Poroplasticity with Cosserat effects

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We prove existence of solutions to a non-monotone and non-gradient type quasi-static model of poroplasticity with Cosserat effects. It is shown that this model possesses global in time solutions, where the inelastic constitutive equation is satisfied in the sense of Young measures. The methods of proof are a monotone approximation, energy estimates, the fundamental theorem on Young measures, and a passage to the limit with the monotone approximation.

1 Introduction

Soil consolidation processes have a great importance in many geophysical applications, e.g. in the prediction of landslides. First models describing the elastic coupling between the solid matrix and the pore fluid behaviour have been obtained by Biot and Therzaghi [3,4]. In this paper we study a model from Ehlers, where in addition to poroelasticity permanent plastic effects take place [12]. The general framework is to couple the Biot poroelasticity model with a flow rule for the plastic strain. This kind of modelling is well known in approaches to metal plasticity. There, usually, the flow rule is monotone and of gradient type, e.g. in Prandtl-Reuss plasticity. Nevertheless, the Prandtl-Reuss model is ill-posed and one is led to investigate certain regularization procedures, as are additional hardening or viscosity.

In the classical metal perfect plasticity models at infinitesimal strain it has been shown that a coupling with Cosserat elasticity may also regularize the ill-posedness of the Prandtl-Reuss plasticity. Perfect plasticity, however, is characterized by a monotone flow rule of gradient type (associated plasticity). In that case, powerful methods from convex analysis can be used in the mathematical treatment.

On the contrary, in the model from Ehlers the flow rule is much more involved: it is non-monotone and not of gradient type. Therefore the powerful solution concept of an energetic solution, introduced by Mielke [21] is not applicable. The existence result for a quasi-static, non-monotone model of poroplasticity has been studied in [24]. There, the author proposed a monotone approximation for this non-monotone model. The author also used the coercive approximation (for more details we refer to [5]) and he passed to the limit with these approximations. It was shown that there exists a global in time solution, where the inelastic constitutive equation is satisfied in the sense of Young measures, which is a very weak notion of solution.

Therefore, the question arises naturally, whether adding Cosserat microrotations to the model is still enough to regularize the problem in the way to satisfy the flow rule not in an averaged sense, but in a pointwise sense. From a modelling perspective, adding microrotations means to consider a material made up of individual particles which can rotate and interact with each other [16, 17, 19]

The extension of the poroplasticity model to include Cosserat effects follows the lines proposed in [9], where the authors added the Cosserat effect to the classical elasto-plasticity model with a monotone flow rule. It was proved that the new model is thermodynamically admissible and that there exists a unique, global in time solution to Cosserat elasto-plasticity. Moreover, in [11] a H^1_{loc}-regularity of the stresses and strains was proved, cf. [20]. The dynamic Cosserat plasticity was also studied in [10].

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In the present article we apply the results for the model from [9] to the model of poroplasticity and we will study the existence of solution for our new model.

Our paper is organized as follows: first, we formulate the Cosserat poroplasticity model. Then we propose the approximations of the considered model. Using the Yosida approximation scheme we prove the existence of the solution to the approximate system. Finally, we pass to the limit to obtain a solution for the Cosserat poroplasticity model, where the inelastic constitutive equation will be satisfied again in the sense of Young measures. This partially answers the question raised above: adding Cosserat effects does not seem to provide enough regularization to give the flow rule a pointwise meaning.

2 The Cosserat poroplasticity model and the main result

In this section we formulate the model with Cosserat effects and we give the main result of this paper.

Using the mechanical results for Cosserat plasticity we can write that the equations for Cosserat poroplasticity have the form: we are looking for the displacement field $u : \Omega \times [0,T] \to \mathbb{R}^3$, the pore pressure of the fluid $p : \Omega \times [0,T] \to \mathbb{R}$, the microrotation matrix $A : \Omega \times [0,T] \to \mathfrak{so}(3)$ ($\mathfrak{so}(3)$ is the set of skew-symmetric 3×3 matrices) and the plastic strain tensor $\varepsilon^p : \Omega \times [0,T] \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ ($\mathbb{R}^{3\times 3}_{\mathrm{sym}}$ denotes the set of symmetric 3×3 - matrices) satisfying the following system of equations

$$\begin{aligned}
\operatorname{div}_x(\sigma - p\mathbb{1}) &= -F\,, \\
\sigma &= 2\mu\Big(\operatorname{sym}(\nabla_x u - A - \varepsilon^p)\Big) + 2\mu_c \operatorname{skew}(\nabla_x u - A - \varepsilon^p) + \lambda \operatorname{tr}(\nabla_x u - A - \varepsilon^p)\mathbb{1}\,, \\
c\Delta_x p - \operatorname{div}_x u_t &= f\,, \\
-l_c \Delta_x \operatorname{axl}(A) &= \mu_c \operatorname{axl}(\operatorname{skew}(\nabla_x u - A - \varepsilon^p))\,, \\
\varepsilon^p_t &= \mathcal{F}(Y(T_E))\,\frac{\partial P}{\partial T}(T_E)\,, \\
T_E &= 2\mu\Big(\operatorname{sym}(\nabla_x u - A - \varepsilon^p)\Big) + \lambda \operatorname{tr}(\nabla_x u - A - \varepsilon^p)\mathbb{1} = \operatorname{sym}\sigma\,.
\end{aligned} \tag{2.1}$$

The above equations are studied for $x \in \Omega \subset \mathbb{R}^3$ and $t \in [0,T]$, where Ω is a bounded domain with smooth boundary $\partial\Omega$ and t denotes time.

In the system (2.1) T_E is the Eshelby stress tensor (in the usual inelastic deformation theory, this is a special stress tensor). The effective stress is $\sigma - p\mathbb{1}$ and the solid matrix elastic distortion is given by $\nabla u - A - \varepsilon^p$.

$F : \Omega \times [0,T] \to \mathbb{R}^3$ and $f : \Omega \times [0,T] \to \mathbb{R}$ are given functions describing density of the applied body forces and a forced fluid extraction or injection process, respectively. The parameters μ, λ are positive Lamé constants (the elastic constitutive equation can be generalized in the obvious way to anisotropic case), $\mu_c > 0$ is the Cosserat couple modulus and $l_c > 0$ is a material parameter describing a length scale of the model due to the Cosserat effects. $c > 0$ is a constant, which represents the permeability of the porous medium and the viscosity of the fluid. The operators ”sym” and ”skew” denote the symmetric and skew-symmetric parts of a 3×3 tensor, respectively. The operator $\operatorname{axl} : \mathfrak{so}(3) \to \mathbb{R}^3$ is the identification of the skew-symmetric matrix with vectors in $\mathbb{R}^3$. This means that if we take $A \in \mathfrak{so}(3)$, which is in the form $A = ((0,\alpha,\beta),(-\alpha,0,\gamma),(-\beta,-\gamma,0))$ then $\operatorname{axl}(A) = (\alpha,\beta,\gamma)$.

