The Reissner-Mindlin plate is the Γ-limit of Cosserat elasticity.

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Abstract

We show that the Reissner-Mindlin membrane-bending plate model can be exactly obtained as the rigorous Γ-limit for zero thickness of a linear isotropic Cosserat bulk model with symmetric curvature. For this result we use the natural nonlinear scaling for the displacements u and the linear scaling for the infinitesimal microrotations $\overline{A} \in \mathfrak{so}(3)$. We also provide formal calculations for other combinations of scalings whereby be retrieve other plate models previously proposed in the literature by formal asymptotic methods as corresponding Γ-limits. No boundary conditions on the microrotations are prescribed.

Key words: micropolar, thin-plate, Reissner-Mindlin, Gamma convergence.

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1 Introduction

The relation between three-dimensional elasticity and theories for lower-dimensional objects such as rods, beams, membranes, plates and shells has been an outstanding question since the very beginning of the research in elasticity. Recently there has been substantial progress in the rigorous understanding of this relation through the use of variational methods, in particular Γ-convergence. This notion of convergence assures, roughly speaking, that absolute minimizers of the three-dimensional theory (subject to suitable boundary conditions and applied loads) converge to absolute minimizers of the limiting two-dimensional theory.

Variational convergence is not the only way to proceed to obtain lower dimensional models. Since the dimensional reduction of a given continuum-mechanical model is already an old subject it has seen many other "solutions". Another way to proceed is the so called derivation approach, i.e., reducing a given three-dimensional model via physically reasonable constitutive assumptions on the kinematics to a two-dimensional model. This is opposed to either the intrinsic approach which views the plate/shell from the onset as a two-dimensional surface and invokes concepts from differential geometry or the asymptotic methods which try to establish two-dimensional equations by formal expansion of the three-dimensional solution in power series in terms of a small non-dimensional thickness parameter, the aspect ratio $h > 0$. The intrinsic approach is closely related to the direct approach which takes the shell to be a two-dimensional medium with additional extrinsic directors in the sense of a restricted Cosserat surface [10]. For further information together with more references let us refer to the introduction in [24, 26, 25, 27, 29].

It is well known, that Γ-convergence also needs assumptions which concern the scaling of fields and energies. A first major breakthrough in finite elasticity was the justification of a nonlinear membrane model in [12]. Later, a hierarchy of limiting theories based on Γ-convergence, distinguished by different scaling-exponents of the energy as a function of the aspect ratio h is developed in [18, 17, 16, 19]. There the different scaling exponents can be put into effect by corresponding scaling assumptions on the applied forces. A typical feature of Γ-limit models based on classical elasticity is their decoupling into either membrane or bending problems, depending on the regime for the energy. For example, the Kirchhoff-Love plate bending problem appears as Γ-limit but is restricted to inextensible deformations. Similarly, one may obtain a membrane energy with no bending term, having no resistance in compression [12]. But in a given three-dimensional problem the different regimes are hardly separated and one wishes to have a model comprising of membrane and bending contributions simultaneously.

Let us restrict ourselves to linear elasticity in the following. In that case, using Γ-convergence in the weak topology in $H^1(\Omega_1)$, together with a certain linear scaling, Ciarlet [5] arrives at justifying the membrane plate. This result can be, without problems, extended to the strong $L^2 - \Gamma$-limit, see the appendix. Remarkable is that the limit problem is not completely two-dimensional since the admissible set is the space V_{KL}, see Definition 7.2.

In [4] basically the nonlinear scaling of the displacement is considered. Compactness can only be assured by assuming that $\frac{1}{h^2} I_h^\sharp(u^\sharp)$ is bounded independent of the thickness h. In that case, it is easy to see that the limit is purely two-dimensional and the energy coincides with the one previously given. Using the linear scaling in a finite strain setting is known to lead to inconsistencies [15]. A formal deduction of plate models by scaling can be found in [23].

A very prominent model for combined membrane and bending behavior of plates is the Reissner-Mindlin model, see (7.1). But in [3, p.17] we read: "For plate bending, the asymptotic approach leads to the Kirchhoff-Love or biharmonic plate equation, rather than to the Reissner-Mindlin model. .... To the best of our knowledge there is no way to obtain Reissner-Mindlin type models of plate bending from the asymptotic approach." Similarly, Ciarlet writes [8, p.27]: "Open problems: finding a rigorous justification of the Reissner-Mindlin equations." With this contribution we want to fill this gap.[1] Our main idea in this respect is to use extended continuum models, more specifically the linear Cosserat model as a starting point for the application of Γ-convergence methods. The use of Cosserat elasticity as a "parent" model is quite recent, it initiated presumably with [28] immediately for the finite strain case using the nonlinear scaling for deformations and exact rotations $(\varphi, \overline{R}) \in \mathbb{R}^3 \times \mathrm{SO}(3)$. The result is a kind of Reissner-Mindlin model, but not exactly. In [2, 1] a linear Cosserat model is taken as a starting point and the asymptotic development (not the Γ-limit) is given based on the nonlinear scaling for displacement and infinitesimal microrotation $(u, \overline{A}) \in \mathbb{R}^3 \times \mathfrak{so}(3)$. The result is comparable to the previous one in [28]. A precursor to that is [13] where the author used also the asymptotic expansion method but with linear scaling for both $(u, \overline{A}) \in \mathbb{R}^3 \times \mathfrak{so}(3)$. His result is comparable to a formal deduction given much earlier in [14]. Neither of these methods, however, reproduced the Reissner-Mindlin model exactly.

While our method is methodologically rather standard, we want to exhibit the different limit functionals depending on the assumed choice of scaling for the displacement and the infinitesimal microrotation. The major difference is in the coupling term after dimensional reduction. On specific choice of scaling recovers exactly the Reissner-Mindlin membrane bending model, another choice recovers the Tambaca/Neff model and still another choice decouples the problems. It is interesting to note that for the scaling we have in mind, only the symmetric curvature case leads to a local formula for the Γ-limit: the Reissner-Mindlin model. Central to our development is therefore the notion of Γ-convergence, a powerful theory originally initiated by De Giorgi [20] and especially suited for a variational framework on which in turn the numerical treatment with finite elements is based.

Outline of this contribution: We introduce first the underlying "parent" three-dimensional linear isotropic Cosserat model with rotational substructure embodied by the infinitesimal **Cosserat rotations** $\overline{A} \in \mathfrak{so}(3)$. Next we specialize the model to a thin domain in Section 3. The two basically different scalings: linear and nonlinear, are introduced in Section 4. Then we perform the transformation of the bulk model in physical space to a non-dimensional thin domain and introduce the further scaling to a fixed reference domain Ω_1 with constant thickness on which the Γ-convergence procedure is finally based. In Section 5 the Γ-limit model is presented and Section 6 furnishes the proofs. The notation is found at the end of the paper. In the appendix we recall the Reissner-Mindlin model, the Koiter-model and two other proposals based on different scalings. Korn's inequality for different scalings together with a recall on the Γ-limit for classical linear elasticity finishes this work.

2 The linear elastic Cosserat model in variational form

This section does not contain any new results, rather it serves to accommodate the widespread notations used in Cosserat elasticity and to introduce the problem. It is assumed that the microrotation field is kinematically independent from the material rotation (continuum rotation). In the micropolar continuum theory not only forces but also moments can be transmitted across the surface of a material element. The very concept of a micropolar theory involves, in a certain way, the substructure response into the continuum media.

For the **displacement** $u : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ and the **skew-symmetric infinitesimal microrotation** $\overline{A} : \Omega \subset \mathbb{R}^3 \mapsto \mathfrak{so}(3)$ we consider the **two-field** minimization problem

$$I(u, \overline{A}) = \int_\Omega W_{\mathrm{mp}}(\overline{\varepsilon}) + W_{\mathrm{curv}}(\nabla \operatorname{axl} \overline{A}) - \langle f, u \rangle \, \mathrm{dx} \mapsto \quad \min . \text{ w.r.t. } (u, \overline{A}), \tag{2.1}$$

[1]When finishing this paper we have learned that a related justification of the Reissner-Mindlin model based on Γ-convergence has been already given in [30, 31]. Since the authors considered a second-gradient "parent" linear elasticity model instead of our first order Cosserat "parent" model we still believe in the interest of our approach.

under the following constitutive requirements and boundary conditions

$$
\begin{aligned}
\overline{\varepsilon} &= \nabla u - \overline{A}, \quad \textbf{first Cosserat stretch tensor} \\
u_{|\Gamma} &= u_{\rm d}\,, \quad \textbf{essential displacement boundary conditions} \\
W_{\rm mp}(\overline{\varepsilon}) &= \mu\, \|\,\mathrm{sym}\, \overline{\varepsilon}\|^2 + \mu_c\, \|\,\mathrm{skew}\, \overline{\varepsilon}\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\,[\mathrm{sym}\, \overline{\varepsilon}]^2 \qquad \textbf{strain energy} \\
&= \mu\, \|\,\mathrm{dev}\,\mathrm{sym}\, \nabla u\|^2 + \mu_c\, \|\,\mathrm{skew}(\nabla u - \overline{A})\|^2 + \frac{K}{2}\,\mathrm{tr}\,[\mathrm{sym}\, \nabla u]^2 \\
\theta &:= \mathrm{axl}\, \overline{A} \in \mathbb{R}^3, \quad \mathfrak{K} = \nabla \theta\,, \quad \|\,\mathrm{curl}\, \theta\|^2_{\mathbb{R}^3} = 4\|\,\mathrm{axl}\,\mathrm{skew}\, \nabla\theta\|^2_{\mathbb{R}^3} = 2\|\,\mathrm{skew}\, \nabla\theta\|^2_{\mathbb{M}^{3\times 3}}\,, \\
W_{\rm curv}(\nabla\theta) &= \frac{\gamma+\beta}{2}\|\,\mathrm{sym}\, \nabla\theta\|^2 + \frac{\gamma-\beta}{2}\|\,\mathrm{skew}\, \nabla\theta\|^2 + \frac{\alpha}{2}\mathrm{tr}\,[\nabla\theta]^2 \qquad \textbf{curvature energy} \\
&= \frac{\gamma+\beta}{2}\|\,\mathrm{dev}\,\mathrm{sym}\, \nabla\theta\|^2 + \frac{\gamma-\beta}{2}\|\,\mathrm{skew}\, \nabla\theta\|^2 + \frac{k_c}{2}\mathrm{tr}\,[\nabla\theta]^2\,.
\end{aligned}
$$

Here, f are given volume forces while $u_{\rm d}$ are Dirichlet boundary conditions for the displacement at $\Gamma \subset \partial\Omega$. Surface tractions, volume couples and surface couples can be included in the standard way. The strain energy $W_{\rm mp}$ and the curvature energy $W_{\rm curv}$ are the most general isotropic quadratic forms in the **infinitesimal non-symmetric first Cosserat strain tensor** $\overline{\varepsilon} = \nabla u - \overline{A}$ and the **micropolar curvature tensor** $\mathfrak{K} = \nabla\, \mathrm{axl}\, \overline{A} = \nabla\theta$ (curvature-twist tensor). The parameters μ, λ[MPa] are the classical Lamé moduli and α, β, γ are additional micropolar moduli with dimension $[\mathrm{Pa}\cdot\mathrm{m}^2] = [\mathrm{N}]$ of a force. Here, the bulk modulus and curvature bulk modulus are defined by

$$
K = \frac{2\mu + 3\lambda}{3}\,, \qquad k_c := \frac{(\beta+\gamma) + 3\alpha}{3}\,. \tag{2.2}
$$

The additional parameter $\mu_c \geq 0$[MPa] in the strain energy is the **Cosserat couple modulus**. For $\mu_c = 0$ the two fields of displacement and microrotations decouple and one is left formally with classical linear elasticity for the displacement u. The reader should note that even for very weak curvature requirements ($\gamma + \beta > 0, \gamma - \beta \geq 0, k_c \geq 0$) the model is well-posed. This is a new result, proved in [21] making use of a new coercive inequality for formally positive quadratic forms. For our dimension reduction procedure we focuss on the **symmetric-curvature case** with $\beta = \gamma$ and $k_c \geq 0$.

3 The Cosserat bulk problem on a thin flat domain

The basic task of any shell theory is a consistent reduction of some presumably "exact" 3D-theory to 2D. The three-dimensional problem (2.1) defined on the physical space $\mathbb{E}^3$ including units of measurement will now be adapted to a plate-like theory. Let us therefore assume that the problem is already transformed in **non-dimensional** form. This means we are given a three-dimensional (non-dimensional) **thin domain** $\Omega_h \subset \mathbb{R}^3$

$$
\Omega_h := \omega \times [-\frac{h}{2}, \frac{h}{2}], \quad \omega \subset \mathbb{R}^2\,, \tag{3.1}
$$

with **transverse boundary** $\partial\Omega_h^{\rm trans} = \omega \times \{-\frac{h}{2}, \frac{h}{2}\}$ and **lateral boundary** $\partial\Omega_h^{\rm lat} = \partial\omega \times [-\frac{h}{2}, \frac{h}{2}]$, where ω is a bounded open domain[2] in $\mathbb{R}^2$ with smooth boundary $\partial\omega$ and $h > 0$ is the **non-dimensional relative characteristic thickness (aspect ratio)**, $h \ll 1$. Moreover, assume we are given a deformation u and microrotation $\overline{A}$,

$$
u: \quad \Omega_h \subset \mathbb{R}^3 \mapsto \mathbb{R}^3\,, \qquad \overline{A}: \quad \Omega_h \subset \mathbb{R}^3 \mapsto \mathfrak{so}(3)\,, \tag{3.2}
$$

[2] For definiteness, one can think of $\omega = [0,1]] \times [0,1]] \subset \mathbb{R}^2$ without units of length.

solving the minimization problem on the thin domain Ω_h:

$$I(u,\overline{A}) = \int_{\Omega_h} W_{\mathrm{mp}}(\overline{\varepsilon}) + W_{\mathrm{curv}}(\nabla\,\mathrm{axl}\,\overline{A}) - \langle f,u\rangle\,\mathrm{dV} - \int_{\partial\Omega_h^{\mathrm{trans}}\cup\{\gamma_s\times[-\frac{h}{2},\frac{h}{2}]\}} \langle N,u\rangle\,\mathrm{dS} \mapsto \min .\ \text{w.r.t.}\ (u,\overline{A}),$$
$$\overline{\varepsilon} = \nabla u - \overline{A}, \quad u_{|_{\Gamma_0^h}} = u_{\mathrm{d}}(x,y,z), \quad \Gamma_0^h = \gamma_0\times[-\frac{h}{2},\frac{h}{2}], \quad \gamma_0\subset\partial\omega, \quad \gamma_s\cap\gamma_0=\emptyset,$$
$$\overline{A}: \quad \text{free on } \partial\omega\times[-\frac{h}{2},\frac{h}{2}], \quad \text{Neumann-type boundary condition}, \tag{3.3}$$
$$W_{\mathrm{mp}}(\overline{\varepsilon}) = \mu\,\|\mathrm{sym}\,\overline{\varepsilon}\|^2 + \mu_c\,\|\mathrm{skew}\,\overline{\varepsilon}\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\,[\overline{\varepsilon}]^2,$$
$$W_{\mathrm{curv}}(\mathfrak{K}) = \mu\frac{\widehat{L}_c^2}{2}\left(\alpha_1\,\|\mathrm{sym}\,\nabla\,\mathrm{axl}\,\overline{A}\|^2 + \alpha_2\,\|\mathrm{skew}\,\nabla\,\mathrm{axl}\,\overline{A}\|^2 + \frac{\alpha_3}{2}\,\mathrm{tr}\left[\nabla\,\mathrm{axl}\,\overline{A}\right]^2\right),$$

Here, $\alpha_1,\alpha_2,\alpha_3\geq 0$ are non-dimensional parameters. Moreover, the parameter $\widehat{L}_c$ has the form $\widehat{L}_c = \frac{L_c^{RVE}}{L}$, where L_c^{RVE} is a characteristic size of the microstructure and L is a characteristic value of the in-plane elongation of the original, relatively thin domain $\Omega^{\mathrm{rel.thin.}} = [0,L[\mathrm{m}]]\times[0,\mathrm{L}[\mathrm{m}]]\times[-\frac{\mathrm{h}}{2}\,\mathrm{L}[\mathrm{m}],\frac{\mathrm{h}}{2}\,\mathrm{L}[\mathrm{m}]]\subset\mathbb{E}^3$.[3] The "real" thickness of the plate is accordingly $d = h\,L[\mathrm{m}]$. Since for some constant $C_1>0$

$$C_1\cdot L_c^{RVE} = d = h\,L, \tag{3.4}$$

which says that the "real" thickness of the plate is $C_1\times$ "real" dimensions of the microstructure L_c^{RVE}, we obtain the important relation

$$C_1\cdot\widehat{L}_c = C_1\cdot\frac{L_c^{RVE}}{L} = h. \tag{3.5}$$

We want to find a reasonable approximation $(u_h,\overline{A}_h)$ of $(u,\overline{A})$ involving only two-dimensional quantities. Considering in the following $h\to 0$ we see that this weakens the curvature contribution and corresponds formally to $\widehat{L}_c\to 0$. However, $\widehat{L}_c\to 0$ and natural boundary conditions for the infinitesimal microrotations approach in the limit classical linear elasticity. So we might already expect a limit model which is closely related to classical plate models.

4 Scaling of fields

Scaling of independent and/or dependent variables is the usual first step when performing a dimensional reduction asymptotic analysis for a relatively thin domain. The employed scaling is decisive for the application of the Γ-convergence framework. The major justification of the employed scalings comes with the final convergence result.

