-qualitative treatment, they have never been successfully applied in general. Such simplicity for certain problems as may result from a particular strain measure is bought at the cost of complexity for other problems."*

Truesdell, Toupin: The Classical Field Theories

The isotropic Hencky strain energy

Definition (Isotropic Hencky energy [2])

The isotropic Hencky energy is

$$W_H(F) = \mu \,\|\operatorname{dev}\log U\|^2 + \frac{\kappa}{2}[\operatorname{tr}(\log U)]^2 = \mu\,\|\operatorname{dev}\log U\|^2 + \frac{\kappa}{2}(\log\det F)^2\,,$$

where

- $F = \nabla\varphi$ *is the deformation gradient,*
- $U = \sqrt{F^T F}$ *is the symmetric right Biot-stretch tensor,*
- $\mu > 0$ *is the shear modulus,*
- $\kappa > 0$ *is the bulk modulus,*
- $\log U$ *is the principal matrix logarithm of* U *and*
- $\operatorname{dev}\log U = \log U - \frac{\operatorname{tr}\log U}{n}\,1\!\!1$ *is the deviatoric part of* $\log U$.

Heinrich Hencky, 1885-1951, Ph.D. - TH Darmstadt

The isotropic Hencky strain energy

Advantageous properties of the Hencky strain energy:

- ✓ $W_H \to \infty$ as $\det F \to 0$ (infinite energy for infinite compression)
- ✓ $W_H(F) = W_H(F^{-1})$ (tension-compression-symmetry)
- ✓ only 2 Lamé-constants, uniquely determined in infinitesimal range
- ✓ fulfils Baker-Ericksen inequality and Hill's inequality
- ✓ describes Poynting effect: a circular cylinder lengthens under torsion

Tension-compression-symmetry: $W(F) = W(F^{-1})$

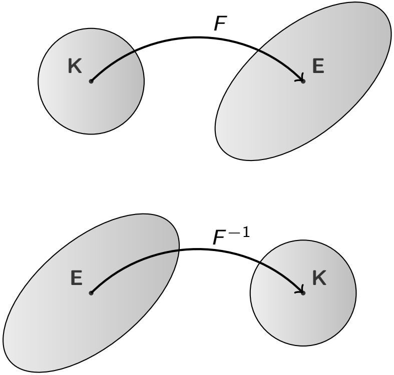


Figure: Homogeneous deformations inverse to each other

Consider a homogeneous deformation of the body K.

- "Freeze" the deformed body
- Take it as a new, stress free reference configuration
- Apply the inverse of the original deformation.

Energy per unit volume is the same in both deformations:

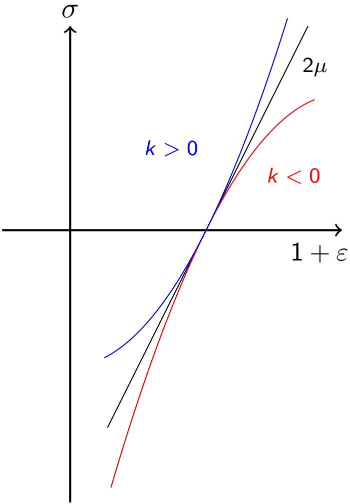
$$\frac{1}{|K|} \int_K W(F)\,\mathrm{dx} = W(F)$$

$$\frac{1}{|E|} \int_E W(F^{-1})\,\mathrm{dx} = W(F^{-1})$$

The isotropic Hencky strain energy

More advantageous properties of the Hencky strain energy:

- ✓ W_H has subquadratic growth (consistent with Stillinger-Weber potential, atomistics, possibility of cavities and fracture)
- ✓ good fit to experimental data for moderately large strains
- ✓ for moderate strains, W_H captures the geometrically nonlinear behaviour correctly
- ✓ replace W_H with new physics for large deformation: plasticity, phase transition
- ✓ good fit also for anisotropy, correct third order elastic constants



Uniaxial stress response:

$$W(\varepsilon) = \mu\varepsilon^2 + \frac{k}{3}\varepsilon^3 + \ldots$$

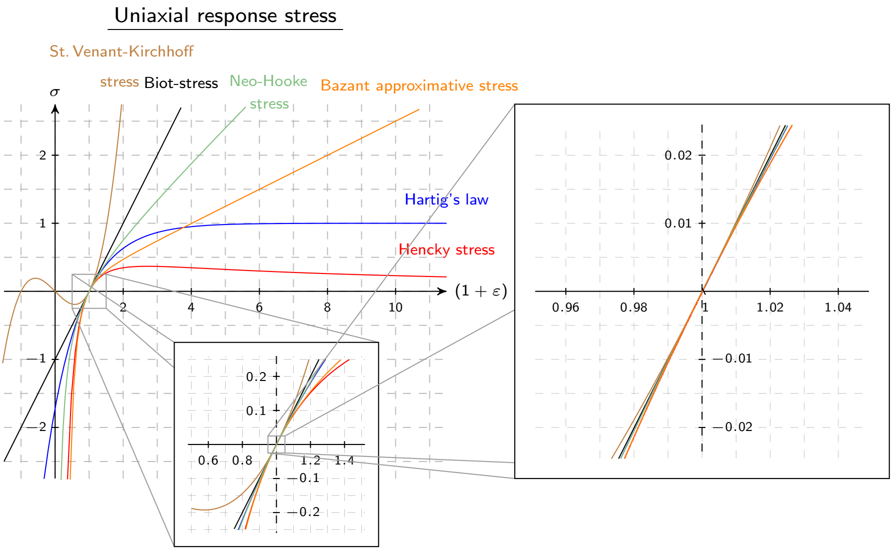
$$\sigma(\varepsilon) = W'(\varepsilon) = 2\mu\varepsilon + k\varepsilon^2 + \ldots$$

All experimental measurements suggest negative third order constants $k < 0$.

$$W_H(\varepsilon) \sim \mu|\log(1+\varepsilon)|^2 \Rightarrow k = -3\mu < 0,$$

$$W_{\mathrm{SVK}}(\varepsilon) \sim \frac{\mu}{4}|(1+\varepsilon)^2 - 1|^2 \Rightarrow k = 3\mu > 0.$$

Stress response and nonlinear behaviour for infinitesimal strains



The isotropic Hencky strain energy

Mathematical challenges associated with the Hencky strain energy:

- ✗ W_H is not polyconvex, not quasiconvex and not rank-one-elliptic [Neff2000].
- ✗ W_H is not Legendre-Hadamard-elliptic:

  $$D^2 W_H(F).(\xi \otimes \eta, \xi \otimes \eta) \geq c^+ \cdot |\xi|^2 \cdot |\eta|^2. \qquad (\to \text{real wave speeds})$$

  However, W_H is LH-elliptic in a large neighbourhood of $\mathbb{1}$ (with admissible stretches $\lambda_i \in (0.21, 1.4)$).
- ✗ W_H has subquadratic growth for large deformations.
- ✗ No general existence result is known for elasticity formulation based on W_H, apart from implicit function theorem in the neighbourhood of $\mathbb{1}$.
- ✗ W_H is difficult to calculate: computation of second derivatives requires spectral representation.

Take on the challenge. . .

A conjecture for ideal elastic materials

The Hencky energy W_H is the best overall isotropic energy up to moderate strains.

- Plan: Understand principal properties singling out the Hencky strain energy
- What makes other well known strain measures and strain energies stand out?

Geometrically linear strain measure as a matrix nearness problem

In linearized elasticity, one considers $\varphi(x) = x + u(x)$ with the displacement $u : \Omega \to \mathbb{R}^3$. The classical linearized strain measure is

$$\boxed{\varepsilon = \operatorname{sym} \nabla u.}$$

The strain measure ε appears through a matrix-nearness problem in the euclidean distance:

$$\operatorname{dist}^2_{\mathrm{euclid}}(\nabla u, \mathfrak{so}(3)) := \min_{W \in \mathfrak{so}(3)} \|\nabla u - W\|^2 = \|\operatorname{sym} \nabla u\|^2 \,,$$

where

- $\|M\| = \sqrt{\operatorname{tr} M^T M} = \sqrt{\sum_{i,j=1}^n M_{ij}^2}$ denotes the Frobenius matrix norm,
- $\operatorname{dist}_{\mathrm{euclid}}(A, B) = \|A - B\|$ denotes the euclidean distance and
- $\mathfrak{so}(3)$ is the set of all skew symmetric matrices in $\mathbb{R}^{3\times 3}$.

The infinitesimal strain tensor $\varepsilon = \operatorname{sym} \nabla u$ is indeed a strain measure:

$$\operatorname{sym} \nabla u = 0 \quad \Longrightarrow \quad \operatorname{dist}^2_{\mathrm{euclid}}(\nabla u(x), \mathfrak{so}(3)) = 0 \quad \Longrightarrow \quad \nabla u(x) = W(x) \in \mathfrak{so}(3)$$

$$\Longrightarrow \quad \operatorname{Curl} W(x) = \operatorname{Curl} \nabla u(x) = 0\,,$$

which implies that $W(x)$ is constant.

Then $u(x) = W.x + b$ is a linearized rigid movement.