The scalar valued functions $Y : \mathbb{R}^{3\times 3}_{\mathrm{sym}} \to \mathbb{R}$ and $P : \mathbb{R}^{3\times 3}_{\mathrm{sym}} \to \mathbb{R}$ are given (for a special example we refer to [12]). We assume that they are convex homogeneous polynomials of the same growth. Moreover, $\mathcal{F} \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}_+)$ and $\mathcal{F}$ is a monotone function with polynomial growth, which means that there exist $\alpha > 1$ and constants m, $M > 0$ such that

$$m\,|s|^\alpha \le \mathcal{F}(s) \le M\,|s|^\alpha \quad \text{for large } |s|.$$

We also assume that $\mathcal{F}(0) = 0$ and $\mathcal{F}(/) = /$, which means that the inelastic response of the material is zero when the Eshelby stress is equal to zero. The fifth equation in (2.1) is called inelastic constitutive equation and it was proposed in the article [12]. It is often used in practice; for the physical explanation of this equation we also refer to [12].

The standard poroplasticity model is built from the balance of momentum with a generalization of the Hook law (the first and the second equation of (2.1) without the second term on the right hand side of $(2.1)_2$), the combination of the Darcy law with the fluid mass conservation (the third equation) and the inelastic constitutive equation. It does not contain the equation for microrotation $(2.1)_4$ (compare with [24]).

Using the properties of the unknowns we can rewrite the system (2.1) in the following form

$$\begin{aligned} &\mathrm{div}_x\sigma - \nabla_x p = -F\,,\\ &\qquad \sigma = 2\mu(\varepsilon(u)-\varepsilon^p) + 2\mu_c(\mathrm{skew}(\nabla_x u) - A) + \lambda \mathrm{tr}(\varepsilon(u)-\varepsilon^p)\mathbb{1}\,,\\ &c\Delta_x p - \mathrm{div}_x u_t = f\,,\\ &-l_c\Delta_x \mathrm{axl}(A) = \mu_c \mathrm{axl}(\mathrm{skew}(\nabla_x u) - A)\,,\\ &\qquad \varepsilon^p_t = \mathcal{F}(Y(T_E))\,\frac{\partial P}{\partial T}(T_E), \quad T_E = 2\mu(\varepsilon(u)-\varepsilon^p) + \lambda\mathrm{tr}(\varepsilon(u)-\varepsilon^p)\mathbb{1}\,, \end{aligned} \tag{2.2}$$

where $\varepsilon(u) = \mathrm{sym}(\nabla_x u)$ denotes the symmetric part of the displacement gradient.

The system (2.2) is considered with Dirichlet boundary conditions

$$\begin{aligned} u(x,t) &= g_D(x,t) \quad \text{for} \quad x\in\partial\Omega \quad \text{and} \quad t\ge 0,\\ p(x,t) &= g_P(x,t) \quad \text{for} \quad x\in\partial\Omega \quad \text{and} \quad t\ge 0,\\ A(x,t) &= A_D(x,t) \quad \text{for} \quad x\in\partial\Omega \quad \text{and} \quad t\ge 0 \end{aligned} \tag{2.3}$$

and with initial conditions

$$\mathrm{div}_x u(x,0) = \mathrm{div} u^0(x), \qquad \varepsilon^p(x,0) = \varepsilon^{p,0}(x), \tag{2.4}$$

where the initial condition for the displacement means that we only know the divergence of $u(x,0)$.

The total energy function associated with the system (2.2) is given by the formula

$$\varepsilon(u,\varepsilon,\varepsilon^p,a)(t) = \int\limits_\omega \big(\mu\|\varepsilon(u)-\varepsilon^p\|^2 + \mu_c\|\mathrm{skew}(\nabla_x u) - A\|^2 + \frac{\lambda}{2}\mathrm{tr}^2(\varepsilon(u)-\varepsilon^p) + 2l_c\|\nabla_x \mathrm{axl}(A)\|^2\big)dx.$$

Remark: a) in many models from the theory of inelastic deformation process in metals the trace of the plastic strain is equal to zero. The energy associated with such a model consists of the term $\frac{\lambda}{2}\int\limits_\Omega \mathrm{tr}^2(\varepsilon)dx$, which controls the divergence of the displacement in $L^2(\Omega;\mathbb{R})$ $(\mathrm{tr}(\varepsilon(u)) = \mathrm{div} u)$. Thus, in the Cosserat elasto-plasticity models studied for example in [9]-[11], coerciveness of the energy was obtained. This was a crucial step in the existence theory (in view of this fact, the whole gradient of the displacement in $L^2(\Omega;\mathbb{R}^3)$ and microrotation tensor in $H^1(\Omega;\mathfrak{so}(3))$ is controlled, for the proof we refer to [9]). In our model of poroplasticity $\mathrm{tr}(\varepsilon^p)$ is not equal to zero and the total energy has the term $\frac{\lambda}{2}\int\limits_\Omega \mathrm{tr}^2(\varepsilon-\varepsilon^p)dx$, which does not allow to control $\mathrm{div} u$ in $L^2(\Omega;\mathbb{R})$. Therefore, the model of poroplasticity with Cosserat effects is still non-coercive in the sense that the total energy does not control the gradient of the displacement.

b) It is not difficult to note that the inelastic constitutive function in (2.2) is not monotone (this follows from the assumptions on Y and P). Therefore, the model (2.2) is also non-monotone.

Let us assume that our data F, f, g_D, g_P, A_D, $\mathrm{div}u^0$, $\varepsilon^{p,0}$ have the following regularity

$$F\in W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^3)), \quad f\in H^1(0,T;L^2(\Omega;\mathbb{R})), \tag{2.5}$$

$$g_D\in W^{3,\infty}(0,T;H^{\frac12}(\partial\Omega;\mathbb{R}^3)), \quad g_P\in W^{2,\infty}(0,T;H^{\frac32}(\partial\Omega;\mathbb{R})), \tag{2.6}$$

$$A_D\in W^{3,\infty}(0,T;H^{\frac32}(\partial\Omega;\mathfrak{so}(3))), \tag{2.7}$$

$$\mathrm{div}u^0\in H^2(\Omega;\mathbb{R}), \quad \varepsilon^{p,0}\in L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}}), \quad \mathrm{div}\,\varepsilon^{p,0}\in H^1(\Omega;\mathbb{R}^3). \tag{2.8}$$

Additionally let us assume that the initial data satisfy

$$F(0)\in H^1(\Omega;\mathbb{R}^3),\ g_D(0)\in H^{\frac52}(\partial\Omega;\mathbb{R}^3),\ A_D(0)\in H^{\frac52}(\partial\Omega;\mathfrak{so}) \tag{2.9}$$

and

$$\int\limits_{\partial\Omega} g_D(x,0)n(x)\,dS(x) = \int\limits_\Omega \mathrm{div}u^0(x)\,dx\,, \tag{2.10}$$

where $n(x)$ is the exterior normal vector to the boundary $\partial\Omega$ at the point $x\in\partial\Omega$. Now we formulate the definition of the solution of the system (2.2) and the main result of this paper.