There are basically two scalings at hand, one which we call the **nonlinear or natural scaling** and one which we refer to as the **linear elasticity scaling**. See [15] for an in-depth discussion of the differences generated by these scalings in classical linear/nonlinear elasticity. The **nonlinear or natural scaling** for a vectorfield $z:\Omega_h\subset\mathbb{R}^3\mapsto\mathbb{R}^3$ is just that one, which defines $z^\sharp:\eta\in\Omega_1\mapsto\mathbb{R}^3$ as the "same" field on the domain $\Omega_1=\omega\times[-1/2,1/2]$ (see (4.4)), only the independent variables are scaled as

$$\xi_1=\eta_1, \quad \xi_2=\eta_2, \quad \xi_3=h\,\eta_3,$$
$$z^\sharp(\xi_1,\xi_2,\frac{1}{h}\xi_3) := z(\xi_1,\xi_2,\xi_3), \qquad \textbf{nonlinear scaling}$$
$$\nabla_\xi z(\xi_1,\xi_2,\xi_3) = \left(\partial_{\eta_1}z^\sharp(\eta_1,\eta_2,\eta_3)|\partial_{\eta_2}z^\sharp(\eta_1,\eta_2,\eta_3)|\frac{1}{h}\partial_{\eta_3}z^\sharp(\eta_1,\eta_2,\eta_3)\right)$$
$$= \begin{pmatrix}\partial_{\eta_1}z_1^\sharp(\eta) & \partial_{\eta_2}z_1^\sharp(\eta) & \frac{1}{h}\partial_{\eta_3}z_1^\sharp(\eta)\\ \partial_{\eta_1}z_2^\sharp(\eta) & \partial_{\eta_2}z_2^\sharp(\eta) & \frac{1}{h}\partial_{\eta_3}z_2^\sharp(\eta)\\ \partial_{\eta_1}z_3^\sharp(\eta) & \partial_{\eta_2}z_3^\sharp(\eta) & \frac{1}{h}\partial_{\eta_3}z_3^\sharp(\eta)\end{pmatrix} =: \nabla_\eta^h u^\sharp(\eta). \tag{4.1}$$

In linear elasticity, in contrast, it is customary [8, 13] to use a simultaneous scaling of independent and dependent variables for the vectorfield $z:\Omega_h\subset\mathbb{R}^3\mapsto\mathbb{R}^3$ by defining $z^\flat:\eta\in\Omega_1\mapsto\mathbb{R}^3$

[3] This is an immediate consequence of the non-dimensionalization procedure.

in the form

$$\begin{aligned}&\xi_1 = \eta_1\,, \quad \xi_2 = \eta_2\,, \quad \xi_3 = h\,\eta_3\,,\\ &\begin{pmatrix} z_1^\flat(\xi_1,\xi_2,\frac{1}{h}\,\xi_3)\\ z_2^\flat(\xi_1,\xi_2,\frac{1}{h}\xi_3)\\ z_3^\flat(\xi_1,\xi_2,\frac{1}{h}\xi_3)\end{pmatrix} := \begin{pmatrix} z_1(\xi_1,\xi_2,\xi_3)\\ z_2(\xi_1,\xi_2,\xi_3)\\ h\,z_3(\xi_1,\xi_2,\xi_3)\end{pmatrix}\,, \qquad \textbf{linear scaling}\,. \end{aligned} \tag{4.2}$$

Here, the in-plane components z_1, z_2 of the vectorfield are treated differently from the out of plane (transverse) component z_3.[4] The corresponding relation between the gradient is expressed as

$$\nabla_\xi z(\xi_1,\xi_2,\xi_3) = \begin{pmatrix} \partial_{\eta_1} z_1^\flat(\eta) & \partial_{\eta_2} z_1^\flat(\eta) & \frac{1}{h}\partial_{\eta_3} z_1^\flat(\eta)\\ \partial_{\eta_1} z_2^\flat(\eta) & \partial_{\eta_2} z_2^\flat(\eta) & \frac{1}{h}\partial_{\eta_3} z_2^\flat(\eta)\\ \frac{1}{h}\partial_{\eta_1} z_3^\flat(\eta) & \frac{1}{h}\partial_{\eta_2} z_3^\flat(\eta) & \frac{1}{h^2}\partial_{\eta_3} z_3^\flat(\eta)\end{pmatrix} =: \widehat{\nabla}_\eta^h z^\flat(\eta)\,. \tag{4.3}$$

The scaling of the dependent variable corresponds to an additional ad-hoc assumption on the assumed response. In our case, we deal with the displacement field $u : \Omega_h \mapsto \mathbb{R}^3$ and the microrotation field $\overline{A} : \Omega_h \mapsto \mathfrak{so}(3)$. For the displacement field we propose not to take any scaling of the dependent variables into account. Thus we do not restrict the modeling to vertical deflections in the order of the plate thickness.[5] Rather we expect large bending terms. In the axial representation $\theta = \operatorname{axl}\overline{A} \in \mathbb{R}^3$ of the infinitesimal microrotation the component $\theta_i, i = 1,2,3$ corresponds to the infinitesimal rotation with axis e_i. Thus the in-plane rotation contribution is mapped by θ_3. Since the plate is getting very thin, we expect θ_3 to be much smaller than θ_1, θ_2, which themselves correspond to the bending rotations (out of plane rotations) with axis e_1, e_2. In order to reflect this behavior, the linear scaling suggests itself for the microrotations, i.e., $h\,\theta_3(\xi_1,\xi_2,\xi_3) = \theta_3^\flat(\xi_1,\xi_2,\frac{1}{h}\xi_3)$.

4.1 Transformation on a fixed domain with unit thickness

In order to apply standard techniques of Γ-convergence, we transform the problem onto a **fixed domain** Ω_1, independent of the aspect ratio $h > 0$. Define therefore

$$\Omega_1 = \omega \times [-\frac{1}{2},\frac{1}{2}] \subset \mathbb{R}^3\,, \quad \omega \subset \mathbb{R}^2\,. \tag{4.4}$$

The scaling transformation

$$\begin{aligned}\zeta : \eta \in \Omega_1 \subset \mathbb{R}^3 \mapsto \mathbb{R}^3, \quad \zeta(\eta_1,\eta_2,\eta_3) := (\eta_1,\eta_2,h\cdot\eta_3)\,,\\ \zeta^{-1} : \xi \in \Omega_h \subset \mathbb{R}^3 \mapsto \mathbb{R}^3, \quad \zeta^{-1}(\xi_1,\xi_2,\xi_3) := (\xi_1,\xi_2,\xi_3/h)\,,\end{aligned} \tag{4.5}$$

maps Ω_1 into Ω_h and $\zeta(\Omega_1) = \Omega_h$. We consider the correspondingly scaled function (subsequently, nonlinearly scaled functions defined on Ω_1 will be indicated with a superscript $\sharp$ while linearly scaled fields will get a superscript $\flat$) $u^\sharp : \Omega_1 \to \mathbb{R}^3$, defined by

$$u(\xi_1,\xi_2,\xi_3) = u^\sharp(\zeta^{-1}(\xi_1,\xi_2,\xi_3)) \quad \forall\,\xi \in \Omega_h\,; \quad u^\sharp(\eta) = u(\zeta(\eta)) \quad \forall\,\eta \in \Omega_1\,, \tag{4.6}$$

$$\begin{aligned}\nabla_\xi u(\xi_1,\xi_2,\xi_3) &= \Big(\partial_{\eta_1} u^\sharp(\eta_1,\eta_2,\eta_3)|\partial_{\eta_2} u^\sharp(\eta_1,\eta_2,\eta_3)|\frac{1}{h}\partial_{\eta_3} u^\sharp(\eta_1,\eta_2,\eta_3)\Big)\\ &= \begin{pmatrix} \partial_{\eta_1} u_1^\sharp(\eta) & \partial_{\eta_2} u_1^\sharp(\eta) & \frac{1}{h}\partial_{\eta_3} u_1^\sharp(\eta)\\ \partial_{\eta_1} u_2^\sharp(\eta) & \partial_{\eta_2} u_2^\sharp(\eta) & \frac{1}{h}\partial_{\eta_3} u_2^\sharp(\eta)\\ \partial_{\eta_1} u_3^\sharp(\eta) & \partial_{\eta_2} u_3^\sharp(\eta) & \frac{1}{h}\partial_{\eta_3} u_3^\sharp(\eta)\end{pmatrix} =: \nabla_\eta^h u^\sharp(\eta)\,.\end{aligned} \tag{4.7}$$

We define a (linearly) scaled infinitesimal microrotation rotation $\overline{A}^\flat : \Omega_1 \subset \mathbb{R}^3 \mapsto \mathfrak{so}(3)$ by considering the corresponding axial vector $\theta(\xi) := \operatorname{axl}\overline{A}(\xi) \in \mathbb{R}^3$ and its linearly scaled corre-

[4]Since we assume that the un-scaled component z_3 is bounded, the linear scaling implies that the scaled vertical component $z_3^\flat$ should be of the order of h, i.e., the vertical deflection should be in the order of the thickness of the plate (instead of large vertical deflections...)

[5]In Ciarlet [8, p.73]: "Thus, counter to appearance, the linear Kirchhoff-Love theory is strictly a 'small displacement' theory: In order that it be valid, the transverse displacement should remain of the order of the thickness of the plate." And in Fonseca et al. [15, p.552]: " ....the limit kinematics that are imposed by the scaling are too stringent: they force the transverse limit displacement to be 0."

spondence $\theta^\flat(\eta)$ through

$$\theta(\xi_1,\xi_2,\xi_3)=\theta^\flat(\zeta^{-1}(\xi_1,\xi_2,\xi_3))\quad\forall\,\xi\in\Omega_h\,;\quad \theta^\flat(\eta)=\theta(\zeta(\eta))\quad\forall\,\eta\in\Omega_1\,,$$

$$\nabla_\xi\theta(\xi_1,\xi_2,\xi_3)=\begin{pmatrix}\partial_{\eta_1}\theta_1^\flat(\eta) & \partial_{\eta_2}\theta_1^\flat(\eta) & \frac{1}{h}\partial_{\eta_3}\theta_1^\flat(\eta)\\ \partial_{\eta_1}\theta_2^\flat(\eta) & \partial_{\eta_2}\theta_2^\flat(\eta) & \frac{1}{h}\partial_{\eta_3}\theta_2^\flat(\eta)\\ \frac{1}{h}\partial_{\eta_1}\theta_3^\flat(\eta) & \frac{1}{h}\partial_{\eta_2}\theta_3^\flat(\eta) & \frac{1}{h^2}\partial_{\eta_3}\theta_3^\flat(\eta)\end{pmatrix}=:\widehat{\nabla}_\eta^h\theta^\flat(\eta)\,. \tag{4.8}$$

This allows us to define scaled nonsymmetric stretches $\overline{\varepsilon}_h^\sharp\in\mathfrak{gl}(3)$ and the scaled second order curvature tensor $\mathfrak{K}_h^\flat:\Omega_1\mapsto\mathfrak{gl}(3)$

$$\overline{\varepsilon}_h^\sharp:=\nabla_\eta^h u^\sharp-\overline{A}^\flat\,,\qquad \widehat{\nabla}_\eta^h\theta^\flat(\eta)=:\mathfrak{K}_h^\flat(\eta)\,, \tag{4.9}$$

where

$$\nabla_\eta^{h_j}u^\sharp-\overline{A}^\flat=\begin{pmatrix}\partial_{\eta_1}u_1^\sharp(\eta) & \partial_{\eta_2}u_1^\sharp(\eta) & \frac{1}{h}\partial_{\eta_3}u_1^\sharp(\eta)\\ \partial_{\eta_1}u_2^\sharp(\eta) & \partial_{\eta_2}u_2^\sharp(\eta) & \frac{1}{h}\partial_{\eta_3}u_2^\sharp(\eta)\\ \partial_{\eta_1}u_3^\sharp(\eta) & \partial_{\eta_2}u_3^\sharp(\eta) & \frac{1}{h}\partial_{\eta_3}u_3^\sharp(\eta)\end{pmatrix}-\begin{pmatrix}0 & -\frac{1}{h}\theta_3^\flat & \theta_2^\flat\\ \frac{1}{h}\theta_3^\flat & 0 & -\theta_1^\flat\\ -\theta_2^\flat & \theta_1^\flat & 0\end{pmatrix} \tag{4.10}$$

for $\theta^\flat:=\operatorname{axl}\overline{A}^\flat$. Moreover, we define nonlinearly scaled functions by setting

$$f^\sharp(\eta):=f(\zeta(\eta)),\qquad u_{\mathrm{d}}^\sharp(\eta)=u_{\mathrm{d}}(\zeta(\eta))\,,\qquad N^\sharp(\eta):=N(\zeta(\eta))\,. \tag{4.11}$$

In terms of the introduced nonlinearly scaled displacement and the linearly scaled infinitesimal microrotations $u^\sharp:\quad \Omega_1\subset\mathbb{R}^3\mapsto\mathbb{R}^3\,,\qquad \overline{A}^\flat:\Omega_1\subset\mathbb{R}^3\mapsto\mathfrak{so}(3)$, the scaled problem solves the following two-field minimization problem on the fixed domain Ω_1:

$$\begin{aligned}
I^{\sharp,\flat}(u^\sharp,\nabla_\eta^h u^\sharp,\overline{A}^\flat,\widehat{\nabla}_\eta^h\operatorname{axl}\overline{A}^\flat)&=\int_{\eta\in\Omega_1}\Big[W_{\mathrm{mp}}(\overline{\varepsilon}_h^\sharp)+W_{\mathrm{curv}}(\mathfrak{K}_h^\flat)-\langle f^\sharp,u^\sharp\rangle\Big]\,\det[\nabla\zeta(\eta)]\,\mathrm{dV}_\eta\\
&\quad-\int_{\partial\Omega_1^{\mathrm{trans}}\cup\{\gamma_s\times[-\frac12,\frac12]\}}\langle N^\sharp,u^\sharp\rangle\,\|\operatorname{Cof}\nabla\zeta(\eta).e_3\|\,\mathrm{dS}_\eta\\
&=h\int_{\eta\in\Omega_1}W_{\mathrm{mp}}(\overline{\varepsilon}_h^\sharp)+W_{\mathrm{curv}}(\mathfrak{K}_h^\flat)-\langle f^\sharp,u^\sharp\rangle\,\mathrm{dV}_\eta\\
&\quad-\int_{\partial\Omega_1^{\mathrm{trans}}}\langle N^\sharp,u^\sharp\rangle\,1\,\mathrm{dS}_\eta-\int_{\gamma_s\times[-\frac12,\frac12]}\langle N^\sharp,u^\sharp\rangle\,h\,\mathrm{dS}_\eta\mapsto\min.\ \text{w.r.t.}\ (u^\sharp,\overline{A}^\flat)\,.
\end{aligned}\tag{4.12}$$

4.2 The rescaled variational Cosserat bulk problem

Since the energy $\frac{1}{h}I^{\sharp,\flat}$ would not be finite for $h\to0$ if tractions $N^\sharp$ on the transverse boundary were present, the investigations are in principle restricted to the case of $N^\sharp=0$ on $\partial\Omega_1^{\mathrm{trans}}$.[6] For conciseness we investigate the following simplified and rescaled ($N^\sharp, f^\sharp=0,\ u_{\mathrm{d}}(\xi_1,\xi_2,\xi_3):=u_{\mathrm{d}}(\xi_1,\xi_2)$) two-field minimization problem on Ω_1 with respect to Γ-convergence (without the factor $h>0$ now), i.e. we are interested in the limiting behavior of the scaled energy per unit aspect ratio h:

$$\begin{aligned}
&I_h^{\sharp,\flat}(u^\sharp,\nabla_\eta^h u^\sharp,\overline{A}^\flat,\widehat{\nabla}_\eta^h\operatorname{axl}\overline{A}^\flat)=\int_{\eta\in\Omega_1}W_{\mathrm{mp}}(\overline{\varepsilon}_h^\sharp)+W_{\mathrm{curv}}(\mathfrak{K}_h^\flat)\,\mathrm{dV}_\eta\mapsto\min.\ \text{w.r.t.}\ (u^\sharp,\overline{A}^\flat)\,,\\
&\overline{\varepsilon}_h^\sharp=\nabla_\eta^h u^\sharp-\overline{A}^\flat,\quad u^\sharp_{|_{\Gamma_0^1}}(\eta)=u_{\mathrm{d}}^\sharp(\eta)=u_{\mathrm{d}}(\zeta(\eta))=u_{\mathrm{d}}(\eta_1,\eta_2,h\cdot\eta_3)=u_{\mathrm{d}}(\eta_1,\eta_2,0)\,,\\
&\Gamma_0^1=\gamma_0\times[-\frac12,\frac12],\quad \gamma_0\subset\partial\omega,\qquad \mathfrak{K}_h^\flat=\widehat{\nabla}_\eta^h\operatorname{axl}\overline{A}^\flat(\eta)\,,\\
&\overline{A}^\flat:\quad\text{free on }\partial\omega\times[-\frac12,\frac12],\quad\text{Neumann-type boundary condition}\,.
\end{aligned}\tag{4.13}$$

Here we assume for simplicity that the bulk boundary condition u_{d} is already independent of the transverse variable and we restrict attention to the weakest response, the **Neumann boundary conditions** on the Cosserat rotations $\overline{A}^\flat$.

[6] The thin plate limit $h\to0$ obviously cannot support non-vanishing transverse surface loads.

4.3 Recall on Γ-convergence

Let us briefly recapitulate the notions involved by using Γ-convergence. For a detailed treatment we refer to [22, 6]. The notion of Γ-convergence depends strongly on the topology of the space X, which in our discussion is assumed to be metrizable. In the following, therefore, X will always denote a metric space such that sequential compactness and compactness coincide. Moreover, we set $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. We consider a sequence of energy functionals $I_{h_j} : X \mapsto \overline{\mathbb{R}}\,, h_j \to 0$.

Definition 4.1 (Γ-convergence)
Let X be a metric space. We say that a sequence of functionals $I_{h_j} : X \mapsto \overline{\mathbb{R}}$ Γ-converges in X to the limit functional $I_0 : X \mapsto \overline{\mathbb{R}}$, if for all $x \in X$ we have

$$\forall\, x \in X : \;\forall\, x_{h_j} \to x : \quad I_0(x) \le \liminf_{h_j \to 0} I_{h_j}(x_{h_j})\,, \qquad \textbf{(lim inf - inequality)}$$
$$\forall\, x \in X : \;\exists\, x_{h_i} \to x : \quad I_0(x) \ge \limsup_{h_i \to 0} I_{h_i}(x_{h_i})\,, \qquad \textbf{(recovery sequence)}\;. \qquad \blacksquare$$

Γ-convergence corresponds to convergence of the energy along minimizing sequences for a family of functionals and all continuous perturbations.