Note: $\|\operatorname{sym} \nabla u\|^2 = \|\operatorname{sym}(-\nabla u)\|^2$ (infinitesimal tension-compression-symmetry ✓)

Geometrically nonlinear strain measure as a matrix nearness problem

In nonlinear elasticity, one assumes that $\nabla\varphi(x) \in \mathrm{GL}^+(3)$ (no local self-interpenetration of matter) and may consider the Biot strain tensor

$$\boxed{U - 1\!\!1 = \sqrt{\nabla\varphi^T \nabla\varphi} - 1\!\!1.}$$

The strain measure $U - 1\!\!1$ appears naturally through a matrix-nearness problem in the euclidean distance:

$$\mathrm{dist}^2_{\mathrm{euclid}}(\nabla\varphi, \mathrm{SO}(3)) := \min_{Q \in \mathrm{SO}(3)} \|\nabla\varphi - Q\|^2 = \min_{Q \in \mathrm{SO}(3)} \|Q^T \nabla\varphi - 1\!\!1\|^2$$

$$= \|\sqrt{\nabla\varphi^T \nabla\varphi} - 1\!\!1\|^2 = \|U - 1\!\!1\|^2$$

by a well known optimality result characterizing the polar decomposition

$$F = RU\,, \quad R \in \mathrm{SO}(n)\,,\ U \in \mathrm{PSym}(n) \quad \Longrightarrow \quad \min_{Q \in \mathrm{SO}(n)} \|Q^T F - 1\!\!1\| = \|U - 1\!\!1\|\,.$$

The Biot strain tensor $U - 1\!\!1$ is a geometrically nonlinear Lagrangian strain measure:

$$\sqrt{\nabla\varphi^T\nabla\varphi} = 0 \quad\Longrightarrow\quad \mathrm{dist}^2_{\mathrm{euclid}}(\nabla\varphi, \mathrm{SO}(3)) = 0 \quad\Longrightarrow\quad \nabla\varphi(x) = Q(x) \in \mathrm{SO}(3)$$

$$\Longrightarrow\quad \mathrm{Curl}\, Q(x) = \mathrm{Curl}\,\nabla\varphi(x) = 0\,,$$

which implies that $Q(x)$ is constant, since

$$\|\,\mathrm{Curl}\, Q\|^2 \geq c^+\|\nabla Q\|^2\,,$$

c.f. Neff, Münch: Curl bounds Grad on SO(3), ESAIM 2008.

Then $\varphi(x) = Q.x + b$ is a rigid movement.

Lagrangian or Eulerian, that is the question!

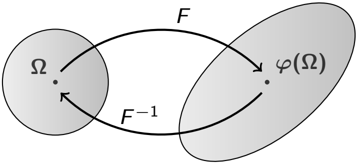
Lagrangian view:

$$\mathrm{dist}^2_{\mathrm{euclid}}(F, \mathrm{SO}(3)) = \|U - \mathbb{1}\|^2 .$$

Eulerian view:

$$\mathrm{dist}^2_{\mathrm{euclid}}(F^{-1}, \mathrm{SO}(3)) = \|\mathbb{1} - V^{-1}\|^2 = \|U^{-1} - \mathbb{1}\|^2 .$$

Who decides whether to take the Lagrangian or the Eulerian point of view?



Lagrangian frame Eulerian frame

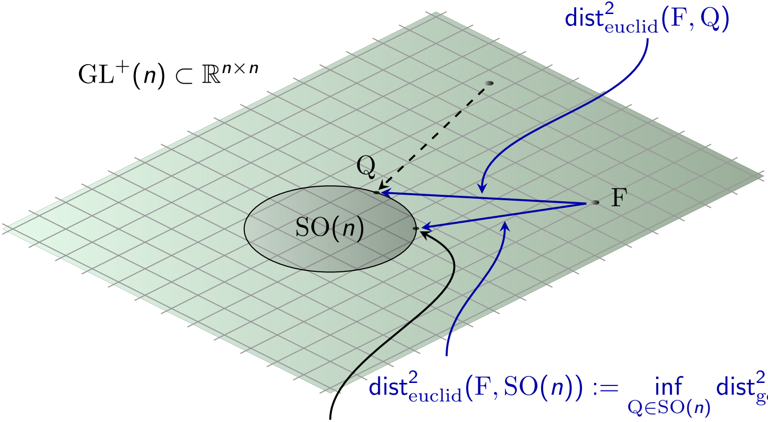
The euclidean distance on $\mathrm{GL}^+(n)$: only an extrinsic distance

Reconsider the euclidean distance $\mathrm{dist}_{\mathrm{euclid}}(A, B) = \|A - B\|$ on $\mathrm{GL}^+(n)$.