Definition 2.1 (solution concept)
Suppose that the given data satisfy (2.5)–(2.10) and let
$F, F_t \in L^\infty(0,T;L^3(\Omega;\mathbb{R}^3))$. Let $\beta > 1$, then we say that a vector $u \in L^{1+\frac{1}{\beta}}(0,T;W^{1,1+\frac{1}{\beta}}(\Omega;\mathbb{R}^3))$, the function $p \in L^2(0,T;H^1(\Omega;\mathbb{R}))$, the inelastic deformation tensor $\varepsilon^p \in W^{1,1+\frac{1}{\beta}}(0,T;L^{1+\frac{1}{\beta}}(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{sym}}))$, the stress tensor $\sigma = 2\mu(\varepsilon(u)-\varepsilon^p) + 2\mu_c(\mathrm{skew}(\nabla u) - A) + \lambda \mathrm{tr}(\varepsilon(u)-\varepsilon^p)\mathbb{1} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^9))$ and the microrotation tensor $A \in L^\infty(0,T;H^2(\Omega;\mathfrak{so}(3)))$ solve the problem (2.2)–(2.4) if

1. the functions u, A, and p are in the form $u(x,t) = v(x,t) + w(x,t)$, $A(x,t) = \hat{A}(x,t) + \hat{w}(x,t)$, $p(x,t) = \tilde{p}(x,t) + \tilde{w}(x,t)$, where $w \in W^{2,\infty}(0,T;H^1(\Omega;\mathbb{R}^3))$, $\hat{w} \in W^{1,\infty}(0,T;H^2(\Omega;\mathfrak{so}(3)))$, and $\tilde{w} \in W^{1,\infty}(0,T;H^2(\Omega;\mathbb{R}))$ are such functions that $w_{|\partial\Omega} = g_D$, $\hat{w}_{|\partial\Omega} = A_D$ and $\tilde{w}_{|\partial\Omega} = g_P$. Moreover the functions $v \in L^{1+\frac{1}{\beta}}(0,T;W_0^{1,1+\frac{1}{\beta}}(\Omega;\mathbb{R}^3))$, $\hat{A} \in L^\infty(0,T;H_0^2(\Omega;\mathfrak{so}(3)))$ and $\tilde{p} \in L^2(0,T;H_0^1(\Omega;\mathbb{R}))$ satisfy the system of equations:

$$\int_\Omega \Big(2\mu(\varepsilon(v)-\varepsilon^p) + 2\mu_c(\mathrm{skew}(\nabla v) - A) + \lambda\mathrm{tr}(\varepsilon(v)-\varepsilon^p)\mathbb{1}\Big)\cdot\nabla\bar{v}\,dx - \int_\Omega p\,\mathrm{div}\bar{v}\,dx$$
$$= \int_\Omega F\bar{v}\,dx - \int_\Omega \Big(2\mu\varepsilon(w) + 2\mu_c\mathrm{skew}(\nabla w) + \lambda\mathrm{tr}(\varepsilon(w))\mathbb{1}\Big)\cdot\nabla\bar{v}\,dx,$$
$$c\int_{\Omega\times[0,T]} \nabla\tilde{p}\nabla\phi\,dxdt - \int_{\Omega\times[0,T]} \mathrm{div}u\,\phi_t\,dxdt + \int_\Omega \mathrm{div}u^0\phi(0)\,dx = -\int_{\Omega\times[0,T]} f\phi\,dxdt - \int_{\Omega\times[0,T]} \nabla\tilde{w}\nabla\phi\,dxdt\,,$$
$$l_c\int_\Omega \nabla\mathrm{axl}(\hat{A})\nabla\mathrm{axl}(\check{A})\,dx = \mu_c\int_\Omega \mathrm{axl}\Big(\mathrm{skew}(\nabla u) - A\Big)\mathrm{axl}(\check{A})\,dx - l_c\int_\Omega \nabla\mathrm{axl}(\hat{w})\nabla\mathrm{axl}(\check{A})\,dx,$$

   where the first equation is satisfied for all $\bar{v} \in H_0^1(\Omega;\mathbb{R}^3)$ and for almost all $t \in [0,T]$, the second is satisfied for all $\phi \in C_0^\infty(\Omega\times[0,T))$ and the third is satisfied for all $\check{A} \in H_0^1(\Omega;\mathfrak{so}(3))$ and for almost all $t \in [0,T]$.

2. The fifth equation in (2.2) is satisfied in the sense of Young measures, i.e.

$$\varepsilon_t^p(x,t) = \int_{\mathbb{R}^{3\times 3}_{\mathrm{sym}}} \mathcal{F}(Y(S))\,\frac{\partial P}{\partial T}(S)d\nu_{(x,t)}(S),$$

   where $\nu_{(x,t)}$ is a Young measure satisfying $T_E(x,t) = \int_{\mathbb{R}^{3\times 3}_{\mathrm{sym}}} S d\nu_{(x,t)}(S)$ a.e. in $\Omega\times(0,T)$.
3. $\mathrm{div}_x u(x,0) = \mathrm{div}u^0(x)$, $\varepsilon^p(x,0) = \varepsilon^{p,0}(x)$ for a.e $x \in \Omega$.

Remark: The poroplasticity model with Cosserat effect is still non-coercive and non-monotone. Therefore, the Definition 2.1 is similar to the definition of the solution for the quasi-static model of poroplasticity formulated in [24]. The linear partial differential equations (part 1) for the displacement and for the microrotation have to be satisfied in the usual weak sense with respect to the space variable and the equation for the pressure in the weak sense with respect to the space and time due to the presence of $\mathrm{div}\,u_t$. The nonlinear system of ordinary differential equations (part 2) has to be satisfied in an averaged sense. This solution concept is not new. In the literature we can find examples of solutions in the sense given in Definition 2.1 (see for example [8, 13, 14, 18]).

Let us denote by $\deg(P)$ the degree of the polynomial P. The following theorem is the main result of this paper:

Theorem 2.2 (Main existence result)
Under the assumptions of the Definition 2.1 *for all* $\beta > r' = (\alpha-1)\deg(Y)(\deg(Y)-1)^2$ *there exists a global in time solution (in the sense of Definition* 2.1*) of the system* (2.2) *with boundary condition* (2.3) *and initial condition* (2.4)*.*

Before we start with an approximation procedure, we give a sketch of the proof of existence and regularity results for Biot model coupled with Cosserat effects. The next section will present first existence result for such model because it was not studied before.