5 The "two-field" Cosserat Γ-limit

5.1 The spaces and admissible sets

We proceed to the investigation of the Γ-limit for the rescaled problem (4.13). We do not use $I_h^{\sharp,\flat}$ directly in our investigation of Γ-convergence, since this would imply working with the weak topology of $H^{1,2}(\Omega_1, \mathbb{R}^3) \times H^{1,2}(\Omega_1, \mathfrak{so}(3))$, which does not give rise to a metric space. Instead, we define suitable "bulk" spaces X, X' and suitable "two-dimensional" spaces X_ω, X'_ω. Now define the spaces

$$\begin{aligned} X &:= \{(u, \overline{A}) \in L^2(\Omega_1, \mathbb{R}^3) \times L^2(\Omega_1, \mathfrak{so}(3))\}\,,\\ X' &:= \{(u, \overline{A}) \in H^{1,2}(\Omega_1, \mathbb{R}^3) \times H^{1,2}(\Omega_1, \mathfrak{so}(3))\}\,,\\ X_\omega &:= \{(u, \overline{A}) \in L^2(\omega, \mathbb{R}^3) \times L^2(\omega, \mathfrak{so}(3))\}\,,\\ X'_\omega &:= \{(u, \overline{A}) \in H^{1,2}(\omega, \mathbb{R}^3) \times H^{1,2}(\omega, \mathfrak{so}(3))\}\,, \end{aligned} \tag{5.1}$$

and the admissible sets

$$\begin{aligned} \mathcal{A}' &:= \{(u, \overline{A}) \in H^{1,2}(\Omega_1, \mathbb{R}^3) \times H^{1,2}(\Omega_1, \mathfrak{so}(3))\,, \quad u_{|_{\Gamma_0^1}}(\eta) = u_{\mathrm{d}}^\sharp(\eta) \quad \}\,, \quad (5.2)\\ \mathcal{A}'_\omega &:= \{(u, \overline{A}) \in H^{1,2}(\omega, \mathbb{R}^3) \times H^{1,2}(\omega, \mathfrak{so}(3))\,, \quad u_{|_{\gamma_0}}(\eta_1, \eta_2) = u_{\mathrm{d}}^\sharp(\eta_1, \eta_2, 0) \quad \}\,, . \end{aligned}$$

We note the compact embedding $X' \subset X$ and the natural inclusions $X_\omega \subset X$ and $X'_\omega \subset X'$. Now we extend the rescaled energies to the space X through redefining

$$I_h^{\sharp,\flat}(u^\sharp, \nabla_\eta^h u^\sharp, \overline{A}^\flat, \widehat{\nabla}_\eta^h \operatorname{axl} \overline{A}^\flat) = \begin{cases} I_h^{\sharp,\flat}(u^\sharp, \nabla_\eta^h u^\sharp, \overline{A}^\flat, \widehat{\nabla}_\eta^h \operatorname{axl} \overline{A}^\flat) & \text{if } (u^\sharp, \overline{A}^\flat) \in \mathcal{A}' \\ +\infty & \text{else in } X\,, \end{cases} \tag{5.3}$$

by abuse of notation. This is a classical trick used in applications of Γ-convergence. It has the virtue of incorporating the boundary conditions already in the energy functional. In the following, Γ-convergence results will be shown with respect to the encompassing metric space X.

5.2 The Γ-limit variational problem

Our main result is the Γ-limit for symmetric curvature $\alpha_2 = 0$ and strictly positive curvature bulk modulus $k_c > 0$.

Theorem 5.1 (Γ-limit for $k_c > 0$ and $\alpha_2 = 0$)
For strictly positive curvature bulk modulus $k_c > 0$ and symmetric curvature $\alpha_2 = 0$ the Γ-limit for problem (4.13) in the setting of (5.3) is given by the limit energy functional $I_0^{\sharp,\flat} : X \mapsto \overline{\mathbb{R}}$,

$$I_0^{\sharp,\flat}(v, \overline{A}) := \begin{cases} \int_\omega W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla v, \operatorname{axl} \overline{A}) + W_{\mathrm{curv}}^{\mathrm{hom}}(\nabla \operatorname{axl} \overline{A}) - \langle f, v\rangle \,\mathrm{d}\omega & (v, \overline{A}) \in \mathcal{A}'_\omega \\ +\infty & \textit{else in } X\,, \end{cases} \tag{5.4}$$

with $W^{\rm hom}_{\rm mp}$ and $W^{\rm hom}_{\rm curv}$ defined below.

The proof of this statement will be given in Section 6. The limit functions are independent of the transverse variable η_3. This Γ-limit determines in fact a purely two-dimensional minimization problem for the deflection of the midsurface $v:\omega\subset\mathbb{R}^2\mapsto\mathbb{R}^3$ and the infinitesimal microrotation of the plate (shell) $\overline{A}:\omega\subset\mathbb{R}^2\mapsto\mathfrak{so}(3)$ on ω under the boundary conditions of place for the midsurface deflection v on the Dirichlet part of the lateral boundary $\gamma_0\subset\partial\omega$,

$$v_{|_{\gamma_0}} = u_{\rm d}(x,y,0)\,, \qquad \text{simply supported (fixed, welded)}\,. \tag{5.5}$$

The boundary conditions for the microrotations $\overline{A}$ are automatically determined in the variational process. The dimensionally homogenized local density is

$$\begin{aligned} W^{\rm hom}_{\rm mp}(\nabla v,\theta) :=& \mu \underbrace{\|\operatorname{sym}\nabla_{\eta_1,\eta_2}(v_1,v_2)\|^2}_{\text{homogenized shear-stretch energy}} + \underbrace{2\mu\,\frac{\mu_c}{\mu+\mu_c}\,\|\nabla_{\eta_1,\eta_2}v_3 - \begin{pmatrix}-\theta_2\\ \theta_1\end{pmatrix}\|^2}_{\text{homogenized transverse shear energy}} \\ &+ \frac{\mu\lambda}{2\mu+\lambda}\underbrace{\operatorname{tr}\left[\nabla_{\eta_1,\eta_2}(v_1,v_2)\right]^2}_{\text{homogenized elongational stretch energy}}\,. \end{aligned}$$

The homogenized curvature density is given by

$$W^{\rm hom}_{\rm curv}(\nabla\theta) := \mu\frac{\widehat{L}_c^2}{2}\left(\alpha_1\,\|\operatorname{sym}\nabla_{\eta_1,\eta_2}(\theta_1,\theta_2)\|^2 + \frac{\alpha_1\alpha_3}{2\alpha_1+\alpha_3}\operatorname{tr}\left[\nabla_{\eta_1,\eta_2}(\theta_1,\theta_2)\right]^2\right)\,.$$

It is clear that the limit functional $I_0^{\sharp,\flat}$ is weakly lower semicontinuous in the topology of $X' = H^{1,2}(\Omega,\mathbb{R}^3)\times H^{1,2}(\Omega,\mathfrak{so}(3))$ by simple convexity arguments. Note the appearance of the **harmonic mean** $\mathcal{H}$,

$$\frac{1}{2}\mathcal{H}(\mu,\frac{\lambda}{2}) = \frac{\mu\lambda}{2\mu+\lambda}\,, \qquad \mathcal{H}(\mu,\mu_c) = 2\mu\,\frac{\mu_c}{\mu+\mu_c}\,, \qquad \frac{1}{2}\mathcal{H}(\alpha_1,\frac{\alpha_3}{2}) = \frac{\alpha_1\,\alpha_3}{2\alpha_1+\alpha_3}\,. \tag{5.6}$$

5.3 Descaled Γ-limit - Reissner-Mindlin membrane-bending model

After descaling the Γ-limit minimization problem turns into

$$\begin{aligned} &\int_\omega h\left(\mu\,\|\operatorname{sym}\nabla(v_1,v_2)\|^2 + \frac{2\mu\,\mu_c}{\mu+\mu_c}\,\|\nabla v_3 - \begin{pmatrix}-\theta_2\\ \theta_1\end{pmatrix}\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\nabla(v_1,v_2)\right]^2\right) \\ &+\mu\frac{\widehat{L}_c^2}{2}\,h\left(\alpha_1\,\|\operatorname{sym}\nabla(\theta_1,\theta_2)\|^2 + \frac{\alpha_1\alpha_3}{2\alpha_1+\alpha_3}\operatorname{tr}\left[\nabla(\theta_1,\theta_2)\right]^2\right) - \langle f,v\rangle\,{\rm d}\omega \mapsto \min.\ \text{w.r.t. } (v,\theta), \\ &\quad v_{|_{\gamma_0}} = u^{\rm d}(x,y,0)\,. \end{aligned} \tag{5.7}$$

Taking into account that $C_1\,\widehat{L}_c = h$ (3.5) and abbreviating $\kappa = \frac{4\mu_c}{\mu+\mu_c}$ yields the classical Reissner-Mindlin model (7.1) with appropriate re-definitions of constants.

6 Proof for positive curvature bulk modulus $k_c > 0$

We continue by proving Theorem 5.1, i.e., the claim on the form of the Γ-limit for strictly positive curvature bulk modulus by considering micropolar curvature energies having the form

$$W_{\rm curv}(\nabla\operatorname{axl}\overline{A}) = \mu\,\frac{\widehat{L}_c^2}{2}\left(\|\operatorname{dev}\operatorname{sym}\nabla\operatorname{axl}\overline{A}\|^2 + \frac{k_c}{2}\operatorname{tr}\left[\nabla\operatorname{axl}\overline{A}\right]^2\right) \tag{6.1}$$

for $k_c > 0$. Note, however, that the Cosserat bulk problem is well-posed for $k_c = 0$, see [21]. The proof of Γ-convergence is subsequently split into several steps.

6.1 Compactness

Theorem 6.1 (Compactness of $I^{\sharp,\flat}_{h_j}$)
Consider a sequence $(u^\sharp_{h_j},\overline{A}^\flat_{h_j})\in\mathcal{A}'\subset X$ such that $\|\overline{A}^\flat_{h_j}\|_{L^2(\Omega_1,\mathfrak{so}(3))}\le K_1$ and $I^{\sharp,\flat}_{h_j}(u^\sharp_{h_j},\overline{A}^\flat_{h_j})\le K_2$, with constants K_1,K_2 independent of $h_j>0$. Then, for positive curvature bulk modulus $k_c>0$ it holds

$$\|u^\sharp_{h_j}\|_{H^{1,2}(\Omega_1,\mathbb{R}^3)}\le K_3\,,\qquad \|\overline{A}^\flat_{h_j}\|_{H^{1,2}(\Omega_1,\mathfrak{so}(3))}\le K_4\,,\tag{6.2}$$

with constants K_3,K_4 independent of $h_j>0$. The sequence $(u^\sharp_{h_j},\overline{A}^\flat_{h_j})\in\mathcal{A}'$ admits weakly convergent subsequences (not relabeled) $(u^\sharp_{h_j},\overline{A}^\flat_{h_j})\rightharpoonup(u^\sharp_0,\overline{A}^\flat_0)\in X$. In addition, the weak limit

$$(u^\sharp_0,\overline{A}^\flat_0)\in\mathcal{A}'_\omega\tag{6.3}$$

is independent of the transverse variable η_3 and $(\operatorname{axl}\overline{A}^\flat_0)_3=0$ (no in-plane drill rotation).

Proof. Along the sequence $(u^\sharp_{h_j},\overline{A}^\flat_{h_j})\in\mathcal{A}'\subset X$ we have

$$\begin{aligned}\infty>K_2>I^{\sharp,\flat}_{h_j}(u^\sharp_{h_j},\overline{A}^\flat_{h_j})&=\int_{\Omega_1}W_{\mathrm{mp}}(\overline{\varepsilon}^\sharp_{h_j})+W_{\mathrm{curv}}(\mathfrak{K}^\flat_{h_j})\,\mathrm{dV}_\eta\ge\int_{\Omega_1}W_{\mathrm{mp}}(\overline{\varepsilon}^\sharp_{h_j})\,\mathrm{dV}_\eta\\&\ge\int_{\Omega_1}\min(\mu_c,\mu,\frac{K}{2})\,\|\nabla^{h_j}_\eta u^\sharp_{h_j}-\overline{A}^\flat_{h_j}\|^2\,\mathrm{dV}_\eta\,.\end{aligned}\tag{6.4}$$

But with (4.10) we obtain

$$\nabla^{h_j}_\eta u^\sharp_{h_j}-\overline{A}^\flat_{h_j}=\begin{pmatrix}\partial_{\eta_1}u^\sharp_1(\eta)&\partial_{\eta_2}u^\sharp_1(\eta)&\frac{1}{h}\partial_{\eta_3}u^\sharp_1(\eta)\\\partial_{\eta_1}u^\sharp_2(\eta)&\partial_{\eta_2}u^\sharp_2(\eta)&\frac{1}{h}\partial_{\eta_3}u^\sharp_2(\eta)\\\partial_{\eta_1}u^\sharp_3(\eta)&\partial_{\eta_2}u^\sharp_3(\eta)&\frac{1}{h}\partial_{\eta_3}u^\sharp_3(\eta)\end{pmatrix}-\begin{pmatrix}0&-\frac{1}{h_j}\theta^\flat_{h_j,3}&\theta^\flat_{h_j,2}\\\frac{1}{h_j}\theta^\flat_{h_j,3}&0&-\theta^\flat_{h_j,1}\\-\theta^\flat_{h_j,2}&\theta^\flat_{h_j,1}&0\end{pmatrix},\tag{6.5}$$

$$\begin{aligned}\|\nabla^{h_j}_\eta u^\sharp_{h_j}-\overline{A}^\flat_{h_j}\|^2=&\|\operatorname{sym}\begin{pmatrix}\partial_{\eta_1}u^\sharp_1(\eta)&\partial_{\eta_2}u^\sharp_1(\eta)\\\partial_{\eta_1}u^\sharp_2(\eta)&\partial_{\eta_2}u^\sharp_2(\eta)\end{pmatrix}\|^2+\frac{1}{{h_j}^2}\|\partial_{\eta_3}u^\sharp_3(\eta)\|^2\\&+\|\operatorname{skew}\begin{pmatrix}\partial_{\eta_1}u^\sharp_1(\eta)&\partial_{\eta_2}u^\sharp_1(\eta)\\\partial_{\eta_1}u^\sharp_2(\eta)&\partial_{\eta_2}u^\sharp_2(\eta)\end{pmatrix}-\begin{pmatrix}0&-\frac{1}{h_j}\theta^\flat_{h_j,3}\\\frac{1}{h_j}\theta^\flat_{h_j,3}&0\end{pmatrix}\|^2\\&+\|\begin{pmatrix}\partial_{\eta_1}u^\sharp_3(\eta)\\\partial_{\eta_2}u^\sharp_3(\eta)\end{pmatrix}-\begin{pmatrix}-\theta^\flat_{h_j,2}\\\theta^\flat_{h_j,1}\end{pmatrix}\|^2+\|\begin{pmatrix}\frac{1}{h}\partial_{\eta_3}u^\sharp_1(\eta)\\\frac{1}{h}\partial_{\eta_3}u^\sharp_2(\eta)\end{pmatrix}-\begin{pmatrix}\theta^\flat_{h_j,2}\\-\theta^\flat_{h_j,1}\end{pmatrix}\|^2.\end{aligned}\tag{6.6}$$

Combining (6.4) with (6.6) and using the assumption that $\theta^\flat_{h_j}=\operatorname{axl}\overline{A}^\flat_{h_j}$ is bounded in $L^2(\Omega_1,\mathbb{R}^3)$ independent of h_j we obtain easily an h_j-independent bound for

$$\begin{aligned}\infty>K_5>&\int_{\Omega_1}\|\operatorname{sym}\begin{pmatrix}\partial_{\eta_1}u^\sharp_1(\eta)&\partial_{\eta_2}u^\sharp_1(\eta)\\\partial_{\eta_1}u^\sharp_2(\eta)&\partial_{\eta_2}u^\sharp_2(\eta)\end{pmatrix}\|^2+\frac{1}{{h_j}^2}\|\partial_{\eta_3}u^\sharp_3(\eta)\|^2\\&+\|\begin{pmatrix}\partial_{\eta_1}u^\sharp_3(\eta)\\\partial_{\eta_2}u^\sharp_3(\eta)\end{pmatrix}\|^2+\|\begin{pmatrix}\frac{1}{h}\partial_{\eta_3}u^\sharp_1(\eta)\\\frac{1}{h}\partial_{\eta_3}u^\sharp_2(\eta)\end{pmatrix}\|^2\,\mathrm{dV}_\eta\\&=\int_{\Omega_1}\|\operatorname{sym}\begin{pmatrix}\partial_{\eta_1}u^\sharp_1(\eta)&\partial_{\eta_2}u^\sharp_1(\eta)\\\partial_{\eta_1}u^\sharp_2(\eta)&\partial_{\eta_2}u^\sharp_2(\eta)\end{pmatrix}\|^2+\frac{1}{{h_j}^2}\|\partial_{\eta_3}u^\sharp_3(\eta)\|^2\\&+\|\begin{pmatrix}\partial_{\eta_1}u^\sharp_3(\eta)\\\partial_{\eta_2}u^\sharp_3(\eta)\end{pmatrix}\|^2+\frac{1}{{h_j}^2}\|\begin{pmatrix}\partial_{\eta_3}u^\sharp_1(\eta)\\\partial_{\eta_3}u^\sharp_2(\eta)\end{pmatrix}\|^2\,\mathrm{dV}_\eta\\&\ge\int_{\Omega_1}\|\operatorname{sym}\begin{pmatrix}\partial_{\eta_1}u^\sharp_1(\eta)&\partial_{\eta_2}u^\sharp_1(\eta)\\\partial_{\eta_1}u^\sharp_2(\eta)&\partial_{\eta_2}u^\sharp_2(\eta)\end{pmatrix}\|^2+\|\partial_{\eta_3}u^\sharp_3(\eta)\|^2\\&+\|\begin{pmatrix}\partial_{\eta_1}u^\sharp_3(\eta)\\\partial_{\eta_2}u^\sharp_3(\eta)\end{pmatrix}\|^2+\|\begin{pmatrix}\partial_{\eta_3}u^\sharp_1(\eta)\\\partial_{\eta_3}u^\sharp_2(\eta)\end{pmatrix}\|^2\,\mathrm{dV}_\eta\\&\ge1\int_{\Omega_1}\|\operatorname{sym}\begin{pmatrix}\partial_{\eta_1}u^\sharp_1(\eta)&\partial_{\eta_2}u^\sharp_1(\eta)&\partial_{\eta_3}u^\sharp_1(\eta)\\\partial_{\eta_1}u^\sharp_2(\eta)&\partial_{\eta_2}u^\sharp_2(\eta)&\partial_{\eta_3}u^\sharp_2(\eta)\\\partial_{\eta_1}u^\sharp_3(\eta)&\partial_{\eta_2}u^\sharp_3(\eta)&\partial_{\eta_3}u^\sharp_3(\eta)\end{pmatrix}\|^2\,\mathrm{dV}_\eta\,,\end{aligned}\tag{6.7}$$

(for $h_j > 0$ small enough). Korn's first inequality and the Dirichlet-boundary condition on $u^\sharp_{h_j}$ show the h_j-independent H^1-bound on $u^\sharp_{h_j}$. Thus we may extract a weakly convergent subsequence (not relabeled) $u^\sharp_{h_j} \rightharpoonup u^\sharp_0$ and the weak limit must be independent of η_3 on account of $(6.7)_2$.