Problems:

- The Euclidean distance is an arbitrary choice for a distance measure.
- The euclidean distance cannot be weighted.
- $\mathrm{dist}_{\mathrm{euclid}}(F, \mathrm{SO}(n)) \neq \mathrm{dist}_{\mathrm{euclid}}(F^{-1}, \mathrm{SO}(n))$
  Lagrangian measure $\neq$ Eulerian measure
- $\mathrm{dist}_{\mathrm{euclid}}$ is not an intrinsic distance measure on $\mathrm{GL}^+(n)$: because, in general, $A - B \notin \mathrm{GL}^+(n)$, the term $\|A - B\|$ depends on the underlying linear structure of $\mathbb{R}^{n\times n}$.
- Generally $\mathrm{dist}_{\mathrm{euclid}}(CA, CB) \neq \mathrm{dist}_{\mathrm{euclid}}(A, B)$, i.e. $\mathrm{dist}_{\mathrm{euclid}}$ does not respect the algebraic Lie-group structure of $\mathrm{GL}^+(n)$.
- $\mathrm{GL}^+(n)$ is not closed in $\mathbb{R}^{n\times n}$ under $\mathrm{dist}_{\mathrm{euclid}}$ and thus $\mathrm{GL}^+(n)$ is not complete in the euclidean metric.
- $A, B \in \mathrm{GL}^+(n) \not\Rightarrow A + t(B - A) \in \mathrm{GL}^+(n)$, thus $\mathrm{dist}_{\mathrm{euclid}}$ can not be characterized as the length of a connecting line in $\mathrm{GL}^+(n)$.
- Thus $\mathrm{dist}_{\mathrm{euclid}}$ is only an <u>extrinsic</u> distance measure on $\mathrm{GL}^+(n)$.

The euclidean distance on $GL^+(n)$: only an extrinsic distance



$$\text{dist}^2_{\text{euclid}}(F, SO(n)) := \inf_{Q \in SO(n)} \text{dist}^2_{\text{geod}}(F, Q)$$

$$= \|U - 1\|^2$$

polar(F)

$\mathrm{GL}^+(n)$ as a Riemannian manifold

We view $\mathrm{GL}^+(n)$ as a Riemannian manifold and consider the geodesic distance on $\mathrm{GL}^+(n)$:

- Let g be a left-invariant Riemannian metric g on $\mathrm{GL}(n)$ of the form

$$g_A : \begin{cases} T_A\,\mathrm{GL}(n) \times T_A\,\mathrm{GL}(n) \to \mathbb{R} \\ g_A(X,Y) = \langle A^{-1}X, A^{-1}Y\rangle_g, \quad A \in \mathrm{GL}(n)\,, \end{cases}$$

  with a fixed inner product $\langle\cdot,\cdot\rangle_g$ on the tangent space $T_{\mathbb{1}}\,\mathrm{GL}(n) = \mathfrak{gl}(n) = \mathbb{R}^{n\times n}$.

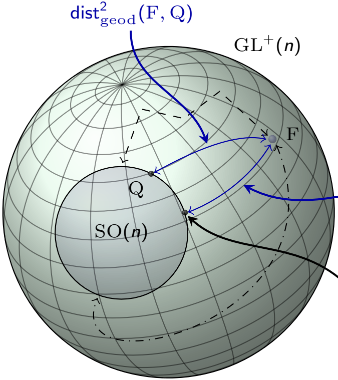
- The length of a curve $\gamma \in C^1([0,1];\mathrm{GL}^+(n))$ is

$$L(\gamma) = \int_0^1 g_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))\,\mathrm{dt} = \int_0^1 \langle \gamma^{-1}\dot\gamma, \gamma^{-1}\dot\gamma\rangle_g\,\mathrm{dt}\,.$$

- The geodesic distance between $P, F \in \mathrm{GL}^+(n)$ is defined as

$$\mathrm{dist}_{\mathrm{geod}}(P,F) = \inf\{L(\gamma)\,|\,\gamma \in C^1([0,1];\mathrm{GL}^+(n)),\ \gamma(0) = P,\ \gamma(1) = F\}.$$

$GL^+(n)$ as a Riemannian manifold: intrinsic distance



$$\mathrm{dist}^2_{\mathrm{geod}}(F, SO(n)) = \inf_{Q \in SO(n)} \mathrm{dist}^2_{\mathrm{geod}}(F, Q)$$
$$\leq \mathrm{dist}^2_{\mathrm{geod}}(F, Q)$$

$\mathrm{dist}_{\mathrm{geod}}(F, SO(n)) = ?$

$\underset{Q \in SO(n)}{\mathrm{argmin}} \ \mathrm{dist}^2_{\mathrm{geod}}(F, Q) = ?$

Figure: Intuitive sketch of the manifold $GL^+(n)$ and $SO(n)$

Left-invariant, right O(n)-invariant Riemannian metrics

We consider Riemannian metrics that are left invariant:

$$g_{BA}(BX, BY) = g_A(X, Y) \qquad \text{for all } B \in \mathrm{GL}(n)\,,$$

as well as right O(n)-invariant:

$$g_{AQ}(XQ, YQ) = g_A(X, Y) \qquad \text{for all } Q \in \mathrm{O}(n)\,.$$

- right O(n)-invariance $\cong$ isotropy of the material
- left SO(n)-invariance $\cong$ frame-indifference
- left GL(n)-invariance $\cong$ $\mathrm{dist}_{\mathrm{geod}}(AF, AP) = \mathrm{dist}_{\mathrm{geod}}(F, P) \quad \forall A \in \mathrm{GL}(n)$