3 Biot model with Cosserat effects

The Biot model with Cosserat effects has the following form

$$\begin{aligned} \operatorname{div}_x\sigma &= -F,\\ \sigma &= 2\mu\varepsilon + 2\mu_c(\operatorname{skew}(\nabla_x u) - A) + \lambda\operatorname{tr}(\varepsilon)1\!\!1 - p1\!\!1,\\ c\Delta_x p - \operatorname{div}_x u_t &= f,\\ -l_c\Delta_x \operatorname{axl}(A) &= \mu_c \operatorname{axl}(\operatorname{skew}(\nabla_x u) - A), \end{aligned} \tag{3.1}$$

where $x \in \Omega$ and $t \in [0,T]$. The system of linear partial differential equations is considered with homogeneous Dirichlet boundary conditions

$$\begin{aligned} u(x,t) &= 0 \quad \text{for} \quad x \in \partial\Omega \quad \text{and} \quad t \geq 0,\\ p(x,t) &= 0 \quad \text{for} \quad x \in \partial\Omega \quad \text{and} \quad t \geq 0,\\ A(x,t) &= 0 \quad \text{for} \quad x \in \partial\Omega \quad \text{and} \quad t \geq 0 \end{aligned} \tag{3.2}$$

and initial condition

$$\operatorname{div}_x u(x,0) = \operatorname{div} u^0(x). \tag{3.3}$$

Definition 3.1 *We say that a vector $u \in C([0,T]; H^1_0(\Omega;\mathbb{R}^3))$, the function $p \in H^1(0,T; H^1_0(\Omega;\mathbb{R}))$ and the microrotation tensor $A \in C([0,T]; H^1_0(\Omega;\mathfrak{so}(3)))$ such that $u_t \in L^\infty(0,T; H^1_0(\Omega;\mathbb{R}^3))$ and $A_t \in L^\infty(0,T; H^1_0(\Omega;\mathfrak{so}(3)))$ are weak solutions of the problem (3.1) - (3.3) if for almost all $t \in [0,T]$*

$$\int_\Omega \Big(2\mu\varepsilon(u) + 2\mu_c(\operatorname{skew}(\nabla u) - A) + \lambda\operatorname{tr}(\varepsilon(u))1\!\!1\Big)\nabla v dx - \int_\Omega p \operatorname{div} v dx = \int_\Omega F v dx,$$

$$c\int_\Omega \nabla p \nabla w dx + \int_\Omega \operatorname{div} u_t w dx = -\int_\Omega f w dx,$$

$$l_c \int_\Omega \nabla \operatorname{axl}(A)\nabla\operatorname{axl}(\hat{A}) dx = \mu_c \int_\Omega \operatorname{axl}\Big(\operatorname{skew}(\nabla u) - A\Big)\operatorname{axl}(\hat{A}) dx,$$

where the first equation is satisfied for all $v \in H^1_0(\Omega;\mathbb{R}^3)$, the second is satisfied for all $w \in H^1_0(\Omega;\mathbb{R})$ and the third is satisfied for all $\hat{A} \in H^1_0(\Omega;\mathfrak{so}(3))$. Moreover $\operatorname{div}_x u(x,0) = \operatorname{div} u^0(x)$.

Theorem 3.2 *Suppose that for all $T > 0$ the external forces $F,\ f$ satisfy*

$$F \in C^1([0,T]; L^2(\Omega;\mathbb{R}^3)),\ F_{tt} \in L^2(0,T; L^2(\Omega;\mathbb{R}^3)), \quad f \in H^1([0,T]; L^2(\Omega;\mathbb{R}))$$

and

$$F(0) \in H^1(\Omega;\mathbb{R}^3),\ \operatorname{div} u^0 \in H^2(\Omega;\mathbb{R}),\ \int_\Omega \operatorname{div} u^0 dx = 0.$$

Additionally suppose that the function $f(0)$ satisfies a compatibility condition, which will be specified in the proof of this theorem. Then for all $T > 0$ the system (3.1) with initial - boundary conditions (3.2) and (3.3) possesses a unique weak solution.

Proof: The proof of this theorem is divided into five steps.

Step 1: (Solutions for $t = 0$)
Let us consider the following system of equations

$$\begin{aligned} \operatorname{div}\sigma(0) - \nabla p(0) &= -F(0)\,,\\ \operatorname{div} u(x,0) &= \operatorname{div} u^0(x)\,,\\ -l_c\Delta\operatorname{axl}(A(0)) &= \mu_c\operatorname{axl}(\operatorname{skew}(\nabla u(0)) - A(0))\,, \end{aligned} \tag{3.4}$$

$$u(0)_{|\partial\Omega} = 0, \quad A(0)_{|\partial\Omega} = 0, \qquad p(0)_{|\partial\Omega} = 0\,,$$

where

$$\sigma(0) = 2\mu\varepsilon(u(0)) + 2\mu_c(\mathrm{skew}(\nabla u(0)) - A(0)) + \lambda \mathrm{tr}(\varepsilon(u(0)))1\!\!1.$$

The system (3.4) is the Stokes problem coupled with a linear elliptic system for the microrotation. The bilinear form $B : \big(V \times H_0^1(\Omega; \mathfrak{so}(3))\big) \times \big(V \times H_0^1(\Omega; \mathfrak{so}(3))\big) \to \mathbb{R}$ associated with the system (3.4) is defined by the formula

$$B[(u, A), (v, w)] = 2\mu \int_\Omega \varepsilon(u)\varepsilon(v)dx + 2\mu_c \int_\Omega (\mathrm{skew}(\nabla u) - A)(\mathrm{skew}(\nabla v) - w)dx$$
$$+\lambda \int_\Omega \mathrm{tr}(\varepsilon(u))\mathrm{tr}(\varepsilon(v)) + 4l_c \int_\Omega \nabla \mathrm{axl}(A)\nabla \mathrm{axl}(w)dx,$$

where $u, v \in V = \{v \in H_0^1(\Omega; \mathbb{R}^3) : \ \mathrm{div} v = 0\}$ and $A, w \in H_0^1(\Omega; \mathfrak{so}(3))$. From the Lax-Milgram theorem and the regularity of the initial data we conclude the existence of a unique solution $u(0) \in H_0^1(\Omega; \mathbb{R}^3)$, $A(0) \in H_0^1(\Omega; \mathfrak{so}(3))$ and $p(0) \in L^2(\Omega; \mathbb{R})$. Using standard methods from regularity theory for elliptic systems (difference quotients) we also get that $u(0) \in H^3(\Omega; \mathbb{R}^3)$, $A(0) \in H^3(\Omega; \mathfrak{so}(3))$ and $p(0) \in H^2(\Omega; \mathbb{R})$. Next we study the following system of equations

$$\begin{aligned}
\mathrm{div}\sigma_t(0) - \nabla p_t(0) &= -F_t(0)\,,\\
\mathrm{div} u_t(0) &= c\Delta p(0) - f(0)\,,\\
-l_c\Delta \mathrm{axl}(A_t(0)) &= \mu_c \mathrm{axl}\Big(\mathrm{skew}(\nabla u_t(0)) - A_t(0)\Big)\,,\\
u_t(0)_{|\partial\Omega} = 0, \quad A_t(0)_{|\partial\Omega} &= 0, \qquad p_t(0)_{|\partial\Omega} = 0\,,
\end{aligned} \tag{3.5}$$

where

$$\sigma_t(0) = 2\mu\varepsilon(u_t(0)) + 2\mu_c\Big(\mathrm{skew}(\nabla u_t(0)) - A_t(0)\Big) + \lambda \mathrm{tr}(\varepsilon(u_t(0)))1\!\!1$$

and $p(0)$ is the solution of the problem (3.4). The system (3.5) is again the Stokes problem with microrotations. Let us assume that the initial data $f(0)$ is chosen such that the following compatibility condition

$$0 = \int_\Omega \Big(c\Delta p(0) - f(0)\Big)\, dx$$

holds. Then, from the Lax-Milgram theorem we obtain a unique solution $u_t(0) \in H_0^1(\Omega; \mathbb{R}^3)$, $A_t(0) \in H_0^1(\Omega; \mathfrak{so}(3))$ and $p_t(0) \in L^2(\Omega; \mathbb{R})$.