Next, combine (6.4) and (6.6) and the boundedness of the in-plane skew-symmetric deflection to see that the boundedness of

$$\int_{\Omega_1} \| \operatorname{skew} \begin{pmatrix} \partial_{\eta_1} u^\sharp_1(\eta) & \partial_{\eta_2} u^\sharp_1(\eta) \\ \partial_{\eta_1} u^\sharp_2(\eta) & \partial_{\eta_2} u^\sharp_2(\eta) \end{pmatrix} - \begin{pmatrix} 0 & -\frac{1}{h_j}\theta^\flat_{h_j,3} \\ \frac{1}{h_j}\theta^\flat_{h_j,3} & 0 \end{pmatrix} \|^2 \, \mathrm{dV}_\eta \tag{6.8}$$

implies the boundedness of

$$\int_{\Omega_1} \| \begin{pmatrix} 0 & -\frac{1}{h_j}\theta^\flat_{h_j,3} \\ \frac{1}{h_j}\theta^\flat_{h_j,3} & 0 \end{pmatrix} \|^2 \, \mathrm{dV}_\eta \tag{6.9}$$

showing that $\|\theta^\flat_{h_j,3}\|_{L^2(\Omega_1,\mathbb{R})} \to 0$ for $h_j \to 0$. For the (similar) treatment of the curvature energy we note that

$$\begin{aligned} \infty > K_2 > I^{\sharp,\flat}_{h_j}(u^\sharp_{h_j}, \overline{A}^\flat_{h_j}) &= \int_{\Omega_1} W_{\mathrm{mp}}(\overline{\varepsilon}^\sharp_{h_j}) + W_{\mathrm{curv}}(\mathfrak{K}^\flat_{h_j}) \, \mathrm{dV}_\eta \geq \int_{\Omega_1} W_{\mathrm{curv}}(\mathfrak{K}^\flat_{h_j}) \, \mathrm{dV}_\eta \\ &\geq \int_{\Omega_1} \mu \frac{L_c^2}{2} \min(1, \frac{3k_c}{2}) \, \| \operatorname{sym} \widehat{\nabla}^h_\eta \theta^\flat_{h_j}(\eta)\|^2 \, \mathrm{dV}_\eta \, . \end{aligned} \tag{6.10}$$

Now use Theorem 7.1 to get the h_j-independent H^1-bound on $\theta^\flat_{h_j}$ together with the existence of a weakly convergent subsequence $\theta^\flat_{h_j} \rightharpoonup \theta^\flat_0$ and the claim that the weak limit is independent of the transverse variable η_3 and $\theta^\flat_{0,3} = 0$. ■

Remark 6.2
In linear Cosserat models Korn's first inequality is usually not needed in showing coercivity.

6.2 Lower bound - the $\liminf$-condition

If $I^{\sharp,\flat}_0$ is the Γ-limit of the sequence of energy functionals $I^{\sharp,\flat}_{h_j}$ then we must have ($\liminf$-inequality) that

$$I^{\sharp,\flat}_0(u^\sharp_0, \overline{A}^\flat_0) \leq \liminf_{h_j} I^{\sharp,\flat}_{h_j}(u^\sharp_{h_j}, \overline{A}^\flat_{h_j}) \, , \tag{6.11}$$

whenever

$$u^\sharp_{h_j} \to u^\sharp_0 \quad \text{in } L^2(\Omega_1, \mathbb{R}^3) \, , \quad \overline{A}^\flat_{h_j} \to \overline{A}^\flat_0 \quad \text{in } L^2(\Omega_1, \mathfrak{so}(3)) \, , \tag{6.12}$$

for arbitrary $(u^\sharp_0, \overline{A}^\flat_0) \in X$. Observe that we can restrict attention to sequences $(u^\sharp_{h_j}, \overline{A}^\flat_{h_j}) \in X$ such that $I^{\sharp,\flat}_{h_j}(u^\sharp_{h_j}, \overline{A}^\flat_{h_j}) < \infty$ since otherwise the statement is true anyway. Sequences with $I^{\sharp,\flat}_{h_j}(u^\sharp_{h_j}, \overline{A}^\flat_{h_j}) < \infty$ are uniformly bounded in the space X', as seen previously. This implies weak convergence of a subsequence in X'. But we know already that the original sequences converge strongly in X to the limit $(u^\sharp_0, \overline{A}^\flat_0) \in X$. Hence we must have as well weak convergence to $u^\sharp_0 \in H^{1,2}(\omega, \mathbb{R}^3)$ and $\overline{A}^\flat_0 \in H^{1,2}(\omega, \mathfrak{so}(3))$, independent of the transverse variable η_3.

In a first step we consider now the **local energy contribution**: along sequences $(u^\sharp_{h_j}, \overline{A}^\flat_{h_j}) \in X$ with finite energy $I^{\sharp,\flat}_{h_j}$, the third column of $\nabla^{h_j}_\eta u^\sharp_{h_j}$ remains bounded but otherwise indetermined. Therefore, a really trivial lower bound is obtained by minimizing the effect of the derivative in this direction in the local energy W_{mp}. To continue our development, we need some calculations: for smooth $v : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$, $\overline{A} : \omega \subset \mathbb{R}^2 \mapsto \mathfrak{so}(3)$ define the vector

$(b^*, \tilde{p}_1^*) \in \mathbb{R}^4$ formally through

$$W_{\rm mp}^{\rm hom}(\nabla v, \theta) = W_{\rm mp}\left((\nabla v|b^*) - \begin{pmatrix} 0 & -\tilde{p}_1^* & \theta_2 \\ \tilde{p}_1^* & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{pmatrix}\right)$$
$$:= \inf_{b\in\mathbb{R}^3, \tilde{p}_1\in\mathbb{R}} W_{\rm mp}\left((\nabla v|b) - \begin{pmatrix} 0 & -\tilde{p}_1 & \theta_2 \\ \tilde{p}_1 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{pmatrix}\right). \tag{6.13}$$

The vector $(b^*, \tilde{p}_1^*)$, which realizes this infimum, can be explicitly determined. The calculation is lengthy but otherwise straight forward. We obtain

$$\begin{pmatrix} b_1^* \\ b_2^* \\ b_3^* \\ \tilde{p}_1^* \end{pmatrix} = \begin{pmatrix} \frac{\mu_c-\mu}{\mu+\mu_c}\,\partial_{\eta_1} v_3 + \frac{2\mu_c}{\mu+\mu_c}\theta_2 \\ \frac{\mu_c-\mu}{\mu+\mu_c}\,\partial_{\eta_2} v_3 - \frac{2\mu_c}{\mu+\mu_c}\theta_1 \\ -\frac{\lambda}{2\mu+\lambda}\,(\partial_{\eta_1} v_1 + \partial_{\eta_2} v_2) \\ \frac{\partial_{\eta_1} v_2 - \partial_{\eta_2} v_1}{2} \end{pmatrix}. \tag{6.14}$$

Reinserting the result in the energy yields finally for $W_{\rm mp}^{\rm hom}(\nabla v, \operatorname{axl} \overline{A})$

$$W_{\rm mp}^{\rm hom}(\nabla v, \theta) := \mu \,\|\, \operatorname{sym}\left(\nabla_{\eta_1,\eta_2}(v_1, v_2) - \begin{pmatrix} 0 & -\theta_3 \\ \theta_3 & 0 \end{pmatrix}\right) \|^2 + 2\mu\,\frac{\mu_c}{\mu+\mu_c}\,\|\nabla_{\eta_1,\eta_2} v_3 - \begin{pmatrix} -\theta_2 \\ \theta_1 \end{pmatrix}\|^2$$
$$+ \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}\left(\nabla_{\eta_1,\eta_2}(v_1, v_2) - \begin{pmatrix} 0 & -\theta_3 \\ \theta_3 & 0 \end{pmatrix}\right)\right]^2.$$

Note that θ_3 cancels (left for clarity to show the coupling). Consider next

$$\nabla_\eta^{h_j} u_{h_j}^\sharp - \overline{A}_{h_j}^\flat = \begin{pmatrix} \partial_{\eta_1} u_1^\sharp(\eta) & \partial_{\eta_2} u_1^\sharp(\eta) & \frac{1}{h}\partial_{\eta_3} u_1^\sharp(\eta) \\ \partial_{\eta_1} u_2^\sharp(\eta) & \partial_{\eta_2} u_2^\sharp(\eta) & \frac{1}{h}\partial_{\eta_3} u_2^\sharp(\eta) \\ \partial_{\eta_1} u_3^\sharp(\eta) & \partial_{\eta_2} u_3^\sharp(\eta) & \frac{1}{h}\partial_{\eta_3} u_3^\sharp(\eta) \end{pmatrix} - \begin{pmatrix} 0 & -\frac{1}{h_j}\theta_{h_j,3}^\flat & \theta_{h_j,2}^\flat \\ \frac{1}{h_j}\theta_{h_j,3}^\flat & 0 & -\theta_{h_j,1}^\flat \\ -\theta_{h_j,2}^\flat & \theta_{h_j,1}^\flat & 0 \end{pmatrix},$$

where $\theta_{h_j}^\flat := \operatorname{axl} \overline{A}_{h_j}^\flat$. Along the sequence $(u_{h_j}^\sharp, \overline{A}_{h_j}^\flat)$ we have by construction,

$$W_{\rm mp}(\nabla_\eta^{h_j} u_{h_j}^\sharp - \overline{A}_{h_j}^\flat) \geq W_{\rm mp}^{\rm hom}(\nabla u_{h_j}^\sharp, \operatorname{axl} \overline{A}_{h_j}^\sharp). \tag{6.15}$$

Hence, integrating and taking the lim inf also

$$\liminf_{h_j} \int_{\Omega_1} W_{\rm mp}(\nabla_\eta^{h_j} u_{h_j}^\sharp - \overline{A}_{h_j}^\flat)\,{\rm dV}_\eta \geq \liminf_{h_j} \int_{\Omega_1} W_{\rm mp}^{\rm hom}(\nabla u_{h_j}^\sharp, \operatorname{axl} \overline{A}_{h_j}^\flat)\,{\rm dV}_\eta. \tag{6.16}$$

Now we use weak convergence of $(u_{h_j}^\sharp, \overline{A}_{h_j}^\flat) \rightharpoonup (u_0^\sharp, \overline{A}_0^\flat)$, together with the convexity w.r.t. $(\nabla v, \operatorname{axl} \overline{A})$ of $\int_{\Omega_1} W_{\rm mp}^{\rm hom}(\nabla v, \operatorname{axl} \overline{A})\,{\rm dV}_\eta$ to get lower semi-continuity of the right hand side in (6.16) and to obtain altogether

$$\liminf_{h_j} \int_{\Omega_1} W_{\rm mp}(\nabla_\eta^{h_j} u_{h_j}^\sharp - \overline{A}_{h_j}^\flat)\,{\rm dV}_\eta \geq \int_{\Omega_1} W_{\rm mp}^{\rm hom}(\nabla u_0^\sharp, \operatorname{axl} \overline{A}_0^\flat)\,{\rm dV}_\eta. \tag{6.17}$$

Consider next the curvature energy along the sequence $\theta_{h_j}^\flat(\eta) = \operatorname{axl} \overline{A}_{h_j}^\flat(\eta)$ with

$$W_{\rm curv}(\widehat{\nabla}_\eta^h \theta^\flat(\eta)) = W_{\rm curv}\left(\begin{pmatrix} \partial_{\eta_1}\theta_1^\flat(\eta) & \partial_{\eta_2}\theta_1^\flat(\eta) & \frac{1}{h}\partial_{\eta_3}\theta_1^\flat(\eta) \\ \partial_{\eta_1}\theta_2^\flat(\eta) & \partial_{\eta_2}\theta_2^\flat(\eta) & \frac{1}{h}\partial_{\eta_3}\theta_2^\flat(\eta) \\ \frac{1}{h}\partial_{\eta_1}\theta_3^\flat(\eta) & \frac{1}{h}\partial_{\eta_2}\theta_3^\flat(\eta) & \frac{1}{h^2}\partial_{\eta_3}\theta_3^\flat(\eta) \end{pmatrix}\right). \tag{6.18}$$

This motivates to get a trivial lower bound by defining

$$W_{\rm curv}^{\rm hom}(\nabla\theta) := \inf_{\tilde{p}_2,\tilde{p}_3,\tilde{p}_4,\tilde{p}_5,\tilde{p}_6\in\mathbb{R}} W_{\rm curv}\left(\begin{pmatrix} \partial_{\eta_1}\theta_1 & \partial_{\eta_2}\theta_1 & \tilde{p}_2 \\ \partial_{\eta_1}\theta_2 & \partial_{\eta_2}\theta_2 & \tilde{p}_3 \\ \tilde{p}_6 & \tilde{p}_5 & \tilde{p}_4 \end{pmatrix}\right). \tag{6.19}$$

The infimizing values are obtained as

$$\tilde{p}_2^* = 0\,,\ \tilde{p}_3^* = 0\,, \quad \tilde{p}_4^* = -\frac{\alpha_3}{2\alpha_1+\alpha_3}\,(\partial_{\eta_1}\theta_1 + \partial_{\eta_2}\theta_2)\,, \quad \tilde{p}_5^* = 0\,,\ \tilde{p}_6^* = 0\,, \tag{6.20}$$

such that the homogenized reduced curvature density is given by

$$W_{\rm curv}^{\rm hom}(\nabla\theta) := \mu\frac{\widehat{L}_c^2}{2}\left(\alpha_1\,\|\,{\rm sym}\,\nabla_{\eta_1,\eta_2}(\theta_1,\theta_2)\|^2 + \frac{\alpha_1\alpha_3}{2\alpha_1+\alpha_3}\,{\rm tr}\,[\nabla_{\eta_1,\eta_2}(\theta_1,\theta_2)]^2\right).$$

By construction we have along the sequence $\theta^\flat_{h_j}$

$$W_{\rm curv}(\widehat{\nabla}^h_\eta\theta^\flat_{h_j}) \geq W_{\rm curv}^{\rm hom}(\nabla\theta^\flat_{h_j}))\,. \tag{6.21}$$

Integrating the last inequality, taking the lim inf on both sides and using that $W_{\rm curv}^{\rm hom}$ is convex (quadratic) in its argument, together with weak convergence of the two in-plane components of the curvature tensor, i.e.

$$\nabla_{\eta_1,\eta_2}(\theta^\flat_{h_j,1},\theta^\flat_{h_j,2}) \rightharpoonup \nabla_{\eta_1,\eta_2}(\theta^\flat_{0,1},\theta^\flat_{0,2}) = \nabla_{\eta_1,\eta_2}\,{\rm axl}\,\overline{A}^\flat_0 \in L^2(\Omega_1,\mathfrak{gl}(3))\,, \tag{6.22}$$

(see the appendix Theorem 7.1) we obtain

$$\begin{aligned}\liminf_{h_j}\int_{\Omega_1} W_{\rm curv}(\widehat{\nabla}^h_\eta\,{\rm axl}\,\overline{A}^\flat_{h_j})\,{\rm dV}_\eta &= \liminf_{h_j}\int_{\Omega_1} W_{\rm curv}(\widehat{\nabla}^h_\eta\theta^\flat_{h_j})\,{\rm dV}_\eta \\ &\geq \liminf_{h_j}\int_{\Omega_1} W_{\rm curv}^{\rm hom}(\nabla\theta^\flat_{h_j}))\,{\rm dV}_\eta \geq \int_{\Omega_1} W_{\rm curv}^{\rm hom}(\nabla\,{\rm axl}\,\overline{A}^\flat_0)\,{\rm dV}_\eta\,.\end{aligned} \tag{6.23}$$

Then, because $W_{\rm curv}, W_{\rm mp} \geq 0$,

$$\begin{aligned}&\liminf_{h_j}\int_{\Omega_1} W_{\rm mp}(\nabla^h_\eta u^\sharp_{h_j} - \overline{A}^\flat_{h_j}) + W_{\rm curv}(\widehat{\nabla}^h_\eta\,{\rm axl}\,\overline{A}^\flat_{h_j})\,{\rm dV}_\eta \\ &\geq \liminf_{h_j}\int_{\Omega_1} W_{\rm mp}(\nabla^{h_j}_\eta u^\sharp_{h_j} - \overline{A}^\flat_{h_j})\,{\rm dV}_\eta + \liminf_{h_j}\int_{\Omega_1} W_{\rm curv}(\widehat{\nabla}^h_\eta\,{\rm axl}\,\overline{A}^\flat_{h_j})\,{\rm dV}_\eta \\ &\geq \int_{\Omega_1} W_{\rm mp}^{\rm hom}(\nabla u^\sharp_0, {\rm axl}\,\overline{A}^\flat_0)\,{\rm dV}_\eta + \int_{\Omega_1} W_{\rm curv}^{\rm hom}(\nabla\,{\rm axl}\,\overline{A}^\flat_0)\,{\rm dV}_\eta\,,\end{aligned} \tag{6.24}$$

where we used (6.17) and (6.23). Now we use that $u^\sharp_0, \overline{A}^\flat_0$ are both independent of the transverse variable η_3 to obtain altogether the desired lim inf-inequality

$$I_0^{\sharp,\flat}(u^\sharp_0,\overline{A}^\flat_0) \leq \liminf_{h_j} I_{h_j}^{\sharp,\flat}(u^\sharp_{h_j},\overline{A}^\flat_{h_j}) \tag{6.25}$$

for

$$\begin{aligned}I_0^{\sharp,\flat}(u_0,\overline{A}^\flat_0) &:= \int_{-\frac12}^{\frac12}\int_\omega W_{\rm mp}^{\rm hom}(\nabla u_0, {\rm axl}\,\overline{A}^\flat_0) + W_{\rm curv}^{\rm hom}(\nabla\,{\rm axl}\,\overline{A}^\flat_0)\,{\rm dV}_\eta \\ &= \int_\omega W_{\rm mp}^{\rm hom}(\nabla u_0,\overline{A}^\flat_0) + W_{\rm curv}^{\rm hom}(\nabla\,{\rm axl}\,\overline{A}^\flat_0)\,{\rm d}\omega\,.\end{aligned}$$

■

6.3 Global/local minimization

Because of the coupling of the fields together with the scaling of the third component of the microrotation we have to compute a more complicated minimization problem. Looking simultaneously at scaled stretch and scaled curvature we are led to

$$\begin{aligned}&\inf_{b\in\mathbb{R}^3,p\in\mathbb{R}^4}\left[W_{\rm mp}\left((\nabla v|b) - \begin{pmatrix}0 & -p_1 & \theta_2\\ p_1 & 0 & -\theta_1\\ -\theta_2 & \theta_1 & 0\end{pmatrix}\right) + W_{\rm curv}\left(\begin{pmatrix}\partial_{\eta_1}\theta_1 & \partial_{\eta_2}\theta_1 & p_2\\ \partial_{\eta_1}\theta_2 & \partial_{\eta_2}\theta_2 & p_3\\ \partial_{\eta_1}p_1 & \partial_{\eta_2}p_1 & p_4\end{pmatrix}\right)\right] \\ &\quad = W_{\rm mp}^{\rm hom}(\nabla v,\theta) + W_{\rm curv}^{\rm hom}(\nabla\theta)\,.\end{aligned} \tag{6.26}$$

The minimization problem is in principle a global PDE-problem, since $\nabla_{\eta_1,\eta_2}p_1$ appears in the curvature energy. However, (6.26) is the correct result in precisely the case where the curvature energy depends only on the symmetric part.