Definition

The <u>*isotropic inner product*</u> $\langle \cdot, \cdot \rangle_{\mu,\mu_c,\kappa}$ *on* $\mathfrak{gl}(n) = \mathbb{R}^{n\times n}$ *is*

$$\langle X, Y\rangle_{\mu,\mu_c,\kappa} := \mu\langle \operatorname{dev}\operatorname{sym} X, \operatorname{dev}\operatorname{sym} Y\rangle + \mu_c\langle \operatorname{skew} X, \operatorname{skew} Y\rangle + \frac{\kappa}{2}\operatorname{tr} X \operatorname{tr} Y\,,$$

where

- $\langle X, Y\rangle = \operatorname{tr}(X^T Y)$ *is the canonical inner product on* $\mathfrak{gl}(n)$*,*
- $\operatorname{dev}\operatorname{sym} X = \operatorname{sym} X - \frac{1}{n}\operatorname{tr}[\operatorname{sym} X]\cdot \mathbb{1}$ *is the deviatoric part of* $\operatorname{sym} X$*,*
- $\mu > 0$ *is the shear modulus,*
- $\mu_c > 0$ *is the spin modulus and*
- $\kappa > 0$ *is the bulk modulus.*

The unique family of left invariant, right O(n)-invariant metrics on GL(n)

Every left invariant, right O(n)-invariant Riemannian metric on GL(n) has the form [3]

$$\boxed{\begin{aligned} g_A(X,Y) &= \langle A^{-1}X, A^{-1}Y\rangle_{\mu,\mu_c,\kappa} \\ &= \mu\langle \operatorname{dev}\operatorname{sym} X, \operatorname{dev}\operatorname{sym} Y\rangle + \mu_c\langle \operatorname{skew} X, \operatorname{skew} Y\rangle + \tfrac{\kappa}{2}\operatorname{tr} X \operatorname{tr} Y\,. \end{aligned}}$$

The invariances imply

$$\operatorname{dist}_{\mathrm{geod}}(F,Q) = \operatorname{dist}_{\mathrm{geod}}(F^{-1},Q^T)\,, \quad Q \in \mathrm{SO}(n)\,,$$

thus we obtain

$$\boxed{\begin{array}{l} \underset{\text{(Lagrangian measure)}}{\operatorname{dist}_{\mathrm{geod}}(F,\mathrm{SO}(n))} = \min\limits_{Q\in\mathrm{SO}(n)} \operatorname{dist}_{\mathrm{geod}}(F,Q) = \underset{\text{(Eulerian measure)}}{\operatorname{dist}_{\mathrm{geod}}(F^{-1},\mathrm{SO}(n))} \end{array}}$$

without computing the result.

Shortest geodesics on $\mathrm{GL}^+(n)$

Every geodesic curve γ connecting $F, P \in \mathrm{GL}^+(n)$ is of the form [4, 5]

$$\boxed{\gamma(t) = F\ \exp(t(\operatorname{sym}\xi - \tfrac{\mu_c}{\mu}\operatorname{skew}\xi))\ \exp(t(1+\tfrac{\mu_c}{\mu})\operatorname{skew}\xi\,,} \tag{1}$$

with $\xi \in \mathfrak{gl}(n)$ such that

$$\boxed{P = \gamma(1) = F\ \exp(\operatorname{sym}\xi - \tfrac{\mu_c}{\mu}\operatorname{skew}\xi)\ \exp((1+\tfrac{\mu_c}{\mu})\operatorname{skew}\xi\,.} \tag{2}$$

Here:

- $\exp : \mathfrak{gl}(n) \to \mathrm{GL}^+(n)$ is the matrix exponential,
- $\operatorname{sym}\xi = \frac{1}{2}(\xi + \xi^T)$ is the symmetric part and
- $\operatorname{skew}\xi = \frac{1}{2}(\xi - \xi^T)$ is the skew symmetric part of ξ

No closed form solution to (2) for given P, F is known, but (1) can be used to obtain a lower bound for $\mathrm{dist}_{\mathrm{geod}}(F, \mathrm{SO}(n)) = \min\limits_{Q \in \mathrm{SO}(n)} \mathrm{dist}_{\mathrm{geod}}(F, Q)$.