Step 2: (Preparation for the Galerkin method)
Let $\{w_k\}_{k=1}^\infty$ be any basis in $H_0^1(\Omega; \mathbb{R}^3)$, $\{\tilde{w}_k\}_{k=1}^\infty$ be any basis in $H_0^1(\Omega, \mathbb{R})$ and $\{\hat{w}_k\}_{k=1}^\infty$ be any basis in $H_0^1(\Omega, \mathfrak{so}(3))$, where $w_k = w_k(x)$, $\tilde{w}_k = \tilde{w}_k(x)$, $\hat{w}_k = \hat{w}_k(x)$ and $w_k, \tilde{w}_k, \hat{w}_k$ are smooth inside Ω for all $k = 1, \ldots$.

Fix a positive integer m. We will find functions $u_m : [0, T] \to H_0^1(\Omega, \mathbb{R}^3)$, $p_m : [0, T] \to H_0^1(\Omega, \mathbb{R})$ and $A_m : [0, T] \to H_0^1(\Omega, \mathfrak{so}(3))$ in the form:

$$u_m(t) = \sum_{k=1}^m g_m^k(t) w_k, \tag{3.6}$$

$$p_m(t) = \sum_{k=1}^m \tilde{g}_m^k(t) \tilde{w}_k, \tag{3.7}$$

$$A_m(t) = \sum_{k=1}^m \hat{g}_m^k(t) \hat{w}_k \tag{3.8}$$

which satisfy

$$\int_\Omega \Big(2\mu\varepsilon(u_m(t)) + 2\mu_c\Big(\text{skew}(\nabla u_m(t)) - A_m(t)\Big) + \lambda\text{tr}(\varepsilon(u_m(t)))1\!\!1\Big)\nabla w_k dx - \int_\Omega p_m(t)\text{div}w_k dx = \int_\Omega F w_k dx, \tag{3.9}$$

$$c\int_\Omega \nabla p_m(t)\nabla\tilde{w}_k dx + \int_\Omega \text{div}u_m{}'(t)\tilde{w}_k dx = -\int_\Omega f\tilde{w}_k dx, \tag{3.10}$$

$$l_c\int_\Omega \nabla\text{axl}(A_m(t))\nabla\text{axl}(\hat{w}_k)dx = \mu_c\int_\Omega \text{axl}\Big(\text{skew}(\nabla u_m(t)) - A_m(t)\Big)\text{axl}(\hat{w}_k)dx \tag{3.11}$$

for all $k = 1,\ldots,m$. We choose $u_m(0)$ and $A_m(0)$ such that $u_m(0) \to u(0)$ in $H_0^1(\Omega;\mathbb{R}^3)$ and $A_m(0) \to A(0)$ in $H_0^1(\Omega;\mathfrak{so}(3))$ as $m\to\infty$, where $u(0)$ i $A(0)$ are the solutions of system (3.4).

Step 3: (Existence for each Galerkin approximation step)

Theorem 3.3 *Let us suppose that our data has the regularity required in Theorem* 3.2*. Then for all natural* $m = 1,2,\ldots$ *there exist unique functions* $u_m(t)$*,* $A_m(t)$ *and* $p_m(t)$ *of the form (*3.6*), (*3.7*), and (*3.8*) respectively which satisfy (*3.9*), (*3.10*), and (*3.11*).*

Proof of the Theorem 3.3 follows from the theory of ordinary differential equations (for a similar result we refer to [22]). The details are left to the reader.

Step 4: (Energy estimates)

Theorem 3.4 *Under the assumptions of Theorem* 3.2*, there exist positive constants* $C(T)$*,* $\tilde{C}(T)$*, not depending on* m*, such that for all* $T > 0$ *and* $t \le T$ *the following inequalities are satisfied*

$$\|u_m(t)\|^2_{H_0^1(\Omega;\mathbb{R}^3)} + \|\text{axl}(\text{skew}(\nabla u_m(t)) - A_m(t))\|^2_{L^2(\Omega;\mathbb{R}^3)} + \lambda\|\text{tr}(\varepsilon(u_m(t)))\|^2_{L^2(\Omega;\mathbb{R})} + \|\text{axl}(A_m(t))\|^2_{H_0^1(\Omega;\mathbb{R}^3)} + \int_0^t \|p_m(\tau)\|^2_{H_0^1(\Omega;\mathbb{R})}d\tau \le C(T), \tag{3.12}$$

$$\|u'_m(t)\|^2_{H_0^1(\Omega;\mathbb{R}^3)} + \|\text{axl}\big(\text{skew}(\nabla u'_m(t)) - A'_m(t)\big)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \lambda\|\text{tr}(\varepsilon(u'_m(t)))\|^2_{L^2(\Omega;\mathbb{R})} + \|\text{axl}(A'_m(t))\|^2_{H_0^1(\Omega;\mathbb{R}^3)} + \int_0^t \|p'_m(\tau)\|^2_{H_0^1(\Omega;\mathbb{R})}d\tau \le \tilde{C}(T). \tag{3.13}$$

P r o o f (of Theorem 3.4). It is sufficient to prove inequality (3.13). From the definition of approximate solutions, Theorem 3.3 and some standard operations we can obtain the following equality

$$\begin{aligned}&\mu\|\varepsilon(u'_m(t))\|^2_{L^2(\Omega;\mathbb{R}^3)} + \mu_c\|\text{axl}\big(\text{skew}(\nabla u'_m(t)) - A'_m(t)\big)\|^2_{L^2(\Omega;\mathbb{R}^3)}\\ &\quad + \frac{\lambda}{2}\|\text{tr}(\varepsilon(u'_m(t)))\|^2_{L^2(\Omega;\mathbb{R})} + 2l_c\|\nabla\text{axl}A'_m(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^t \|\nabla p'_m(\tau)\|^2_{L^2(\Omega;\mathbb{R}^3)}d\tau\\ &= \mu\|\varepsilon(u'_m(0))\|^2_{L^2(\Omega;\mathbb{R}^3)} + \mu_c\|\text{axl}\big(\text{skew}(\nabla u'_m(0)) - A'_m(0)\big)\|^2_{L^2(\Omega;\mathbb{R}^3)}\\ &\quad + \frac{\lambda}{2}\|\text{tr}(\varepsilon(u'_m(0)))\|^2_{L^2(\Omega;\mathbb{R})} + 2l_c\|\nabla\text{axl}A'_m(0)\|^2_{L^2(\Omega;\mathbb{R}^9)}\\ &\quad + \int_0^t\int_\Omega F_t u_m{}''(\tau)dxd\tau - \int_0^t\int_\Omega f_t p_m{}'(\tau)dxd\tau.\end{aligned} \tag{3.14}$$

We can also assume that $u_m{}'(0)$ and $A_m{}'(0)$ are selected from the space span$\{w_1,\ldots,w_m\}$ so that $u_m{}'(0) \to u_t(0)$ in $H_0^1(\Omega;\mathbb{R}^3)$ and $A'_m(0) \to A_t(0)$ in $H_0^1(\Omega;\mathfrak{so}(3))$, where $u_t(0)$ and $A_t(0)$ are solutions of the system (3.5). Therefore the initial terms occuring on the right hand side of (3.14) are bounded. Integrating by parts with respect to time in the fifth integral and using a weighted Cauchy inequality we finish the proof of Theorem 3.4. □

Step 5: (Existence and uniqueness)

Theorem 3.5 *Let us assume that the given data satisfies all requirements of Theorem* 3.2. *Then there exists a unique weak solution of the system* (3.1).