Let us use the precise form of the energy to see what is going on. We write

$$\left[W_{\mathrm{mp}}\left((\nabla v|b)-\begin{pmatrix}0 & -p_1 & \theta_2\\ p_1 & 0 & -\theta_1\\ -\theta_2 & \theta_1 & 0\end{pmatrix}\right)+W_{\mathrm{curv}}\left(\begin{pmatrix}\partial_{\eta_1}\theta_1 & \partial_{\eta_2}\theta_1 & p_2\\ \partial_{\eta_1}\theta_2 & \partial_{\eta_2}\theta_2 & p_3\\ \partial_{\eta_1}p_1 & \partial_{\eta_2}p_1 & p_4\end{pmatrix}\right)\right]$$

$$= \mu \,\|\,\mathrm{sym}(\nabla v|b)\|^2 + \mu_c \,\|\,\mathrm{skew}(\nabla v|b) - \begin{pmatrix}0 & -p_1 & \theta_2\\ p_1 & 0 & -\theta_1\\ -\theta_2 & \theta_1 & 0\end{pmatrix}\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[(\nabla v|b)\right]^2$$

$$+ \alpha_1 \,\|\,\mathrm{sym}\begin{pmatrix}\partial_{\eta_1}\theta_1 & \partial_{\eta_2}\theta_1 & p_2\\ \partial_{\eta_1}\theta_2 & \partial_{\eta_2}\theta_2 & p_3\\ \partial_{\eta_1}p_1 & \partial_{\eta_2}p_1 & p_4\end{pmatrix}\|^2 + \alpha_2 \,\|\,\mathrm{skew}\begin{pmatrix}\partial_{\eta_1}\theta_1 & \partial_{\eta_2}\theta_1 & p_2\\ \partial_{\eta_1}\theta_2 & \partial_{\eta_2}\theta_2 & p_3\\ \partial_{\eta_1}p_1 & \partial_{\eta_2}p_1 & p_4\end{pmatrix}\|^2 + \frac{\alpha_3}{2}\,(\partial_{\eta_1}\theta_1+\partial_{\eta_2}\theta_2+p_4)^2$$

$$= \mu \,\|\,\mathrm{sym}\begin{pmatrix}\partial_{\eta_1}v_1 & \partial_{\eta_2}v_1 & b_1\\ \partial_{\eta_1}v_2 & \partial_{\eta_2}v_2 & b_2\\ \partial_{\eta_1}v_3 & \partial_{\eta_2}v_3 & b_3\end{pmatrix}\|^2 + \mu_c \,\|\,\mathrm{skew}(\nabla v|b) - \begin{pmatrix}0 & -p_1 & \theta_2\\ p_1 & 0 & -\theta_1\\ -\theta_2 & \theta_1 & 0\end{pmatrix}\|^2$$

$$+ \frac{\lambda}{2}\,(\partial_{\eta_1}v_1+\partial_{\eta_2}v_2+b_3)^2$$

$$+ \alpha_1 \,\|\,\mathrm{sym}\begin{pmatrix}\partial_{\eta_1}\theta_1 & \partial_{\eta_2}\theta_1 & 0\\ \partial_{\eta_1}\theta_2 & \partial_{\eta_2}\theta_2 & 0\\ 0 & 0 & p_4\end{pmatrix}\|^2 + \frac{\alpha_1}{2}\left[(p_2+\partial_{\eta_1}p_1)^2+(p_3+\partial_{\eta_2}p_1)^2\right]$$

$$+ \alpha_2 \,\|\,\mathrm{skew}\begin{pmatrix}\partial_{\eta_1}\theta_1 & \partial_{\eta_2}\theta_1 & p_2\\ \partial_{\eta_1}\theta_2 & \partial_{\eta_2}\theta_2 & p_3\\ \partial_{\eta_1}p_1 & \partial_{\eta_2}p_1 & p_4\end{pmatrix}\|^2 + \frac{\alpha_3}{2}\,(\partial_{\eta_1}\theta_1+\partial_{\eta_2}\theta_2+p_4)^2$$

$$= \mu \,\|\,\mathrm{sym}\begin{pmatrix}\partial_{\eta_1}v_1 & \partial_{\eta_2}v_1 & 0\\ \partial_{\eta_1}v_2 & \partial_{\eta_2}v_2 & 0\\ 0 & 0 & b_3\end{pmatrix}\|^2 + \frac{\mu}{2}\left[(b_1+\partial_{\eta_1}v_3)^2+(b_2+\partial_{\eta_2}v_3)^2\right]$$

$$+ \mu_c \,\|\,\mathrm{skew}\begin{pmatrix}\partial_{\eta_1}v_1 & \partial_{\eta_2}v_1\\ \partial_{\eta_1}v_2 & \partial_{\eta_2}v_2\end{pmatrix} - \begin{pmatrix}0 & -p_1\\ p_1 & 0\end{pmatrix}\|^2 + \mu_c \,\|\,\mathrm{skew}\begin{pmatrix}0 & 0 & b_1\\ 0 & 0 & b_2\\ \partial_{\eta_1}v_3 & \partial_{\eta_2}v_3 & 0\end{pmatrix} - \begin{pmatrix}0 & 0 & \theta_2\\ 0 & 0 & -\theta_1\\ -\theta_2 & \theta_1 & 0\end{pmatrix}\|^2$$

$$+ \frac{\lambda}{2}\,(\partial_{\eta_1}v_1+\partial_{\eta_2}v_2+b_3)^2$$

$$+ \alpha_1 \,\|\,\mathrm{sym}\begin{pmatrix}\partial_{\eta_1}\theta_1 & \partial_{\eta_2}\theta_1 & 0\\ \partial_{\eta_1}\theta_2 & \partial_{\eta_2}\theta_2 & 0\\ 0 & 0 & p_4\end{pmatrix}\|^2 + \frac{\alpha_1}{2}\left[(p_2+\partial_{\eta_1}p_1)^2+(p_3+\partial_{\eta_2}p_1)^2\right]$$

$$+ \alpha_2 \,\|\,\mathrm{skew}\begin{pmatrix}\partial_{\eta_1}\theta_1 & \partial_{\eta_2}\theta_1\\ \partial_{\eta_1}\theta_2 & \partial_{\eta_2}\theta_2\end{pmatrix}\|^2 + \frac{\alpha_2}{2}\left[(p_2-\partial_{\eta_1}p_1)^2+(p_3-\partial_{\eta_2}p_1)^2\right]$$

$$+ \frac{\alpha_3}{2}\,(\partial_{\eta_1}\theta_1+\partial_{\eta_2}\theta_2+p_4)^2\,. \tag{6.27}$$

Grouping the different expressions together we see that for the symmetric case $\alpha_2 = 0$ the vector $(b^*, p^*) \in \mathbb{R}^7$, which realizes the infimum, can be explicitly determined. The calculation is lengthy but otherwise straight forward. We obtain

$$\begin{pmatrix}b_1^*\\ b_2^*\\ b_3^*\\ p_1^*\\ p_2^*\\ p_3^*\\ p_4^*\end{pmatrix} = \begin{pmatrix}\frac{\mu_c-\mu}{\mu+\mu_c}\,\partial_{\eta_1}v_3 + \frac{2\mu_c}{\mu+\mu_c}\theta_2\\ \frac{\mu_c-\mu}{\mu+\mu_c}\,\partial_{\eta_2}v_3 - \frac{2\mu_c}{\mu+\mu_c}\theta_1\\ -\frac{\lambda}{2\mu+\lambda}\,(\partial_{\eta_1}v_1+\partial_{\eta_2}v_2)\\ \frac{\partial_{\eta_1}v_2-\partial_{\eta_2}v_1}{2}\\ -\partial_{\eta_1}p_1^*\\ -\partial_{\eta_2}p_1^*\\ -\frac{\alpha_3}{2\alpha_1+\alpha_3}\,(\partial_{\eta_1}\theta_1+\partial_{\eta_2}\theta_2)\end{pmatrix}. \tag{6.28}$$

Reinserting the result in the energy yields the claim in (6.26). The importance of this calculation (while not changing the lower bound trivial limit energy), rests with the determination of the minimizing values (6.28) which are needed in the following reconstruction procedure.[7]

[7] If the curvature energy depends also on the non-symmetric part of the curvature tensor, i.e., if $\alpha_2 > 0$, then the minimization step is truly global and no simple solution can be provided. Moreover, the resulting limit energy would depend on imposed boundary conditions for θ. But any useful effective two-dimensional model should be boundary condition independent! Thus we get a strong motivation to use only curvature energies depending only on the symmetric part of the curvature tensor in the Cosserat bulk model.

6.4 Upper bound - the recovery sequence

Now we show that the lower bound will actually be reached. A sufficient requirement for the recovery sequence is that

$$\begin{aligned}
&\forall\,(u_0,\overrightarrow{A}_0^\flat)\in X=L^2(\Omega_1,\mathbb{R}^3)\times L^2(\Omega_1,\mathfrak{so}(3))\\
&\quad\exists\, u^\sharp_{h_j}\to u_0\quad\text{in } L^2(\Omega_1,\mathbb{R}^3)\,,\quad \overrightarrow{A}^\flat_{h_j}\to\overrightarrow{A}^\flat_0\quad\text{in } L^2(\Omega_1,\mathfrak{so}(3)):\\
&\qquad \limsup I^{\sharp,\flat}_{h_j}(u^\sharp_{h_j},\overrightarrow{A}^\flat_{h_j})\le I^{\sharp,\flat}_0(u_0,\overrightarrow{A}^\flat_0)\,.
\end{aligned}\tag{6.29}$$

Observe that this is now only a condition if $I^{\sharp,\flat}_0(u_0,\overrightarrow{A}^\flat_0)<\infty$. In this case the uniform coercivity of $I^{\sharp,\flat}_{h_j}(u^\sharp_{h_j},\overrightarrow{A}^\flat_{h_j})$ over $X'=H^{1,2}(\Omega_1,\mathbb{R}^3)\times H^{1,2}(\Omega_1,\mathfrak{so}(3))$ implies that we can restrict attention to sequences $(u^\sharp_{h_j},\overrightarrow{A}^\flat_{h_j})$ converging weakly to some $(u_0,\overrightarrow{A}^\flat_0)\in H^{1,2}(\omega,\mathbb{R}^3)\times H^{1,2}(\omega,\mathfrak{so}(3))=X'_\omega$, defined over the two-dimensional domain ω only. Note, however, that finally it is strong convergence in X which matters.

Since

$$u^\sharp_{h_j}(\eta_1,\eta_2,\eta_3)=u^\sharp_{h_j}(\eta_1,\eta_2,0)+\partial_{\eta_3}u^\sharp_{h_j}(\eta_1,\eta_2,0)\,\eta_3+\dots\tag{6.30}$$

and $b^*(\eta_1,\eta_2)$ replaces the term $\frac{1}{h_j}\partial_{\eta_3}u^\sharp_{h_j}(\eta_1,\eta_2,\eta_3)$ the natural candidate for the recovery sequence for the bulk displacement is given by the "reconstruction"

$$\begin{aligned}
u^\sharp_{h_j}(\eta_1,\eta_2,\eta_3)&:=u_0(\eta_1,\eta_2)+h_j\,\eta_3\,b^*(\eta_1,\eta_2)=u_0(\eta_1,\eta_2)+h_j\,\eta_3\,b^*(\eta_1,\eta_2)\\
&=u_0(\eta_1,\eta_2)+h_j\,\eta_3\begin{pmatrix}\frac{(\mu_c-\mu)\,\partial_{\eta_1}u_{0,3}+2\mu_c\,\theta^\flat_{0,2}}{\mu+\mu_c}\\ \frac{(\mu_c-\mu)\,\partial_{\eta_2}u_{0,3}-2\mu_c\,\theta^\flat_{0,1}}{\mu+\mu_c}\\ \frac{-\lambda\,(\partial_{\eta_1}u_{0,1}+\partial_{\eta_2}u_{0,2})}{2\mu+\lambda}\end{pmatrix},
\end{aligned}\tag{6.31}$$

where we have used the definition of b^* given in (6.28). Observe that $b^*\in L^2(\omega,\mathbb{R}^3)$. Convergence of $u^\sharp_{h_j}$ in $L^2(\Omega_1,\mathbb{R}^3)$ to the limit u_0 as $h_j\to 0$ is obvious.

The reconstruction for the infinitesimal rotation $\overrightarrow{A}^\flat_0$ is only slightly more complicated. In terms of the axial representation we write

$$\begin{aligned}
\theta^\flat_{h_j}(\eta_1,\eta_2,\eta_3)&=\begin{pmatrix}\theta^\flat_{0,1}(\eta_1,\eta_2)\\ \theta^\flat_{0,2}(\eta_1,\eta_2)\\ h\,p^*_1(\eta_1,\eta_2)\end{pmatrix}+\begin{pmatrix}h\,\eta_3\,p^*_2(\eta_1,\eta_2)\\ h\,\eta_3\,p^*_3(\eta_1,\eta_2)\\ h^2\,\eta_3\,p^*_4(\eta_1,\eta_2)\end{pmatrix}=\begin{pmatrix}\theta^\flat_{0,1}(\eta_1,\eta_2)\\ \theta^\flat_{0,2}(\eta_1,\eta_2)\\ h\,p^*_1(\eta_1,\eta_2)\end{pmatrix}+\begin{pmatrix}-h\,\eta_3\,\partial_{\eta_1}p^*_1(\eta_1,\eta_2)\\ -h\,\eta_3\,\partial_{\eta_2}p^*_1(\eta_1,\eta_2)\\ h^2\,\eta_3\,p^*_4(\eta_1,\eta_2)\end{pmatrix}\\
&=\begin{pmatrix}\theta^\flat_{0,1}(\eta_1,\eta_2)\\ \theta^\flat_{0,2}(\eta_1,\eta_2)\\ h\,\frac{\partial_{\eta_1}u_{0,2}-\partial_{\eta_2}u_{0,1}}{2}\end{pmatrix}+\begin{pmatrix}-h\,\eta_3\,\partial_{\eta_1}\frac{\partial_{\eta_1}u_{0,2}-\partial_{\eta_2}u_{0,1}}{2}\\ -h\,\eta_3\,\partial_{\eta_2}\frac{\partial_{\eta_1}u_{0,2}-\partial_{\eta_2}u_{0,1}}{2}\\ -h^2\,\eta_3\,\frac{\alpha_3\,(\partial_{\eta_1}\theta^\flat_{0,1}+\partial_{\eta_2}\theta^\flat_{0,2})}{2\alpha_1+\alpha_3}\end{pmatrix},
\end{aligned}\tag{6.32}$$

where we have used (6.28). Again it is clear that $\theta^\flat_{h_j}\to\theta^\flat_0\in L^2(\Omega_1,\mathbb{R}^3)$ as $h_j\to 0$. Both reconstructions are completely given in terms of the two-dimensional functions $(u_0,\theta^\flat_0)$. Since neither b^* nor p^* need be differentiable, we have to consider slightly modified recovery sequences, however. With fixed $\varepsilon>0$ choose $b_\varepsilon\in H^{1,2}(\omega,\mathbb{R}^3)$ such that $\|b_\varepsilon-b^*\|_{L^2(\omega,\mathbb{R}^3)}<\varepsilon$ and similarly for p^* choose $p^*_\varepsilon\in H^{2,2}(\omega,\mathbb{R}^4)$ such that $\|p^*_\varepsilon-p^*\|_{L^2(\omega,\mathbb{R}^4)}<\varepsilon$. This allows us to present finally our **recovery sequence**

$$\begin{aligned}
u^\sharp_{h_j,\varepsilon}(\eta_1,\eta_2,\eta_3)&:=u_0(\eta_1,\eta_2)+h_j\,\eta_3\,b_\varepsilon(\eta_1,\eta_2)\,,\\
\theta^\flat_{h_j,\varepsilon}(\eta_1,\eta_2,\eta_3)&:=\begin{pmatrix}\theta^\flat_{0,1}(\eta_1,\eta_2)\\ \theta^\flat_{0,2}(\eta_1,\eta_2)\\ h_j\,p^*_{1,\varepsilon}(\eta_1,\eta_2)\end{pmatrix}+\begin{pmatrix}-h\,\eta_3\,\partial_{\eta_1}p^*_{1,\varepsilon}(\eta_1,\eta_2)\\ -h\,\eta_3\,\partial_{\eta_2}p^*_{1,\varepsilon}(\eta_1,\eta_2)\\ h^2\,\eta_3\,p^*_{4,\varepsilon}(\eta_1,\eta_2)\end{pmatrix}.
\end{aligned}\tag{6.33}$$

and thus

$$\overline{A}^{\flat}_{h_j,\varepsilon} := \begin{pmatrix} 0 & -h_j\, p^*_{1,\varepsilon}(\eta_1,\eta_2) & \theta^\flat_{0,2}(\eta_1,\eta_2) \\ h_j\, p^*_{1,\varepsilon}(\eta_1,\eta_2) & 0 & -\theta^\flat_{0,1}(\eta_1,\eta_2) \\ -\theta^\flat_{0,2}(\eta_1,\eta_2) & \theta^\flat_{0,1}(\eta_1,\eta_2) & 0 \end{pmatrix} + \begin{pmatrix} 0 & -h_j^2\, \eta_3\, p^*_{4,\varepsilon}(\eta_1,\eta_2) & -h_j\, \eta_3 \partial_{\eta_2}\, p^*_{1,\varepsilon}(\eta_1,\eta_2) \\ h_j^2\, \eta_3\, p^*_{4,\varepsilon}(\eta_1,\eta_2) & 0 & h_j\, \eta_3 \partial_{\eta_1}\, p^*_{1,\varepsilon}(\eta_1,\eta_2) \\ h_j\, \eta_3 \partial_{\eta_2}\, p^*_{1,\varepsilon}(\eta_1,\eta_2) & -h_j\, \eta_3 \partial_{\eta_1}\, p^*_{1,\varepsilon}(\eta_1,\eta_2) & 0 \end{pmatrix},$$

$$\overline{A}^{\flat,\flat}_{h_j,\varepsilon} := \begin{pmatrix} 0 & -\, p^*_{1,\varepsilon}(\eta_1,\eta_2) & \theta^\flat_{0,2}(\eta_1,\eta_2) \\ p^*_{1,\varepsilon}(\eta_1,\eta_2) & 0 & -\theta^\flat_{0,1}(\eta_1,\eta_2) \\ -\theta^\flat_{0,2}(\eta_1,\eta_2) & \theta^\flat_{0,1}(\eta_1,\eta_2) & 0 \end{pmatrix} + \begin{pmatrix} 0 & -h_j\, \eta_3\, p^*_{4,\varepsilon}(\eta_1,\eta_2) & -h_j\, \eta_3 \partial_{\eta_2}\, p^*_{1,\varepsilon}(\eta_1,\eta_2) \\ h_j\, \eta_3\, p^*_{4,\varepsilon}(\eta_1,\eta_2) & 0 & h_j\, \eta_3 \partial_{\eta_1}\, p^*_{1,\varepsilon}(\eta_1,\eta_2) \\ h_j\, \eta_3 \partial_{\eta_2}\, p^*_{1,\varepsilon}(\eta_1,\eta_2) & -h_j\, \eta_3 \partial_{\eta_1}\, p^*_{1,\varepsilon}(\eta_1,\eta_2) & 0 \end{pmatrix},$$

$$\overline{A}^{\flat,\flat}_{0,\varepsilon} := \begin{pmatrix} 0 & -p^*_{1,\varepsilon}(\eta_1,\eta_2) & \theta^\flat_{0,2}(\eta_1,\eta_2) \\ p^*_{1,\varepsilon}(\eta_1,\eta_2) & 0 & -\theta^\flat_{0,1}(\eta_1,\eta_2) \\ -\theta^\flat_{0,2}(\eta_1,\eta_2) & \theta^\flat_{0,1}(\eta_1,\eta_2) & 0 \end{pmatrix},$$

$$\overline{A}^{\flat,\flat}_{0} := \begin{pmatrix} 0 & -p^*_{1}(\eta_1,\eta_2) & \theta^\flat_{0,2}(\eta_1,\eta_2) \\ p^*_{1}(\eta_1,\eta_2) & 0 & -\theta^\flat_{0,1}(\eta_1,\eta_2) \\ -\theta^\flat_{0,2}(\eta_1,\eta_2) & \theta^\flat_{0,1}(\eta_1,\eta_2) & 0 \end{pmatrix},$$

$$\overline{A}^{\flat}_{0} := \begin{pmatrix} 0 & 0 & \theta^\flat_{0,2}(\eta_1,\eta_2) \\ 0 & 0 & -\theta^\flat_{0,1}(\eta_1,\eta_2) \\ -\theta^\flat_{0,2}(\eta_1,\eta_2) & \theta^\flat_{0,1}(\eta_1,\eta_2) & 0 \end{pmatrix} = \mathrm{anti}(\theta^\flat_0)\,. \tag{6.34}$$