The geodesic distance of F to $\mathrm{SO}(n)$

Lower bound: (can be obtained from the geodesic parameterization)

$$\mathrm{dist}^2_{\mathrm{geod}}(F, \mathrm{SO}(n)) = \min_{Q \in \mathrm{SO}(n)} \mathrm{dist}^2_{\mathrm{geod}}(F, Q) \geq \min_{Q \in \mathrm{SO}(n)} \| \operatorname{Log}(Q\, F) \|^2_{\mu, \mu_c, \kappa}$$

Upper bound:

$$\begin{aligned}\mathrm{dist}^2_{\mathrm{geod}}(F, \mathrm{SO}(n)) \leq \mathrm{dist}^2_{\mathrm{geod}}(F, \mathrm{polar}(F)) &\leq \| \log(\mathrm{polar}(F)^T F) \|^2_{\mu, \mu_c, \kappa} \\ &= \| \log U \|^2_{\mu, \mu_c, \kappa} = \mu \| \operatorname{dev} \log U \|^2 + \frac{\kappa}{2} [\operatorname{tr}(\log U)]^2 \,,\end{aligned}$$

where

- $F = RU$ is the polar decomposition,
- $R = \mathrm{polar}(F) \in \mathrm{SO}(n)$ is the orthogonal polar factor of F and
- $U = \sqrt{F^T F} \in \mathrm{PSym}(n)$.

The geodesic distance of F to $\mathrm{SO}(n)$

Theorem (Optimality result, Neff et al. 2013, [6])

Let $\|\,.\,\|$ be the Frobenius matrix norm on $\mathfrak{gl}(n)$, $F \in \mathrm{GL}^+(n)$. Then the minimum

$$\min_{Q\in\mathrm{SO}(n)} \|\operatorname{Log}(Q^TF)\|^2 = \|\log(\operatorname{polar}(F)^TF)\|^2 = \|\log(\sqrt{F^TF})\|^2 = \|\log U\|^2\,,$$

$$\min_{Q\in\mathrm{SO}(n)} \mu\|\operatorname{dev}\operatorname{sym}\operatorname{Log}(Q^TF)\|^2 + \mu_c\|\operatorname{skew}\operatorname{Log}(Q^TF)\|^2 + \frac{\kappa}{2}[\operatorname{tr}(\operatorname{Log}(Q^TF))]^2$$

$$= \mu\|\operatorname{dev}\log(U)\|^2 + \frac{\kappa}{2}[\operatorname{tr}(\log U)]^2$$

is uniquely attained at $Q = \operatorname{polar}(F)$.

The theorem holds for every unitary invariant norm $\|\,.\,\|$ on $\mathfrak{gl}(n,\mathbb{C})$ as well, c.f. [7].

Note that the minimum is taken over all logarithms of Q^TF (including non-symmetric arguments):

$$\min_{Q\in\mathrm{SO}(n)} \|\operatorname{Log}(Q^TF)\|^2 = \min\{\|X\| \,:\, X\in\mathfrak{gl}(n),\ \exp(X) = Q^TF\}\,.$$

Combining this theorem with the upper and lower bound for $\operatorname{dist}_{\mathrm{geod}}(F,\mathrm{SO}(n))$ yields our main result.

Main result

Theorem (Main result [8])

Let g be any left-invariant Riemannian metric on $\mathrm{GL}(n)$ that is also right invariant under $\mathrm{O}(n)$, and let $F \in \mathrm{GL}^+(n)$. Then:

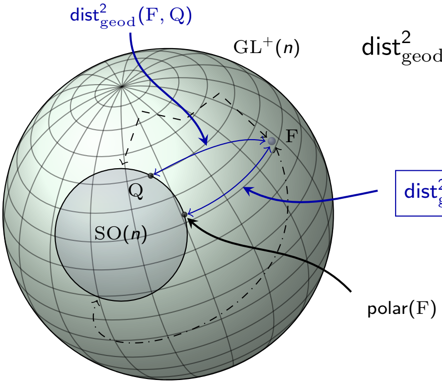
$$\mathrm{dist}^2_{\mathrm{geod}}(F, \mathrm{SO}(n)) = \mathrm{dist}^2_{\mathrm{geod}}(F, \mathrm{polar}(F)) = \mu \| \operatorname{dev} \log(U) \|^2 + \frac{\kappa}{2} [\mathrm{tr}(\log U)]^2 \, .$$

Thus the geodesic distance of the deformation gradient F to $\mathrm{SO}(n)$ is the isotropic Hencky strain energy of F. In particular, the result is independent of the spin modulus $\mu_c > 0$.

For $\mu_c = 0$, the theorem still holds for the resulting pseudometric.

Furthermore, the result is basically identical for any right invariant, left $\mathrm{O}(n)$-invariant metric $g_A(X, Y) = \langle X A^{-1}, Y A^{-1} \rangle_{\mu, \mu_c, \kappa}$.

Main result



$$\mathrm{dist}^2_{\mathrm{geod}}(\mathrm{F}, \mathrm{SO}(n)) = \inf_{\mathrm{Q} \in \mathrm{SO}(n)} \mathrm{dist}^2_{\mathrm{geod}}(\mathrm{F}, \mathrm{Q})$$

$$\leq \mathrm{dist}^2_{\mathrm{geod}}(\mathrm{F}, \mathrm{Q})$$

$$\mathrm{dist}^2_{\mathrm{geod}}(\mathrm{F}, \mathrm{SO}(n)) = \|\log U\|^2$$

Main result: The isotropic Hencky energy of F is the geodesic distance of F to $\mathrm{SO}(n)$.