The energy estimates proved in the last step are sufficient to pass to the limit with $m \to \infty$ in (3.9), (3.10) and (3.11). Thus we obtain a solution of the system (3.1) in the sense of Definition 3.1. This part of the proof is a standard one in the Galerkin method and the details are left to the reader (for more information we refer to [22], where the quasistatic Biot model was studied). The remark above finishes the proof of Theorem 3.2. □

4 Existence for an approximate system

In this section we are going to prove existence of solutions of the system (2.2). We apply here a special monotone approximation, which was already proposed for the quasi-static model of poroplasticity with the Ehlers flow rule in the article [24]. Moreover, we slightly change this approximation, such that the approximate total energy will now be coercive in the sense that it controls the approximate displacement vector in $H^1(\Omega;\mathbb{R}^3)$ and the approximate microrotation tensor in $H^1(\Omega;\mathfrak{so}(3))$.

Let $\eta > 0$ and $\beta > 1$. Then the approximation of the system (2.2) is defined by

$$\begin{aligned}
\operatorname{div}\sigma^\eta - \nabla p^\eta &= -F\,,\\
\sigma^\eta &= 2\mu(\varepsilon^\eta - \varepsilon^{p,\eta}) + 2\mu_c(\operatorname{skew}(\nabla u^\eta) - A^\eta) + \lambda \operatorname{tr}(\varepsilon^\eta - \varepsilon^{p,\eta} + \underline{\eta\varepsilon^\eta})\mathbb{1}\,,\\
c\Delta p^\eta - \operatorname{div} u^\eta_t &= f\,,\\
-l_c\Delta \operatorname{axl}(A^\eta) &= \mu_c \operatorname{axl}\Big(\operatorname{skew}(\nabla u^\eta) - A^\eta\Big)\,,\\
\varepsilon^{p,\eta}_t &= \underline{\eta|T^\eta_E|^\beta \frac{T^\eta_E}{|T^\eta_E|}} + \mathcal{F}(Y(T^\eta_E))\,\frac{\partial P}{\partial T}(T^\eta_E) = G^\eta(T^\eta_E)\,,
\end{aligned} \tag{4.1}$$

where $T^\eta_E = 2\mu(\varepsilon^\eta - \varepsilon^{p,\eta}) + \lambda \operatorname{tr}(\varepsilon^\eta - \varepsilon^{p,\eta})\mathbb{1}$ and the underlined terms are new terms due to the approximation.

The problem (4.1) is considered with Dirichlet boundary conditions (2.3) and initial conditions (2.4), which are the same as for the original system of equations.

The total energy associated with the system (4.1) is now given by the formula

$$\begin{aligned}
\mathcal{E}^\eta(u^\eta,\varepsilon^\eta,\varepsilon^{p,\eta},A^\eta) = \int_\Omega \Big(&\mu\|\varepsilon^\eta - \varepsilon^{p,\eta}\|^2 + \mu_c\|\operatorname{skew}(\nabla u^\eta) - A^\eta\|^2\\
&+\frac{\lambda}{2}\operatorname{tr}^2(\varepsilon^\eta - \varepsilon^{p,\eta}) + \frac{\lambda}{2}\eta\operatorname{tr}^2(\varepsilon^\eta) + 2l_c\|\nabla\operatorname{axl}(A^\eta)\|^2\Big)\,dx.
\end{aligned} \tag{4.2}$$

The modification in $(4.1)_2$ yields the coerciveness of the total energy in the sense from the theorem below. For fixed $\eta > 0$ and $\beta > 0$ large enough the new term in $(4.1)_5$ dominates the Ehlers – vector field and the new flow rule "is not very far" from the class of monotone flow rules.

Theorem 4.1 (coerciveness of the energy)
(a) (*the case with zero boundary data*)
For all $\eta > 0$ *the energy function* (4.2) *is elastically coercive with respect to* ∇u. *This means that* $\exists C_E(\eta) > 0$, $\forall u \in H^1_0(\Omega)$, $\forall A \in H^1_0(\Omega)$, $\forall \varepsilon^p \in L^2(\Omega)$

$$\mathcal{E}^\eta(u,\varepsilon,\varepsilon^p,A) \ge C_E(\eta)(\|u\|^2_{H^1(\Omega)} + \|A\|^2_{H^1(\Omega)}).$$

(b) (*the case with non − zero boundary data*)
Moreover, $\exists C_E(\eta) > 0$, $\forall g_D, A_D \in H^{\frac12}(\partial\Omega)$, $\exists C_d > 0$, $\forall \varepsilon^p \in L^2(\Omega)$, $\forall u \in H^1(\Omega)$, $\forall A \in H^1(\Omega)$ *with boundary conditions* $u_{|\partial\Omega} = g_D$ *and* $A_{|\partial\Omega} = A_D$ *it holds that*

$$\mathcal{E}^\eta(u,\varepsilon,\varepsilon^p,A) + C_d \ge C_E(\eta)(\|u\|^2_{H^1(\Omega)} + \|A\|^2_{H^1(\Omega)}).$$

P r o o f. From the form of the energy (4.2) it is easy to see that for all $\eta > 0$ the term $\mathrm{div}u$ is controlled in $L^2(\Omega)$ (see [9]). Using the Poincaré inequality and the following estimate [[15], p.36]

$$\|\nabla u\|^2_{L^2(\Omega)} \le C^{\mathrm{curl}}_{\mathrm{div}}(\|\mathrm{div}u\|^2_{L^2(\Omega)} + \|\mathrm{curl}u\|^2_{L^2(\Omega)}),$$

(the constant $C^{\mathrm{curl}}_{\mathrm{div}}$ does not depend on u and 'curl' is the rotation operator) we obtain the first statement (the details can be found in [9]). The second one follows from the first inequality for the difference $u - \tilde{u}$ and $A - \tilde{A}$, where $\tilde{u}, \tilde{A} \in H^1(\Omega)$ are such functions that $\tilde{u}_{|\partial\Omega} = g_D$, $\tilde{A}_{|\partial\Omega} = A_D$. □

From the article [24] we know that for all $\eta > 0$ the approximate inelastic constitutive equation belongs to the class $\mathcal{LM}$ (the class of Lipschitz perturbations of monotone vector fields). Therefore, G^η can be written in the form $G^\eta = g^\eta + \mathcal{L}^\eta$, where $g^\eta : \mathbb{R}^{3\times3}_{\mathrm{sym}} \to \mathbb{R}^{3\times3}_{\mathrm{sym}}$ is a monotone field and $\mathcal{L}^\eta : \mathbb{R}^{3\times3}_{\mathrm{sym}} \to \mathbb{R}^{3\times3}_{\mathrm{sym}}$ is a global Lipschitz operator. The class $\mathcal{LM}$ in the theory of inelastic deformations of metals was defined by Chełmiński and Gwiazda in [8]. Intuitively it is clear that the modified flow rule $(4.1)_5$ belongs to $\mathcal{LM}$: for large values of $|T_E|$ the monotone new term dominates and on bounded sets the derivative of this vector field cannot blow up.