The definition (6.33) implies

$$\nabla u^\sharp_{h_j,\varepsilon}(\eta_1,\eta_2,\eta_3) = (\nabla u_0(\eta_1,\eta_2)|h_j\, b_\varepsilon(\eta_1,\eta_2)) + h_j\, \eta_3\, (\nabla b_\varepsilon(\eta_1,\eta_2)|0)\,, \tag{6.35}$$

$$\nabla \theta^\flat_{h_j,\varepsilon}(\eta_1,\eta_2,\eta_3) = \begin{pmatrix} \partial_{\eta_1}\theta^\flat_{0,1}(\eta_1,\eta_2) & \partial_{\eta_2}\theta^\flat_{0,1}(\eta_1,\eta_2) & 0 \\ \partial_{\eta_1}\theta^\flat_{0,2}(\eta_1,\eta_2) & \partial_{\eta_2}\theta^\flat_{0,2}(\eta_1,\eta_2) & 0 \\ h_j\partial_{\eta_1}p^*_{1,\varepsilon}(\eta_1,\eta_2) & h_j\partial_{\eta_2}p^*_{1,\varepsilon}(\eta_1,\eta_2) & 0 \end{pmatrix} + \begin{pmatrix} -h\, \eta_3\partial_{\eta_1}\partial_{\eta_1}p^*_{1,\varepsilon}(\eta_1,\eta_2) & -h\, \eta_3\partial_{\eta_2}\partial_{\eta_1}p^*_{1,\varepsilon}(\eta_1,\eta_2) & -h\, \partial_{\eta_1}p^*_{1,\varepsilon}(\eta_1,\eta_2) \\ -h\, \eta_3\partial_{\eta_1}\partial_{\eta_2}p^*_{1,\varepsilon}(\eta_1,\eta_2) & -h\, \eta_3\partial_{\eta_2}\partial_{\eta_2}p^*_{1,\varepsilon}(\eta_1,\eta_2) & -h\, \partial_{\eta_2}p^*_{1,\varepsilon}(\eta_1,\eta_2) \\ h_j^2\, \eta_3\, \partial_{\eta_1}p^*_{4,\varepsilon}(\eta_1,\eta_2) & h_j^2\, \eta_3\, \partial_{\eta_2}p^*_{4,\varepsilon}(\eta_1,\eta_2) & h_j^2\, p^*_{4,\varepsilon}(\eta_1,\eta_2) \end{pmatrix}.$$

Note that by appropriately choosing $h_j, \varepsilon > 0$ we can arrange that strong convergence of (6.35) to the correct limit still obtains. Now abbreviate further

$$\tilde{\overline{E}}^{\varepsilon}_{h_j} := [(\nabla u_0(\eta_1,\eta_2)|b_\varepsilon(\eta_1,\eta_2)) + h_j\, \eta_3\, (\nabla b_\varepsilon(\eta_1,\eta_2)|0)] - \overline{A}^{\flat,\flat}_{h_j,\varepsilon} \in \mathfrak{gl}(3)\,, \tag{6.36}$$

$$\tilde{\overline{E}}^{\varepsilon}_{0} := (\nabla u_0(\eta_1,\eta_2)|b_\varepsilon(\eta_1,\eta_2)) - \overline{A}^{\flat,\flat}_{0,\varepsilon} \in \mathfrak{gl}(3)\,,$$

$$\tilde{\overline{E}} := (\nabla u_0(\eta_1,\eta_2)|b^*(\eta_1,\eta_2)) - \overline{A}^{\flat,\flat}_{0} \in \mathfrak{gl}(3)\,,$$

$$\tilde{\mathfrak{K}}^\flat_{h_j,\varepsilon} := \begin{pmatrix} \partial_{\eta_1}\theta^\flat_{0,1}(\eta_1,\eta_2) & \partial_{\eta_2}\theta^\flat_{0,1}(\eta_1,\eta_2) & 0 \\ \partial_{\eta_1}\theta^\flat_{0,2}(\eta_1,\eta_2) & \partial_{\eta_2}\theta^\flat_{0,2}(\eta_1,\eta_2) & 0 \\ \partial_{\eta_1}p^*_{1,\varepsilon}(\eta_1,\eta_2) & \partial_{\eta_2}p^*_{1,\varepsilon}(\eta_1,\eta_2) & 0 \end{pmatrix} + \begin{pmatrix} -h\, \eta_3\partial_{\eta_1}\partial_{\eta_1}p^*_{1,\varepsilon}(\eta_1,\eta_2) & -h\, \eta_3\partial_{\eta_2}\partial_{\eta_1}p^*_{1,\varepsilon}(\eta_1,\eta_2) & -\partial_{\eta_1}p^*_{1,\varepsilon}(\eta_1,\eta_2) \\ -h\, \eta_3\partial_{\eta_1}\partial_{\eta_2}p^*_{1,\varepsilon}(\eta_1,\eta_2) & -h\, \eta_3\partial_{\eta_2}\partial_{\eta_2}p^*_{1,\varepsilon}(\eta_1,\eta_2) & -\partial_{\eta_2}p^*_{1,\varepsilon}(\eta_1,\eta_2) \\ h_j\, \eta_3\, \partial_{\eta_1}p^*_{4,\varepsilon}(\eta_1,\eta_2) & h_j\, \eta_3\, \partial_{\eta_2}p^*_{4,\varepsilon}(\eta_1,\eta_2) & p^*_{4,\varepsilon}(\eta_1,\eta_2) \end{pmatrix},$$

$$\tilde{\mathfrak{K}}^\flat_{0,\varepsilon} := \begin{pmatrix} \partial_{\eta_1}\theta^\flat_{0,1}(\eta_1,\eta_2) & \partial_{\eta_2}\theta^\flat_{0,1}(\eta_1,\eta_2) & -\partial_{\eta_1}p^*_{1,\varepsilon}(\eta_1,\eta_2) \\ \partial_{\eta_1}\theta^\flat_{0,2}(\eta_1,\eta_2) & \partial_{\eta_2}\theta^\flat_{0,2}(\eta_1,\eta_2) & -\partial_{\eta_2}p^*_{1,\varepsilon}(\eta_1,\eta_2) \\ \partial_{\eta_1}p^*_{1,\varepsilon}(\eta_1,\eta_2) & \partial_{\eta_2}p^*_{1,\varepsilon}(\eta_1,\eta_2) & p^*_{4,\varepsilon}(\eta_1,\eta_2) \end{pmatrix},$$

$$\tilde{\mathfrak{K}}^\flat_{0} := \begin{pmatrix} \partial_{\eta_1}\theta^\flat_{0,1}(\eta_1,\eta_2) & \partial_{\eta_2}\theta^\flat_{0,1}(\eta_1,\eta_2) & -\partial_{\eta_1}p^*_{1}(\eta_1,\eta_2) \\ \partial_{\eta_1}\theta^\flat_{0,2}(\eta_1,\eta_2) & \partial_{\eta_2}\theta^\flat_{0,2}(\eta_1,\eta_2) & -\partial_{\eta_2}p^*_{1}(\eta_1,\eta_2) \\ \partial_{\eta_1}p^*_{1}(\eta_1,\eta_2) & \partial_{\eta_2}p^*_{1}(\eta_1,\eta_2) & p^*_{4}(\eta_1,\eta_2) \end{pmatrix},$$

$$\mathfrak{K}^\flat_{0} := \begin{pmatrix} \partial_{\eta_1}\theta^\flat_{0,1}(\eta_1,\eta_2) & \partial_{\eta_2}\theta^\flat_{0,1}(\eta_1,\eta_2) & 0 \\ \partial_{\eta_1}\theta^\flat_{0,2}(\eta_1,\eta_2) & \partial_{\eta_2}\theta^\flat_{0,2}(\eta_1,\eta_2) & 0 \\ 0 & 0 & 0 \end{pmatrix} = \nabla\theta^\flat_0 \in \mathfrak{gl}(3)\,.$$

We note that

$$
\begin{aligned}
&\|\tilde{\mathfrak{K}}^{\flat}_{h_j,\varepsilon}-\tilde{\mathfrak{K}}^{\flat}_{0,\varepsilon}\|_{L^2(\Omega_1,\mathbb{M}^{3\times3})}\to 0 \quad \text{if} \quad h_j\to 0\,,\\
&\|\tilde{\mathfrak{K}}^{\flat}_{0,\varepsilon}-\tilde{\mathfrak{K}}^{\flat}_{0}\|_{L^2(\Omega_1,\mathbb{M}^{3\times3})}\to 0 \quad \text{if} \quad \varepsilon\to 0\,,\\
&\|\tilde{\overline{E}}^{\varepsilon}_{h_j}-\tilde{\overline{E}}^{\varepsilon}_{0}\|_{L^2(\Omega_1,\mathbb{M}^{3\times3})}\to 0 \quad \text{if} \quad h_j\to 0\,,\\
&\|\tilde{\overline{E}}^{\varepsilon}_{0}-\tilde{\overline{E}}\|_{L^2(\Omega_1,\mathbb{M}^{3\times3})}\to 0 \quad \text{if} \quad \varepsilon\to 0\,.
\end{aligned}
\tag{6.37}
$$

The abbreviations in (6.36) imply

$$
I^{\sharp,\flat}_{h_j}(u^{\sharp}_{h_j,\varepsilon},\overline{A}^{\flat}_{h_j,\varepsilon})=\int_{\Omega_1} W_{\mathrm{mp}}(\tilde{\overline{E}}^{\varepsilon}_{h_j})+W_{\mathrm{curv}}(\tilde{\mathfrak{K}}^{\flat}_{h_j,\varepsilon})\,\mathrm{dV}_\eta\,, \tag{6.38}
$$

where we used that $h_j\cdot b_\varepsilon$ in the definition of the recovery deformation gradient $(6.35)_1$ is cancelled by the factor $\frac{1}{h_j}$ in the definition of $I^{\sharp,\flat}_{h_j}$, similarly for the other components. Whence, adding and subtracting $W_{\mathrm{mp}}(\tilde{\overline{E}})$

$$
\begin{aligned}
I^{\sharp,\flat}_{h_j}(u^{\sharp}_{h_j,\varepsilon},\overline{A}^{\flat}_{h_j,\varepsilon})&=\int_{\Omega_1} W_{\mathrm{mp}}(\tilde{\overline{E}})+W_{\mathrm{mp}}(\tilde{\overline{E}}^{\varepsilon}_{h_j})-W_{\mathrm{mp}}(\tilde{\overline{E}})+W_{\mathrm{curv}}(\tilde{\mathfrak{K}}^{\flat}_{h_j,\varepsilon})\,\mathrm{dV}_\eta\\
&=\int_{\Omega_1} W_{\mathrm{mp}}(\tilde{\overline{E}})+W_{\mathrm{mp}}(\tilde{\overline{E}}+\tilde{\overline{E}}^{\varepsilon}_{h_j}-\tilde{\overline{E}})-W_{\mathrm{mp}}(\tilde{\overline{E}})+W_{\mathrm{curv}}(\mathfrak{K}_{h_j})\,\mathrm{dV}_\eta
\end{aligned}
$$

since W_{mp} and W_{curv} are both positive, we get from the triangle inequality

$$
\le\int_{\Omega_1} W_{\mathrm{mp}}(\tilde{\overline{E}})+|W_{\mathrm{mp}}(\tilde{\overline{E}}+\tilde{\overline{E}}^{\varepsilon}_{h_j}-\tilde{\overline{E}})-W_{\mathrm{mp}}(\tilde{\overline{E}})|+W_{\mathrm{curv}}(\tilde{\mathfrak{K}}^{\flat}_{h_j,\varepsilon})\,\mathrm{dV}_\eta
$$

expanding the quadratic energy W_{mp} we obtain

$$
\begin{aligned}
&=\int_{\Omega_1} W_{\mathrm{mp}}(\tilde{\overline{E}})+|W_{\mathrm{mp}}(\tilde{\overline{E}})+\langle DW_{\mathrm{mp}}(\tilde{\overline{E}}),\tilde{\overline{E}}^{\varepsilon}_{h_j}-\tilde{\overline{E}}\rangle\\
&\qquad+D^2W_{\mathrm{mp}}(\tilde{\overline{E}}).(\tilde{\overline{E}}^{\varepsilon}_{h_j}-\tilde{\overline{E}},\tilde{\overline{E}}^{\varepsilon}_{h_j}-\tilde{\overline{E}})-W_{\mathrm{mp}}(\tilde{\overline{E}})|+W_{\mathrm{curv}}(\tilde{\mathfrak{K}}^{\flat}_{h_j,\varepsilon})\,\mathrm{dV}_\eta\\
&\le\int_{\Omega_1} W_{\mathrm{mp}}(\tilde{\overline{E}})+\|DW_{\mathrm{mp}}(\tilde{\overline{E}})\|\,\|\tilde{\overline{E}}^{\varepsilon}_{h_j}-\tilde{\overline{E}}\|+C\,\|\tilde{\overline{E}}^{\varepsilon}_{h_j}-\tilde{\overline{E}}\|^2+W_{\mathrm{curv}}(\tilde{\mathfrak{K}}^{\flat}_{h_j,\varepsilon})\,\mathrm{dV}_\eta
\end{aligned}
\tag{6.39}
$$

for $\|\tilde{\overline{E}}^{\varepsilon}_{h_j}-\tilde{\overline{E}}\|\le 1$ we have

$$
\le\int_{\Omega_1} W_{\mathrm{mp}}(\tilde{\overline{E}})+\Big(C+\|DW_{\mathrm{mp}}(\tilde{\overline{E}})\|\Big)\,\|\tilde{\overline{E}}^{\varepsilon}_{h_j}-\tilde{\overline{E}}\|+W_{\mathrm{curv}}(\tilde{\mathfrak{K}}^{\flat}_{h_j,\varepsilon})\,\mathrm{dV}_\eta
$$

since $\|DW_{\mathrm{mp}}(\tilde{\overline{E}})\|\le C_2\,\|\tilde{\overline{E}}\|$ we obtain

$$
\le\int_{\Omega_1} W_{\mathrm{mp}}(\tilde{\overline{E}})+\Big(C+C_2\,\|\tilde{\overline{E}}\|\Big)\,\|\tilde{\overline{E}}^{\varepsilon}_{h_j}-\tilde{\overline{E}}\|+W_{\mathrm{curv}}(\tilde{\mathfrak{K}}^{\flat}_{h_j,\varepsilon})\,\mathrm{dV}_\eta
$$

and by Hölder's inequality we get

$$
\le\int_{\Omega_1} W_{\mathrm{mp}}(\tilde{\overline{E}})+W_{\mathrm{curv}}(\tilde{\mathfrak{K}}^{\flat}_{h_j,\varepsilon})\,\mathrm{dV}_\eta+\Big(C+C_2\,\|\tilde{\overline{E}}\|_{L^2(\Omega_1)}\Big)\,\|\tilde{\overline{E}}^{\varepsilon}_{h_j}-\tilde{\overline{E}}\|_{L^2(\Omega_1)}\,.
$$

Continuing the estimate with regard to $W_{\mathrm{curv}}(\tilde{\mathfrak{K}}^{\flat}_{h_j,\varepsilon})$ and adding and subtracting $\tilde{\overline{E}}^{\varepsilon}_{0}$ we may obtain

$$
\begin{aligned}
I^{\sharp,\flat}_{h_j}(u^{\sharp}_{h_j,\varepsilon},\overline{A}^{\flat}_{h_j,\varepsilon})&\le\int_{\Omega_1} W_{\mathrm{mp}}(\tilde{\overline{E}})+W_{\mathrm{curv}}(\tilde{\mathfrak{K}}^{\flat}_{0})+W_{\mathrm{curv}}(\tilde{\mathfrak{K}}^{\flat}_{h_j,\varepsilon})\\
&\qquad-W_{\mathrm{curv}}(\tilde{\mathfrak{K}}^{\flat}_{0})\,\mathrm{dV}_\eta\\
&\quad+\Big(C+C_2\,\|\tilde{\overline{E}}\|_{L^2(\Omega_1)}\Big)\,\|\tilde{\overline{E}}^{\varepsilon}_{h_j}-\tilde{\overline{E}}^{\varepsilon}_{0}+\tilde{\overline{E}}^{\varepsilon}_{0}-\tilde{\overline{E}}\|_{L^2(\Omega_1)}\\
&\le\int_{\Omega_1} W_{\mathrm{mp}}(\tilde{\overline{E}})+W_{\mathrm{curv}}(\tilde{\mathfrak{K}}^{\flat}_{0})\,\mathrm{dV}_\eta\\
&\quad+\|W_{\mathrm{curv}}(\tilde{\mathfrak{K}}^{\flat}_{h_j,\varepsilon})-W_{\mathrm{curv}}(\tilde{\mathfrak{K}}^{\flat}_{0,\varepsilon})\|_{L^1(\Omega_1)}\\
&\quad+\|W_{\mathrm{curv}}(\tilde{\mathfrak{K}}^{\flat}_{0,\varepsilon})-W_{\mathrm{curv}}(\tilde{\mathfrak{K}}^{\flat}_{0})\|_{L^1(\Omega_1)}\\
&\qquad+\Big(C+C_2\,\|\tilde{\overline{E}}\|_{L^2(\Omega_1)}\Big)\,\Big(\|\tilde{\overline{E}}^{\varepsilon}_{h_j}-\tilde{\overline{E}}^{\varepsilon}_{0}\|_{L^2(\Omega_1)}+\|\tilde{\overline{E}}^{\varepsilon}_{0}-\tilde{\overline{E}}\|_{L^2(\Omega_1)}\Big)\,.
\end{aligned}
\tag{6.40}
$$