Outlook:

- Characterize anisotropic Hencky strain energy $\langle \mathbb{C}.\log U, \log U \rangle$ as a distance in an appropriate anisotropic Riemannian metric?
- Calculate "anisotropic" geodesics?
- Reconsider the well-posedness problem for the Hencky energy (which is unknown).
- Obtain geometric properties of our metric, e.g. the Levi-Civita connection coefficients, the Riemannian or Ricci curvature, preliminary results for $\mu = \mu_c$, $\kappa = \frac{2}{3}\mu$ (Poisson number $\nu = 0$).
- Numerical implementations: Justify tension-compression-symmetry by atomistic calculations for nearly isotropic lattices?

Thank You!

Presentation available at:
http://www.uni-due.de/imperia/md/content/mathematik/ag_neff/neff_hencky13.pdf

Matrix logarithms

Logarithm of a symmetric matrix

The logarithm of a positive definite matrix is defined as

$$\log U = \sum_{i=1}^{3} (\ln \lambda_i) \;\; n_i \otimes n_i \,,$$

where

- λ_i are the (positive) eigenvalues of U,
- n_i are the corresponding (orthonormal) eigenvectors of U and
- ln is the natural logarithm on $\mathbb{R}^+$.

Logarithm of a non-symmetric argument:

$$\log X = (X - 1\!\!1) - \frac{1}{2}(X - 1\!\!1)^2 + \frac{1}{3}(X - 1\!\!1)^3 - \dots$$

The series converges for $\|X - 1\!\!1\| < 1$.

Every nonsingular X has a (perhaps complex) logarithm.

Polar decomposition

- $F = R\,U$ $\quad$ U: Lagrangian (material) stretch tensor,
- $F = V\,R$ $\quad$ V: Eulerian (spatial) stretch tensor,

- $U = \sqrt{F^T F}\,,\quad F^T F, U:\ T\Omega_{\text{ref}} \to T\Omega_{\text{ref}}$ $\quad$ Lagrangian,
- $V = \sqrt{F F^T}\,,\quad F F^T, V:\ T\varphi(\Omega_{\text{ref}}) \to T\varphi(\Omega_{\text{ref}})$ $\quad$ Eulerian,

- $\text{dist}_{\text{euclid}}(F, \text{SO}(3)) = \|U - 1\!\!1\|$ $\quad$ Lagrangian Euclidean distance,
- $\text{dist}_{\text{euclid}}(F^{-1}, \text{SO}(3)) = \|1\!\!1 - V^{-1}\|$ $\quad$ Eulerian Euclidean distance,

- $\text{dist}_{\text{euclid}}(F, \text{SO}(3)) \neq \text{dist}_{\text{euclid}}(F^{-1}, \text{SO}(3))$,
- $\text{dist}_{\text{geod}}(F, \text{SO}(3)) = \text{dist}_{\text{geod}}(F, \text{SO}(3)$,

- Weighted euclidean distance

$$\mu \|\operatorname{dev}\operatorname{sym}(F - R)\|^2 + \mu_c \|\operatorname{skew}(F - R)\|^2 + \frac{\kappa}{2}[\operatorname{tr}(F - R)]^2$$

is tensorially impossible.

Weighted isotropic infinitesimal euclidean distance on $\mathfrak{gl}(n)$

$$\mathrm{dist}^2_{\mathrm{euclid},\mu,\mu_c,\kappa}(X,Y)) := \mu\|\operatorname{dev}\operatorname{sym}(X-Y)\|^2 + \mu_c\|\operatorname{skew}(X-Y)\|^2 + \frac{\kappa}{2}[\operatorname{tr}(X-Y)]^2\,,$$

where

- $\mu > 0$ is the shear modulus,
- $\mu_c > 0$ is the spin modulus,
- $\kappa > 0$ is the bulk modulus.

The distance to the set of skew symmetric matrices (infinitesimal strain energy)

$$\begin{aligned}
&\mathrm{dist}_{\mathrm{euclid},\mu,\mu_c,\kappa}(\nabla u, \mathfrak{so}(3)) \\
&= \min_{W\in\mathfrak{so}(3)} \mu\|\operatorname{dev}\operatorname{sym}(\nabla u - W)\|^2 + \mu_c\|\operatorname{skew}(\nabla u - W)\|^2 + \frac{\kappa}{2}[\operatorname{tr}(\nabla u - W)]^2 \\
&= \mu\|\operatorname{dev}\operatorname{sym}(\nabla u)\|^2 + \frac{\kappa}{2}[\operatorname{tr}(\nabla u)]^2 \;=\; \mu\|\varepsilon\|^2 + \frac{\lambda}{2}[\operatorname{tr}(\varepsilon)]^2\,,
\end{aligned}$$

is independent of the spin modulus $\mu_c \geq 0$.