Theorem 4.2 (global existence for the approximated system)
Let us assume that the given data possesses the regularity as in (2.5)*–*(2.10)*. Moreover, suppose that the initial data is chosen such that for all $\eta > 0$ the initial value $g^\eta(2\mu(\varepsilon^\eta(0) - \varepsilon^{p,0}) + \lambda\mathrm{tr}(\varepsilon^\eta(0) - \varepsilon^{p,0})1\!\!1)$ belongs to $L^2(\Omega; \mathbb{R}^{3\times3}_{\mathrm{sym}})$. Then for all $\eta > 0$, the system* (4.1) *with initial-boundary conditions* (2.3)*–*(2.4) *possesses a global in time, unique solution $(u^\eta, \varepsilon^{p,\eta}, p^\eta, A^\eta)$ with the regularity: for all times $T > 0$*

$$(u^\eta, \varepsilon^{p,\eta}, A^\eta) \in W^{1,\infty}(0, T; H^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^{3\times3}_{\mathrm{sym}}) \times H^2(\Omega; \mathfrak{so}(3))) \quad \textit{and} \quad p^\eta \in H^1(0, T; H^1(\Omega; \mathbb{R})).$$

Proof: The assumption on $g^\eta(2\mu(\varepsilon^\eta(0) - \varepsilon^{p,0}) + \lambda\mathrm{tr}(\varepsilon^\eta(0) - \varepsilon^{p,0})1\!\!1)$ means that the initial value of $T^\eta_E(0)$ belongs to the domain of the maximal monotone operator g^η in the space $L^2(\Omega; \mathbb{R}^{3\times3}_{\mathrm{sym}})$.

Let us fix $\eta > 0$. We use the Yosida approximation (superscript $\nu > 0$) for the monotone part g^η of the constitutive vector field G^η in order to get Lipschitz nonlinearities only. Hence, we obtain the following system of equations

$$\begin{aligned} \mathrm{div}\sigma^{\eta,\nu}(x,t) - \nabla p^{\eta,\nu}(x,t) &= -F(x,t)\,, \\ c\Delta p^{\eta,\nu}(x,t) - \mathrm{div}u^{\eta,\nu}_t(x,t) &= f(x,t)\,, \\ -l_c\Delta\mathrm{axl}(A^{\eta,\nu}(x,t)) &= \mu_c\mathrm{axl}\Big(\mathrm{skew}(\nabla u^{\eta,\nu}(x,t)) - A^{\eta,\nu}(x,t)\Big)\,, \\ \varepsilon^{p,\eta,\nu}_t(x,t) &= G^{\eta,\nu}(T^{\eta,\nu}_E(x,t))\,, \end{aligned} \tag{4.3}$$

where $G^{\eta,\nu} = g^{\eta,\nu} + \mathcal{L}^\eta$, $g^{\eta,\nu} = \nu^{-1}(I - J_\nu)$ and $J_\nu = (I + \nu g^\eta)^{-1}$ is the resolvent of the operator g^η. The system (4.3) is considered with the same boundary and initial conditions as the system (4.1).

The next step in the proof of Theorem 4.2 consists of existence and uniqueness of global in time strong solutions for the system (4.3) with the Lipschitz nonlinearity. We formulate it as a theorem in its own right. We will drop the superscript $\eta > 0$ and write u^ν, $\varepsilon^{p,\nu}$, p^ν, A^ν instead of $u^{\eta,\nu}$, $\varepsilon^{p,\eta,\nu}$, $p^{\eta,\nu}$, $A^{\eta,\nu}$.

Theorem 4.3 (global existence for Lipschitz nonlinearities)
Assume that the given data has the following regularity

$$F \in C^1((0,T]; L^2(\Omega; \mathbb{R}^3)),\ F_{tt} \in L^2(0,T; L^2(\Omega; \mathbb{R}^3)), \quad f \in H^1([0,T]; L^2(\Omega; \mathbb{R})),$$

$$g_D \in C^1([0,T]; H^{\frac12}(\partial\Omega; \mathbb{R}^3)),\ \partial_{tt} g_D \in L^2(0,T; H^{\frac12}(\partial\Omega; \mathbb{R}^3))\,,$$

$$A_D \in C^1([0,T]; H^{\frac32}(\partial\Omega; \mathfrak{so}(3))),\ \partial_{tt} A_D \in L^2(0,T; H^{\frac32}(\partial\Omega; \mathfrak{so}(3)))\,,$$

$$g_P \in H^1(0,T; H^{\frac32}(\partial\Omega; \mathbb{R}))$$

and let the initial data satisfy (2.8) *and* (2.10)*. Additionally suppose that the initial data $\partial_t g_D(0)$ and $f(0)$ satisfy some compatibility condition, which will be specified in the remark below this theorem. Then, for all $\nu > 0$ the approximate problem* (4.3) *has a global in time, unique solution $(u^\nu, \varepsilon^{p,\nu}, p^\nu, A^\nu)$ with regularity*

$$(u^\nu, A^\nu, \varepsilon^{p,\nu}) \in C([0,T]; H^1(\Omega; \mathbb{R}^3) \times H^2(\Omega; \mathfrak{so}(3)) \times L^2(\Omega; \mathbb{R}^{3\times3}_{\mathrm{sym}})),$$

$$(u^\nu_t, A^\nu_t) \in L^\infty(0,T; H^1(\Omega; \mathbb{R}^3) \times H^2(\Omega; \mathfrak{so}(3))),\ \varepsilon^{p,\nu}_t \in C([0,T]; L^2(\Omega; \mathbb{R}^{3\times3}_{\mathrm{sym}})),$$

$p^\nu \in H^1(0,T;H^1(\Omega;\mathbb{R}))$.

P r o o f. The idea of the proof of Theorem 4.3 is the iteration between the Biot poroelasticity model coupled with the elliptic equation for the microrotation and the global Lipschitz evolution equation for inelastic strain $\varepsilon^{p,\nu}$. Using the Banach fixed point theorem we get a unique solution of the system (4.3) with the regularity required in the statement of the Theorem. Here we want to give a sketch of the proof. For a fixed $T>0$ we construct an operator $P: C([0,T];L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})) \to C([0,T];L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}}))$ as follows: for $\varepsilon^\nu \in C([0,T];L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}}))$ let us consider the equation

$$\varepsilon^{p,\nu}(t) = \varepsilon^{p,0} + \int_0^t G^{\eta,\nu}\Big(2\mu(\varepsilon^\nu - \varepsilon^{p,\nu}) + \lambda \mathrm{tr}(\varepsilon^\nu - \varepsilon^{p,\nu})1\!\!1\Big)d\tau. \tag{4.4}$$