Now take $h_j \to 0$ to obtain by the continuity of W_{curv} (the argument is similar to (6.42)) and $(6.37)_3$

$$\begin{aligned}\limsup_{h_j\to 0} I^{\sharp,\flat}_{h_j}(u^\sharp_{h_j,\varepsilon},\overline{A}^\flat_{h_j,\varepsilon}) \le \int_{\Omega_1} & W_{\text{mp}}(\tilde{\overline{E}}) + W_{\text{curv}}(\tilde{\mathfrak{K}}^\flat_0)\,\mathrm{dV}_\eta \\ & + \|W_{\text{curv}}(\tilde{\mathfrak{K}}^\flat_{0,\varepsilon}) - W_{\text{curv}}(\tilde{\mathfrak{K}}^\flat_0)\|_{L^1(\Omega_1)} \\ & + \left(C + C_2\,\|\tilde{\overline{E}}\|_{L^2(\Omega_1)}\right)\,\|\tilde{\overline{E}}^\varepsilon_0 - \tilde{\overline{E}}\|_{L^2(\Omega_1)}\,. \end{aligned} \tag{6.41}$$

Because the curvature energy depends only on the symmetric part, we observe also

$$W_{\text{curv}}(\tilde{\mathfrak{K}}^\flat_{0,\varepsilon}) - W_{\text{curv}}(\tilde{\mathfrak{K}}^\flat_0) = \|p^*_{4,\varepsilon} - p^*_4\|^2\,. \tag{6.42}$$

Since

$$\begin{aligned}\|\tilde{\overline{E}}^\varepsilon_0 - \tilde{\overline{E}}\|^2 &= \|(\nabla u_0(\eta_1,\eta_2)|b_\varepsilon) - (\nabla u_0(\eta_1,\eta_2)|b^*) + (\overline{A}^{\flat,\flat}_{0,\varepsilon} - \overline{A}^{\flat,\flat}_0)\|^2 \\ &\le 2\left(\|b_\varepsilon - b^*\|^2 + \|\overline{A}^{\flat,\flat}_{0,\varepsilon} - \overline{A}^{\flat,\flat}_0\|^2\right) = 2\left(\|b_\varepsilon - b^*\|^2 + 2\|p^*_{1,\varepsilon} - p^*_1\|^2\right)\,, \end{aligned} \tag{6.43}$$

we get, by letting $\varepsilon \to 0$ and using (6.42), the bound

$$\begin{aligned}\limsup_{h_j\to 0} I^{\sharp,\flat}_{h_j}(u^\sharp_{h_j,\varepsilon},\overline{A}^\flat_{h_j,\varepsilon}) &\le \int_{\Omega_1} W_{\text{mp}}(\tilde{\overline{E}}) + W_{\text{curv}}(\tilde{\mathfrak{K}}^\flat_0)\,\mathrm{dV}_\eta \\ &= \int_{\Omega_1} W^{\text{hom}}_{\text{mp}}(\nabla u_0, \overline{A}^\flat_0) + W^{\text{hom}}_{\text{curv}}(\mathfrak{K}^\flat_0)\,\mathrm{dV}_\eta\,. \end{aligned} \tag{6.44}$$

Since $u_0, \overline{A}^\flat_0$ are two-dimensional (independent of the transverse variable), we may write as well

$$\begin{aligned}\limsup_{h_j\to 0} I^{\sharp,\flat}_{h_j}(u^\sharp_{h_j,\varepsilon},\overline{A}^\flat_{h_j,\varepsilon}) &\le \int_{\Omega_1} W^{\text{hom}}_{\text{mp}}(\nabla u_0, \overline{A}^\flat_0) + W^{\text{hom}}_{\text{curv}}(\mathfrak{K}^\flat_0)\,\mathrm{dV}_\eta \\ &= \int_{\omega} W^{\text{hom}}_{\text{mp}}(\nabla u_0, \overline{A}^\flat_0) + W^{\text{hom}}_{\text{curv}}(\mathfrak{K}^\flat_0)\,\mathrm{d}\omega = I^{\sharp,\flat}_0(u_0, \overline{A}^\flat_0)\,, \end{aligned} \tag{6.45}$$

which shows the desired upper bound. This finishes the proof of Theorem 5.1. ■

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Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 second order tensors, written with capital letters and Sym denotes symmetric second orders tensors. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3\times3}$. The identity tensor on $\mathbb{M}^{3\times3}$ will be denoted by $1\!\!1$, so that $\operatorname{tr}[X] = \langle X, 1\!\!1\rangle$. We set $\operatorname{sym}(X) = \frac{1}{2}(X^T + X)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \operatorname{sym}(X) + \operatorname{skew}(X)$. For $X \in \mathbb{M}^{3\times3}$ we set for the deviatoric part $\operatorname{dev} X = X - \frac{1}{3}\operatorname{tr}[X]\, 1\!\!1 \in \mathfrak{sl}(3)$ where $\mathfrak{sl}(3)$ is the Lie-algebra of traceless matrices. The set $\operatorname{Sym}(n)$ denotes all symmetric $n \times n$-matrices. The Lie-algebra of $\mathfrak{so}(3) := \{X \in \operatorname{GL}(3)\ |X^T X = 1\!\!1,\ \det[X] = 1\}$ is given by the set $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times3}\ |\ X^T = -X\}$ of all skew symmetric tensors. The canonical identification of $\mathfrak{so}(3)$ and $\mathbb{R}^3$ is denoted by $\operatorname{axl} \overline{A} \in \mathbb{R}^3$ for $\overline{A} \in \mathfrak{so}(3)$. Note that $(\operatorname{axl} \overline{A}) \times \xi = \overline{A}.\xi$ for all $\xi \in \mathbb{R}^3$, such that

$$\operatorname{axl}\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} := \begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix}, \quad \overline{A}_{ij} = \sum_{k=1}^{3} -\varepsilon_{ijk} \cdot \operatorname{axl} \overline{A}_k ,$$

$$\|\overline{A}\|^2_{\mathbb{M}^{3\times3}} = 2 \, \| \operatorname{axl} \overline{A}\|^2_{\mathbb{R}^3} , \qquad \langle \overline{A}, \overline{B}\rangle_{\mathbb{M}^{3\times3}} = 2\langle \operatorname{axl} \overline{A}, \operatorname{axl} \overline{B}\rangle_{\mathbb{R}^3} ,$$

where ε_{ijk} is the totally antisymmetric permutation tensor. Here, $\overline{A}.\xi$ denotes the application of the matrix $\overline{A}$ to the vector ξ and $a \times b$ is the usual cross-product. Moreover, the inverse of axl is denoted by anti and defined by

$$\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} := \operatorname{anti}\begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix}, \quad \operatorname{axl}(\operatorname{skew}(a \otimes b)) = -\frac{1}{2}\, a \times b ,$$

and

$$2 \operatorname{skew}(b \otimes a) = \operatorname{anti}(a \times b) = \operatorname{anti}(\operatorname{anti}(a).b) .$$

Moreover,

$$\operatorname{curl} u = 2 \operatorname{axl}(\operatorname{skew} \nabla u) .$$

Notation for plates and shells

Let $\omega \subset \mathbb{R}^2$ always be a bounded open domain with Lipschitz boundary $\partial\omega$ and let γ_0 be a smooth subset of $\partial\omega$ with non-vanishing 1-dimensional Hausdorff measure. The aspect ratio of the plate is $h > 0$. We denote by $\mathbb{M}^{m\times n}$ the set of matrices mapping $\mathbb{R}^n \mapsto \mathbb{R}^m$. For $H \in \mathbb{M}^{3\times2}$ and $\xi \in \mathbb{R}^3$ we write $(H|\xi) \in \mathbb{M}^{3\times3}$ for the matrix composed of H and the column ξ. Likewise $(v|\xi|\eta)$ is the matrix composed of the columns v, ξ, η. This allows us to write for $u \in C^1(\mathbb{R}^3, \mathbb{R}^3)$: $\nabla u = (u_x|u_y|u_z) = (\partial_x u|\partial_y u|\partial_z u)$. The identity tensor on $\mathbb{M}^{2\times2}$ is $1\!\!1_2$. The mapping $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ is the deformation of the midsurface, ∇m is the corresponding deformation gradient and $\vec{n}_m$ is the outer unit normal on m. A matrix $X \in \mathbb{M}^{3\times3}$ can now be written as $X = (X.e_2|X.e_2|X.e_3) = (X_1|X_2|X_3)$. We write $v : \mathbb{R}^2 \mapsto \mathbb{R}^3$ for the deflection of the midsurface, such that $m(x, y) = (x, y, 0)^T + v(x, y)$. The standard volume element is $\mathrm{dx\,dy\,dz} = \mathrm{dV} = \mathrm{d\omega\,dz}$.

7 Appendix

7.1 The infinitesimal Reissner-Mindlin membrane/bending model

Abbreviating now $\theta = (\theta_1, \theta_2, 0)^T = -\overline{A}_3$, we are left with the following set of equations for the deflection of the midsurface of the plate $v : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the infinitesimal increment of the 'director', $\theta : \omega \mapsto \mathbb{R}^3$

$$\begin{aligned}
&\int_\omega h \left(\mu \, \| \operatorname{sym} \nabla(v_1, v_2)\|^2 + \underbrace{\frac{\kappa\, \mu}{2} \, \|\nabla v_3 - \theta\|^2}_{\text{transverse shear energy}} + \frac{\mu\lambda}{2\mu + \lambda} \operatorname{tr}\left[\operatorname{sym} \nabla(v_1, v_2)\right]^2 \right) \\
&\qquad + \frac{h^3}{12} \left(\mu \, \| \operatorname{sym} \nabla\theta\|^2 + \frac{\mu\lambda}{2\mu + \lambda} \operatorname{tr}\left[\operatorname{sym} \nabla\theta\right]^2 \right) - \langle f, v\rangle \, \mathrm{d}\omega \mapsto \min . \text{ w.r.t. } (v, \theta), \\
&v_{|\gamma_0} = u^{\mathrm{d}}(x, y, 0) , \qquad \text{simply supported} \\
&-\theta_{|\gamma_0} = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T , \qquad \text{rigid director prescription} .
\end{aligned} \tag{7.1}$$

Here $0 < \kappa \leq 1$ is the so called **shear correction factor**. The model is very popular and can be found, e.g., in [7, p.90].[8]

7.2 The classical infinitesimal-displacement Kirchhoff-Love plate (Koiter model)

For the convenience of the reader we also supply the similar system of equations for the classical infinitesimal-displacement Kirchhoff-Love plate (also the Koiter model). In terms of the midsurface deflection $v : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ we have to find a solution of the minimization problem

$$\begin{aligned}
&\int_\omega h \left(\mu \, \| \operatorname{sym} \nabla(v_1, v_2)\|^2 + \frac{\mu\lambda}{2\mu + \lambda} \operatorname{tr}\left[\operatorname{sym} \nabla(v_1, v_2)\right]^2 \right) \\
&\qquad + \frac{h^3}{12} \left(\mu \, \|D^2 v_3\|^2 + \frac{\mu\lambda}{2\mu + \lambda} \operatorname{tr}\left[D^2 v_3\right]^2 \right) - \langle f, v\rangle \, \mathrm{d}\omega \mapsto \min . \text{ w.r.t. } v, \\
&v_{|\gamma_0} = u^{\mathrm{d}}(x, y, 0) , \qquad \text{simply supported} \\
&-\nabla v_{3|\gamma_0} = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T , \qquad \text{typical rigid prescription of the infinitesimal normal} .
\end{aligned} \tag{7.2}$$

This energy can also be obtained formally from (7.1) by constraining the linearized director to the linearized normal of the plate, i.e., setting $\theta = \nabla v_3$.

[8] Hence the shear correction factor κ is directly determined by the Cosserat couple modulus μ_c. For rather thick plates, it is known that the shear energy in RM_{lin} is overestimated, therefore, one is led to reduce the shear energy contribution a posteriori by taking $\kappa < 1$.

7.3 Aganovic's and Neff's model based on simultaneous nonlinear scaling $u^\sharp, \overline{A}^\sharp$.

In [2] a shell model is proposed based on asymptotic analysis of the linear isotropic micropolar model, the assumption of **nonlinear scaling for displacements $u^\sharp$ and infinitesimal microrotations $\overline{A}^\sharp$** and uniform positivity assumption for the curvature together with homogeneous Dirichlet conditions on the microrotations. We specialize this model from shell to plate, rewrite its weak form into a minimization problem and adapt it to our notation. Then the problem reads: find the deflection of the midsurface $v : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the microrotation vector $\theta : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ such that

$$I_0^{\mathrm{asmp}}(v,\theta) = \int_\omega W_{\mathrm{mp}}^{\mathrm{asymp}}(\nabla v,\theta) + W_{\mathrm{curv}}^{\mathrm{asmp}}(\nabla\theta) - \langle f, v\rangle \,\mathrm{d}\omega \mapsto \min . \text{ w.r.t. } (v,\theta)\,, \tag{7.3}$$

and the boundary conditions of place for the midsurface deflection v on the Dirichlet part of the lateral boundary $\gamma_0 \subset \partial\omega$,

$$v_{|\gamma_0} = u_{\mathrm{d}}(x,y,0) \qquad \text{simply supported}$$

and the homogeneous boundary condition for the microrotation

$$\theta_{|\partial\omega} = 0\,, \qquad \text{completely clamped}\,.$$

The asymptotically reduced local density is

$$\begin{aligned} W_{\mathrm{mp}}^{\mathrm{asmp}}(\nabla v,\theta) := {} & \underbrace{\mu \,\|\operatorname{sym}\left(\nabla_{\eta_1,\eta_2}(v_1,v_2) - \begin{pmatrix} 0 & -\theta_3 \\ \theta_3 & 0\end{pmatrix}\right)\|^2}_{\text{shear-stretch energy}} + \underbrace{\mu_c \,\|\operatorname{skew}\left(\nabla_{\eta_1,\eta_2}(v_1,v_2) - \begin{pmatrix} 0 & -\theta_3 \\ \theta_3 & 0\end{pmatrix}\right)\|^2}_{\text{in-plane drill energy}} \\ & + \underbrace{2\mu\,\frac{\mu_c}{\mu+\mu_c}\,\|\nabla_{\eta_1,\eta_2} v_3 - \begin{pmatrix} -\theta_2 \\ \theta_1\end{pmatrix}\|^2}_{\text{asymptotic transverse shear energy}} + \underbrace{\frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}\left(\nabla_{\eta_1,\eta_2}(v_1,v_2) - \begin{pmatrix} 0 & -\theta_3 \\ \theta_3 & 0\end{pmatrix}\right)\right]^2}_{\text{asymptotic elongational stretch energy}}\,. \end{aligned} \tag{7.4}$$

The asymptotically correct curvature density is given by

$$\begin{aligned} W_{\mathrm{curv}}^{\mathrm{asmp}}(\nabla\theta) := \mu\frac{\widehat{L}_c^2}{2}\Bigg(& \alpha_1 \underbrace{\|\operatorname{sym}\nabla_{\eta_1,\eta_2}(\theta_1,\theta_2)\|^2}_{\text{I-energy}} + \alpha_2 \underbrace{\|\operatorname{skew}\nabla_{\eta_1,\eta_2}(\theta_1,\theta_2)\|^2}_{\text{II-energy}} \\ & + \underbrace{2\alpha_1\,\frac{\alpha_2}{\alpha_1+\alpha_2}\,\|\nabla_{\eta_1,\eta_2}\theta_3\|^2}_{\text{III-energy}} + \frac{\alpha_1\alpha_3}{2\alpha_1+\alpha_3}\underbrace{\operatorname{tr}\left[\nabla_{\eta_1,\eta_2}(\theta_1,\theta_2)\right]^2}_{\text{IV-energy}}\Bigg)\,. \end{aligned} \tag{7.5}$$

While $\alpha_2 = 0$ would give formally the Reissner-Mindlin model, the proof of asymptotic convergence in [2] needs decisively the uniform positive curvature assumption $k_c > 0, \alpha_2 > 0$. The limit model is well-posed for $k_c > 0, \alpha_2 = 0$.

The conformal curvature case is retrieved for $\alpha_1 = 1, \alpha_2 = 0$, $\frac{\alpha_3}{2} = -\frac{1}{3}$ in which case the reduced curvature turns into

$$W_{\mathrm{curv}}^{\mathrm{asmp,conf}}(\nabla\theta) := \mu\frac{\widehat{L}_c^2}{2}\,\|\operatorname{dev}_2\operatorname{sym}\nabla_{\eta_1,\eta_2}(\theta_1,\theta_2)\|^2\,. \tag{7.6}$$

This case is not $2D$-well-posed! It is straightforward to show that this asymptotic limit model coincides with the Γ-limit for simultaneous nonlinear scaling $u^\sharp, \overline{A}^\sharp$ in the strong topology of L^2 for both fields under the conditions $k_c > 0, \alpha_2 > 0$. This asymptotic limit model coincides with the linearization of the Γ-limit for nonlinear Cosserat plates in [28, 29] which was also based on the simultaneous nonlinear scaling of deformations and rotations (note that in the nonlinear regime, dealing with exact rotations, it is difficult to scale the rotations with a linear scaling). Also here, $\alpha_2 > 0$ is implicitly assumed. Thus, the presence of the in-plane drill component θ_3 cannot be avoided and therefore, this is not the Reissner-Mindlin model, for no choice of (derivation-) admissible Cosserat parameters.