Tensor identities

Ψ isotropic scalar-valued function on Sym(3) : $\Psi(Q^T S\, Q) = \Psi(S) \quad \forall Q \in \mathrm{O}(3)$,

$$W(F) = \widehat{W}(C) = \Psi(\log C)\,,$$

$$S_1(F) = D_F[W(F)]\,, \qquad \text{(first Piola-Kirchhoff tensor)}$$

$$S_2(F) = D_C\widehat{W}(C) = F^{-1}\cdot S_1(F)\,, \qquad \text{(second Piola-Kirchhoff tensor)}$$

$$S_1(F) = \det F \cdot T \cdot F^{-T}\,, \qquad \text{(T Cauchy stress tensor)}$$

$$D_C\widehat{W}(C) = D\Psi(\log C)\cdot C^{-1}\,, \qquad \text{while } D_C[\log C] \neq C^{-1} \text{ in general}$$

$$(\det F)\cdot T = D\Psi(\log C)\,, \qquad \text{Hill}$$

$$\langle S_1(F), H\rangle = \langle D\Psi(\log C)\cdot F^{-T}, H\rangle$$

Nonlinear structure of $\mathrm{GL}(n)$

$$\underbrace{\mathrm{GL}^+}_{\text{Lie group}} = \underbrace{\mathrm{SL}(n)}_{\substack{\det F=1\\ \text{Lie group}}} \cdot \underbrace{\mathbb{R}^+ \cdot 1\!\!1}_{\substack{\text{volumetric}\\ \text{Lie group}}} = \underbrace{\mathrm{SO}(n)}_{\substack{\text{rotations}\\ \text{Lie group}}} \cdot \underbrace{\left\{\mathrm{SL}(n)\Big/\mathrm{SO}(n)\right\}}_{\substack{\text{isochoric shears}\\ \text{Not a Lie group!}}} \cdot \underbrace{\mathbb{R}^+ \cdot 1\!\!1}_{\substack{\text{volumetric}\\ \text{Lie group}}}$$

$\left\{\mathrm{SL}(n)/\mathrm{SO}(n)\right\}$, the quotient space of unimodular positive definite symmetric matrices, is not a Lie-group with respect to the matrix multiplication.

Because $\mathrm{PSym}(n)$ is a convex cone, the straight line connecting F with $R = \mathrm{polar}(F)$ lies in $\mathrm{GL}^+(n)$:

$$\det((1-t)F + t\,R) = \det((1-t)R^T F + t\,R^T R) = \det(\underbrace{(1-t)U + t\,1\!\!1}_{\in \mathrm{PSym}(n)}) > 0.$$

However, the line is generally not contained in $\mathrm{SL}(n)$, even if $F \in \mathrm{SL}(n)$.

Geodesic distance on SO(n)

The Riemannian metric induced on the compact Lie group $\mathrm{SO}(n)$

$$g_Q : \begin{cases} T_Q\,\mathrm{SO}(n) \times T_Q\,\mathrm{SO}(n) \to \mathbb{R} \\ g_Q(X,Y) = \mu_c \langle Q^{-1}X, Q^{-1}Y\rangle = \mu_c\langle X,Y\rangle = \mu_c\,\mathrm{tr}(X^TY), \quad Q \in \mathrm{SO}(n) \end{cases}$$

is bi-invariant (left- and right group invariant):

$$\begin{aligned} g_{RQ}(RX,RY) &= g_Q(X,Y)\,, \\ g_{QR}(XR,YR) &= g_Q(X,Y) \qquad \text{for all } Q,R \in \mathrm{SO}(n)\,. \end{aligned}$$

Geodesics on $\mathrm{SO}(n)$ are one-parameter groups:

$$\gamma(t) \;=\; Q\cdot\exp(t\,W), \quad Q \in \mathrm{SO}(n),\ W \in \mathfrak{so}(n)\,.$$

The $\mathrm{SO}(n)$-geodesic distance between $Q_1, Q_2 \in \mathrm{SO}(n)$ is

$$\boxed{\mathrm{dist}^2_{\mathrm{geod},\,\mathrm{SO}(n)}(Q_1,Q_2) \;=\; \mu_c\,\|\log Q_1^TQ_2\|^2\,,}$$

where

- $\|M\| = \sqrt{\mathrm{tr}\,M^TM} = \sqrt{\sum_{i,j=1}^n M_{ij}^2}$ denotes the Frobenius matrix norm and
- $\log$ denotes the principal logarithm on $\mathrm{SO}(n)$.

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More references: http://www.uni-due.de/mathematik/ag_neff/