7.4 A model based on linear scaling of u and nonlinear scaling of $\overline{A}$, i.e., $u^\flat, \overline{A}^\sharp$.

The Γ-limit can be established along the presented lines provided that $\alpha_2 >=$ Note that the local minimization step for linear and nonlinear scaling with respect to the displacement u yields the same homogenized energy[9] since

$$\inf_{b\in\mathbb{R}^3} W((\nabla v|b)) = \inf_{p\in\mathbb{R}^3} W\left(\begin{pmatrix} \partial_{\eta_1} v_1 & \partial_{\eta_2} v_1 & p_1 \\ \partial_{\eta_1} v_2 & \partial_{\eta_2} v_2 & p_2 \\ p_1 & p_2 & p_3 \end{pmatrix}\right) \quad \text{for } W(X) = \mu\|\operatorname{sym} X\|^2 + \frac{\lambda}{2}\operatorname{tr}[X]^2\,. \tag{7.7}$$

The model is: find the deflection of the midsurface $v : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the microrotation vector $\theta : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ such that

$$I_0^{\flat,\sharp}(v,\theta) = \int_\omega W_{\mathrm{mp}}^{\mathrm{asymp}}(\nabla v,\theta) + W_{\mathrm{curv}}^{\mathrm{asmp}}(\nabla\theta) - \langle f, v\rangle \,\mathrm{d}\omega \mapsto \min . \text{ w.r.t. } (v,\theta)\,, \tag{7.8}$$

[9]Not true for the curvature energy depending also on anti-symmetric terms for $\alpha_2 > 0$.

and the boundary conditions of place for the midsurface deflection v on the Dirichlet part of the lateral boundary $\gamma_0 \subset \partial\omega$,

$$v_{|\gamma_0} = u_{\rm d}(x,y,0) = u_{\rm d}(x,y,0)\,, \qquad \text{simply supported (fixed, welded)}\,.$$

and the homogeneous boundary condition for the microrotation

$$\theta_{|\gamma_0} = 0\,, \qquad \text{partially clamped}\,.$$

The asymptotically reduced local density is

$$W^{\rm asmp}_{\rm mp}(\nabla v,\theta) :=\mu\,\|\,\mathrm{sym}\,\nabla_{\eta_1,\eta_2}(v_1,v_2)\|^2 + \mu_c\,\underbrace{\|\,\mathrm{skew}\left(\nabla_{\eta_1,\eta_2}(v_1,v_2) - \begin{pmatrix} 0 & -\theta_3 \\ \theta_3 & 0\end{pmatrix}\right)\|^2}_{\text{in-plane drill energy}} \tag{7.9}$$
$$+ \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\,[\nabla_{\eta_1,\eta_2}(v_1,v_2)]^2\,.$$

The asymptotically correct curvature density is given by

$$W^{\rm asmp}_{\rm curv}(\nabla\theta) := \mu\frac{\widehat{L}_c^2}{2}\left(\alpha_1\,\underbrace{\|\,\mathrm{sym}\,\nabla_{\eta_1,\eta_2}(\theta_1,\theta_2)\|^2}_{\text{I-energy}} + \alpha_2\,\underbrace{\|\,\mathrm{skew}\,\nabla_{\eta_1,\eta_2}(\theta_1,\theta_2)\|^2}_{\text{II-energy}}\right. \tag{7.10}$$
$$\left. + \underbrace{2\alpha_1\,\frac{\alpha_2}{\alpha_1+\alpha_2}\,\|\nabla_{\eta_1,\eta_2}\theta_3\|^2}_{\text{III-energy}} + \frac{\alpha_1\alpha_3}{2\alpha_1+\alpha_3}\,\underbrace{\mathrm{tr}\,[\nabla_{\eta_1,\eta_2}(\theta_1,\theta_2)]^2}_{\text{IE-energy}}\right)\,.$$

The limit model already decouples the bending rotations θ_1,θ_2 from the in-plane deflections v_1,v_2. For the nonlinear scaling of the microrotations it is necessary to have $\alpha_2 > 0$ for the Γ-limit result.

7.5 A model based on linear scaling of u and linear scaling of $\overline{A}$, i.e., $u^\flat, \overline{A}^\flat$.

The problem is: find the deflection of the midsurface $v : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the microrotation vector $\theta : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ such that

$$I_0^{\flat,\flat}(v,\theta) = \int_\omega W^{\rm asymp}_{\rm mp}(\nabla v,\theta) + W^{\rm asmp}_{\rm curv}(\nabla\theta) - \langle f, v\rangle\,\mathrm{d}\omega \mapsto \min\,.\ \text{w.r.t. } (v,\theta)\,, \tag{7.11}$$

and the boundary conditions of place for the midsurface deflection v on the Dirichlet part of the lateral boundary $\gamma_0 \subset \partial\omega$,

$$v_{|\gamma_0} = u_{\rm d}(x,y,0) = u_{\rm d}(x,y,0)\,, \qquad \text{simply supported (fixed, welded)}\,.$$

and the homogeneous boundary condition for the microrotation

$$\theta_{|\gamma_0} = 0\,, \qquad \text{partially clamped}\,.$$

The asymptotically reduced local density is

$$W^{\rm asmp}_{\rm mp}(\nabla v,\theta) := \mu\,\|\,\mathrm{sym}\,\nabla_{\eta_1,\eta_2}(v_1,v_2)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\,[\mathrm{sym}\,\nabla_{\eta_1,\eta_2}(v_1,v_2)]^2\,.$$

The asymptotically correct curvature density is given by

$$W^{\rm asmp}_{\rm curv}(\nabla\theta) := \mu\frac{\widehat{L}_c^2}{2}\left(\alpha_1\,\underbrace{\|\,\mathrm{sym}\,\nabla_{\eta_1,\eta_2}(\theta_1,\theta_2)\|^2}_{\text{I-energy}} + \alpha_2\,\underbrace{\|\,\mathrm{skew}\,\nabla_{\eta_1,\eta_2}(\theta_1,\theta_2)\|^2}_{\text{II-energy}}\right. \tag{7.12}$$
$$\left. + \underbrace{2\alpha_1\,\frac{\alpha_2}{\alpha_1+\alpha_2}\,\|\nabla_{\eta_1,\eta_2}\theta_3\|^2}_{\text{III-energy}} + \frac{\alpha_1\alpha_3}{2\alpha_1+\alpha_3}\,\underbrace{\mathrm{tr}\,[\nabla_{\eta_1,\eta_2}(\theta_1,\theta_2)]^2}_{\text{IV-energy}}\right)\,.$$

If we identify $(\theta_1,\theta_2) = \nabla v_3$ we recover the linear Koiter model (7.2). Erbay writes [13, p.1513]: "An examination of these equations shows that, as in the classical plate theory, the equations governing the flexural (bending) and the extensional (stretching) motions of the plate are independent of each other. " The presented Γ-limit reproduces this decoupling.

7.6 Korn's inequality and the linear scaling for $\overline{A}^\flat$

The major merit of the linear scaling (4.2) is that it respects the infinitesimal strain structure and allows one to derive estimates independent of the scaling parameter $h > 0$ in the case where one controls only symmetrized gradients in the curvature energy. To see this, abbreviate $\theta^\flat := \operatorname{axl} \overline{A}^\flat$ and consider

$$\operatorname{sym} \widehat{\nabla}_\eta^h \theta^\flat(\eta) = \begin{pmatrix} \partial_{\eta_1}\theta_1^\flat(\eta) & \frac{1}{2}[\partial_{\eta_2}\theta_1^\flat(\eta) + \partial_{\eta_1}\theta_2^\flat(\eta)] & \frac{1}{2\,h}[\partial_{\eta_3}\theta_1^\flat(\eta) + \partial_{\eta_1}\theta_3^\flat(\eta)] \\ \frac{1}{2}[\partial_{\eta_1}\theta_2^\flat(\eta) + \partial_{\eta_2}\theta_1^\flat(\eta)] & \partial_{\eta_2}\theta_2^\flat(\eta) & \frac{1}{2\,h}[\partial_{\eta_3}\theta_2^\flat(\eta) + \partial_{\eta_2}\theta_3^\flat(\eta)] \\ \frac{1}{2\,h}[\partial_{\eta_3}\theta_1^\flat(\eta) + \partial_{\eta_1}\theta_3^\flat(\eta)] & \frac{1}{2\,h}[\partial_{\eta_3}\theta_2^\flat(\eta) + \partial_{\eta_2}\theta_3^\flat(\eta)] & \frac{1}{h^2}\partial_{\eta_3}\theta_1^\flat(\eta) \end{pmatrix},$$

$$\begin{aligned} \|\operatorname{sym} \widehat{\nabla}_\eta^h \theta^\flat(\eta)\|^2 &= \|\operatorname{sym} \begin{pmatrix} \partial_{\eta_1}\theta_1^\flat(\eta) & \partial_{\eta_2}\theta_1^\flat(\eta) & \frac{1}{h}\partial_{\eta_3}\theta_1^\flat(\eta) \\ \partial_{\eta_1}\theta_2^\flat(\eta) & \partial_{\eta_2}\theta_2^\flat(\eta) & \frac{1}{h}\partial_{\eta_3}\theta_2^\flat(\eta) \\ \frac{1}{h}\partial_{\eta_1}\theta_3^\flat(\eta) & \frac{1}{h}\partial_{\eta_2}\theta_3^\flat(\eta) & \frac{1}{h^2}\partial_{\eta_3}\theta_1^\flat(\eta) \end{pmatrix}\|^2 \\ &= \|\operatorname{sym} \begin{pmatrix} \partial_{\eta_1}\theta_1^\flat(\eta) & \partial_{\eta_2}\theta_1^\flat(\eta) \\ \partial_{\eta_1}\theta_2^\flat(\eta) & \partial_{\eta_2}\theta_2^\flat(\eta) \end{pmatrix}\|^2 + \frac{1}{h^2}\|\operatorname{sym} \begin{pmatrix} 0 & 0 & \partial_{\eta_3}\theta_1^\flat(\eta) \\ 0 & 0 & \partial_{\eta_3}\theta_2^\flat(\eta) \\ \partial_{\eta_1}\theta_3^\flat(\eta) & \partial_{\eta_2}\theta_3^\flat(\eta) & 0 \end{pmatrix}\|^2 \\ &\quad + \frac{1}{h^4}(\partial_{\eta_3}\theta_3^\flat(\eta))^2 \geq \frac{1}{h^4}\,\|\operatorname{sym} \nabla_\eta \theta^\flat(\eta)\|^2 \geq \|\operatorname{sym} \nabla_\eta \theta^\flat(\eta)\|^2 \,. \end{aligned} \tag{7.13}$$

With this preparation we show

Theorem 7.1 (Scaled Korn's inequality for the micro-rotation vector)
For $h_j \to 0$ as $j \to \infty$ consider the linearly scaled sequence $\theta^\flat_{h_j} : \Omega_1 \mapsto \mathbb{R}^3$ and assume that is satisfies either

1. *$\forall\, h_j > 0 :$* $\quad \|\theta^\flat_{h_j}\|_{L^2(\Omega_1,\mathbb{R}^3)} \leq K_1\,, \quad \|\theta^\flat_{h_j,3}\|_{L^2(\Omega_1,\mathbb{R})} \to 0 \quad$ *as* $\quad h_j \to 0$
2. $\theta^\flat_{h_j}{}_{|_{\Gamma_0^1}} = 0$.

Assume in addition the boundedness of scaled strains along h_j

$$\int_{\Omega_1} \|\operatorname{sym} \widehat{\nabla}_\eta^h \theta^\flat_{h_j}(\eta)\|^2 \,\mathrm{dV}_\eta \leq K_2\,,$$

where the constants K_1, K_2 are independent of h_j. Then

$$\|\theta^\flat_{h_j}\|_{H^{1,2}(\Omega_1,\mathbb{R}^3)} \leq K_3\,,$$

with a constant K_3 independent of h_j and there exists a weakly convergent subsequence in $H^1(\Omega, \mathbb{R}^3)$ (without re-labeling), such that

$$\begin{aligned} \theta^\flat_{h_j} &\rightharpoonup \theta^\flat_0 \in H^1(\Omega, \mathbb{R}^3)\,, \quad h_j \to 0\,, \\ \theta^\flat_{h_j} &\to \theta^\flat_0 \in L^2(\Omega_1, \mathbb{R}^3)\,, \quad h_j \to 0\,. \end{aligned}$$

In the first case we obtain moreover that $\theta^\flat_0(\eta_1, \eta_2, \eta_3) = \theta^\flat_0(\eta_1, \eta_2)$, independent of the transverse variable and for the limit of the third component $\theta^\flat_{0,3} = 0$.

In the second case (Dirichlet-boundary case) we obtain for the weak limit (only) $\theta^\flat_0(\eta_1, \eta_2, \eta_3) \in V_{KL}(\Omega_1)$.

Proof. In the first case, we may use the estimate (7.13) and Korn's second inequality without boundary conditions. In the second case for Dirichlet-boundary conditions, we may use Korn's first inequality with boundary conditions. Then the existence of a weakly convergent subsequence is clear from boundedness in $H^1(\Omega, \mathbb{R}^3)$. Rellich's compact embedding provides us with strong convergence in $L^2(\Omega, \mathbb{R}^3)$. In the second case, the weak limit satisfies the boundary condition $\theta^\flat_0{}_{|_{\Gamma_0^1}} = 0$. Boundedness of scaled strains and weak convergence of $\nabla_\eta \theta^\flat_{h_j}$ implies as well (compare with Ciarlet [8, p.37])

$$\int_{\Omega_1} \|[\operatorname{sym} \nabla_\eta \theta^\flat_{h_j}].e_3\|^2 \leq K\, h_j^2 \to 0 \quad \Rightarrow \quad [\operatorname{sym} \nabla_\eta \theta^\flat_{h_j}].e_3 \rightharpoonup [\operatorname{sym} \nabla_\eta \theta^\flat_0].e_3 = 0\,. \tag{7.14}$$

Thus the weak limit $\theta^\flat_0$ of the scaled micro-rotation vector is found in the space VKL. In the first case we know more, namely that $\langle \theta^\flat_0, e_3 \rangle = 0$ in $L^2(\Omega_1, \mathbb{R})$ which gives the result. ∎

Definition 7.2 (Space of scaled Kirchhoff-Love displacements V_{KL})
Following Ciarlet, we define the space

$$V_{KL}(\Omega_1) := \{\theta \in H^{1,2}(\Omega_1, \mathbb{R}^3) :\ \theta_{|_{\Gamma_0}} = 0\,,\ [\operatorname{sym} \nabla_\eta \theta(\eta)].e_3 = 0 \textit{ for } \eta \in \Omega_1\}\,. \tag{7.15}$$

This space is equivalently characterized by ([8, p. 41] or [15, p.561] or [11, p.12])

$$\begin{aligned} V_{KL}(\Omega_1) := \{\theta \in H^{1,2}(\Omega_1, \mathbb{R}^3) :\ &\theta_{|_{\Gamma_0^1}} = 0\,, \\ &\theta_1(\eta_1, \eta_2, \eta_3) = w_1(\eta_1, \eta_2) - \eta_3\, \partial_{\eta_1}\theta_3(\eta_1, \eta_2)\,, \\ &\theta_2(\eta_1, \eta_2, \eta_3) = w_2(\eta_1, \eta_2) - \eta_3\, \partial_{\eta_2}\theta_3(\eta_1, \eta_2)\,, \\ &\theta_3(\eta_1, \eta_2, \eta_3) = w_3(\eta_1, \eta_2)\,, \\ &w_1, w_2 \in H^1(\omega, \mathbb{R})\,,\ w_3 \in H^2(\omega, \mathbb{R})\,,\ w_1, w_2, w_3{}_{|_{\gamma_0}} = 0\,,\ \partial_\nu w_3{}_{|_{\gamma_0}} = 0\ \}\,. \end{aligned} \tag{7.16}$$

We first remark that the in-plane components θ_1, θ_2 are not necessarily two-dimensional, although they are determined by two-dimensional functions.

7.7 The Γ-limit for linear elasticity and linear scaling $u^\flat_{h_j}$

To put our result into further perspective let us relate it to classical linear elasticity. Using Theorem 7.1 for the scaled displacement $u^\flat_{h_j}$ with Dirichlet boundary conditions allows to establish the suitable bounds. The Γ-limit of $I^\flat_{h_j}(u^\flat_{h_j})$ in the strong topology of $L^2(\Omega_1,\mathbb{R}^3)$ is given by the limit energy functional $I^\flat_0 : L^2(\Omega_1,\mathbb{R}^3) \mapsto \overline{\mathbb{R}}$,

$$I^\flat_0(v) := \begin{cases} \int_{\Omega_1} W^{\text{hom}}(\nabla v) - \langle f, v\rangle \, \mathrm{dV}_\eta & v \in V_{KL}(\Omega_1) \\ +\infty & \text{else in } H^{1,2}(\Omega_1,\mathbb{R}^3)\,, \end{cases} \tag{7.17}$$

with

$$W^{\text{hom}}(\nabla v) := \mu \,\|\operatorname{sym} \nabla_{\eta_1,\eta_2}(v_1,v_2)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\nabla_{\eta_1,\eta_2}(v_1,v_2)\right]^2 .$$

This result is a slight variation of the statements in [8, p.95] and [5]. One might be tempted to think that this defines a membrane model. However, the limit is not truly two-dimensional but in the space V_{KL}. It is therefore possible to insert the limit into the integral and to perform the integration over the thickness analytically. The result is, after descaling, surprisingly, the Kirchhoff-Love membrane-bending plate (7.2) written in the deflection $\overline{v} : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and setting $\overline{v}(\eta_1,\eta_2) := \operatorname{Av}.v$ for $v \in V_{KL}(\Omega_1)$. Note again that the vertical deflection should be of the order of the thickness of the plate for this result to make sense.

7.8 An inequality for linear elasticity with nonlinear scaling $u^\sharp_{h_j}$

Assuming linear elastic behavior and simply considering the nonlinear scaling, the following inequality can be established:

Theorem 7.3 (h_j-dependent Korn's first inequality and nonlinear scaling)
For $h_j \to 0$ consider a sequence $u^\sharp_{h_j} \in \mathcal{A}'$. Then there exists a constant C independent of $h_j > 0$ such that

$$\frac{C}{h_j^2}\int_{\Omega_1} \|\operatorname{sym} \nabla^h_\eta u^\sharp_{h_j}(\eta)\|^2 \, \mathrm{dV}_\eta \geq \|u^\sharp_{h_j}(\eta)\|^2_{H^{1,2}(\Omega_1,\mathbb{R}^3)} .$$

Proof. Can be found in [4, Th.A.1], see also [9, p.176]. ■

Remark 7.4
With this (essentially sharp) inequality, it is difficult to continue the Γ-limit development in classical linear elasticity based on the nonlinear scaling without further assumptions on the scaling of energies. This is one of the reasons, why Ciarlet uses the linear scaling in the case of plates (the inequality can be improved to be independent of h_j in case of a shell with elliptic surface).

Assume, however, that the scaled energy satisfies (this is a strong assumption on the data in disguise)

$$\frac{1}{h_j^2}\, I^\sharp(u^\sharp_{h_j}) \leq C\,. \tag{7.18}$$

Then Theorem 7.3 allows to establish weak compactness of $u^\sharp_{h_j}$ in $H^{1,2}(\Omega_1,\mathbb{R}^3)$. The Γ-limit of $\frac{1}{h_j^2} I^\sharp$ in the strong topology of $L^2(\Omega_1,\mathbb{R}^3)$ is given by the limit energy functional $I^\sharp_0 : L^2(\Omega_1,\mathbb{R}^3) \mapsto \overline{\mathbb{R}}$,

$$I^\sharp_0(v) := \begin{cases} \int_{\omega} W^{\text{hom}}(\nabla v) - \langle f, v\rangle \, \mathrm{d}\omega & v \in H^{1,2}(\omega,\mathbb{R}^3) \\ +\infty & \text{else in } L^2(\Omega_1,\mathbb{R}^3)\,. \end{cases} \tag{7.19}$$

For this result compare to [4, Th.4.2]. For sequences bounded in H^1 it is easy to see that the weak limit is actually independent of η_3 and thus the limit problem is a membrane-plate.

Remark 7.5
In the finite strain setting the assumption $\frac{1}{h_j^2} I^\sharp(\varphi^\sharp_{h_j}) \leq C$ leads to the classical plate-bending problem [18].