$G^{\eta,\nu}$ is global Lipschitz operator so the equation above possesses global in time, unique solution $\varepsilon^{p,\nu} \in C^1([0,T];L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}}))$. Having $\varepsilon^{p,\nu}$ we study the following system of equations

$$\begin{aligned}
&\mathrm{div}\sigma^\nu - \nabla p^\nu = F\,,\\
&\qquad\sigma^\nu = 2\mu(\varepsilon^\nu - \varepsilon^{p,\nu}) + 2\mu_c(\mathrm{skew}(\nabla u^\nu) - A^\nu) + \lambda\mathrm{tr}(\varepsilon^\nu - \varepsilon^{p,\nu})1\!\!1\,,\\
&c\Delta p^\nu - \mathrm{div} u^\nu_t = f\,,\\
&-l_c\Delta \mathrm{axl}(A^\nu) = \mu_c \mathrm{axl}(\mathrm{skew}(\nabla u^\nu - A^\nu)\,,
\end{aligned} \tag{4.5}$$

with boundary conditions and initial conditions as in the system (4.1). The system (4.5) is the Biot poroelasticity model coupled with the elliptic equation for the microrotation, which was considered in the last section (the unknown functions are (u^ν,p^ν,A^ν)). The results in the last section imply that there exists a unique solution with the regularity $u^\nu \in C([0,T];H^1(\Omega;\mathbb{R}^3))$, $p^\nu \in C([0,T];H^1(\Omega;\mathbb{R}))$ and $A^\nu \in C([0,T];H^2(\Omega;\mathfrak{so}))$. By careful analysis of (4.5) we obtain the inequality

$$\|P(\varepsilon^\nu_1)(t) - P(\varepsilon^\nu_2)(t)\|_{L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})} \le C\|\varepsilon^{p,\nu}_1(t) - \varepsilon^{p,\nu}_2(t)\|_{L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})},$$

where $\varepsilon^{p,\nu}_1(t)$ and $\varepsilon^{p,\nu}_2(t)$ are solutions of (4.5) with the input functions $\varepsilon^\nu_1, \varepsilon^\nu_2 \in C([0,T];L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}}))$, respectively and the positive constant C does not depend on these input functions and is independent of t. From (4.4) it is not difficult to get the following inequality

$$\|\varepsilon^{p,\nu}_1(t) - \varepsilon^{p,\nu}_2(t)\|_{L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})} \le \tilde{C}\|\varepsilon^\nu_1(t) - \varepsilon^\nu_2(t)\|_{L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})},$$

where $\tilde{C}$ does not depend on t (it depends only on the Lipschitz constant and on time T). These two inequalities imply that the operator P is a contraction (for similar results we refer to [23]). From the definition of the operator P and Theorem 3.2 we conclude that the solution $(u^\nu, p^\nu, \varepsilon^{p,\nu}, A^\nu)$ has the regularity required in the statement of the Theorem. ☐

Remark The system (4.5) is considered with non-homogeneous Dirichlet boundary conditions. Now the compatibility condition has the form

$$\int_{\partial\Omega} \partial_t g_D(0) n dS = \int_\Omega \Big(c\Delta p^\nu(0) - f(0)\Big)dx,$$

where $p^\nu(0)$ is the solution of system (4.5) for $t=0$.

Let us continue with the proof of Theorem 4.2. We want to pass to the limit in the Yosida approximated system (4.3) with $\nu \to 0^+$. Thus, we need some boundedness (independent of ν) of the solution $(u^\nu, \varepsilon^{p,\nu}, p^\nu, A^\nu)$ and its time derivatives.

From the assumption on the given boundary data g_P, there exists a function $p^* \in W^{1,\infty}(0,T;H^2(\Omega;\mathbb{R}))$ such that $p^*_{|\partial\Omega} = g_{P_{|\partial\Omega}}$. The system (4.3) can now be written in the form:

$$\begin{aligned}
&\mathrm{div}\sigma^\nu(x,t) - \nabla\Big(p^\nu(x,t) - p^*(x,t)\Big) = -F(x,t) + \nabla p^*(x,t)\,,\\
&c\Delta\Big(p^\nu(x,t) - p^*(x,t)\Big) - \mathrm{div}u^\nu_t(x,t) = f(x,t) - c\Delta p^*(x,t)\,,\\
&\qquad -l_c\Delta\mathrm{axl}(A^\nu(x,t)) = \mu_c\mathrm{axl}\Big(\mathrm{skew}(\nabla u^\nu(x,t)) - A^\nu(x,t)\Big)\,,\\
&\qquad\qquad \varepsilon^{p,\nu}_t(x,t) = G^\nu(T^\nu_E(x,t))\,.
\end{aligned}$$

Lemma 4.4 ($L^\infty(L^2)$ estimates for time derivatives)
Let us suppose that all hypotheses of Theorem 4.2 hold. Then for all $T > 0$ and $t \le T$ the solution $(u^\nu, \varepsilon^{p,\nu}, p^\nu, A^\nu)$ satisfies the following inequality

$$\mathcal{E}^\eta(\varepsilon_t^\nu, \varepsilon_t^{p,\nu}, A_t^\nu)(t) + \int_0^t \|\nabla\Big(p_t^\nu(\tau) - p_t^*(\tau)\Big)\|^2_{L^2(\Omega;\mathbb{R})} d\tau \le C(T),$$

where the constant $C(T)$ does not depend on ν.

Lemma 4.5 (strong convergence of the stresses)
Let us suppose that all hypotheses of Theorem 4.2 hold. Then for arbitrary two approximation steps ν, $\mu > 0$ and for all $T > 0$ the inequality

$$\mathcal{E}^\eta(\varepsilon^\nu - \varepsilon^\mu, \varepsilon^{p,\nu} - \varepsilon^{p,\mu}, A^\nu - A^\mu)(t) + \int_0^t \|\nabla\Big(p^\nu(\tau) - p^\mu(\tau)\Big)\|^2_{L^2(\Omega;\mathbb{R})} d\tau \le \frac{1}{2}(\nu + \mu)C(T),$$

holds for $t \le T$, where $C(T)$ is the constant from Lemma 4.4.

For the similar proof of Lemma 4.4 the reader may consult the article [9], where an analogous estimate is shown for the quasi-static elasto-plastic Cosserat model. The estimate from Lemma 4.5 is a standard one in the theory of maximal monotone operators and for the details we refer to [1, 9]. Therefore, the proofs of Lemma 4.4 and Lemma 4.5 can be omitted.

Lemma 4.4 and coercivity of the approximate energy yield that the sequence $(\sigma_t^\nu, \eta\varepsilon_t^\nu, \varepsilon_t^{p,\nu}, \nabla p^\nu, \nabla A_t^\nu)$ is $L^2(L^2)$-bounded. This implies that the sequence $(\sigma^\nu, \eta\varepsilon^\nu, \varepsilon^{p,\nu}, \nabla p^\nu, \nabla A^\nu)$ is $L^\infty(L^2)$-bounded. Hence, there exist the weak-$*$-limits (after passing to a subsequence if needed) satisfying the following system of equations

$$\begin{aligned}
\mathrm{div}\sigma^\eta - \nabla p^\eta &= -F\,,\\
\sigma^\eta &= 2\mu(\varepsilon^\eta - \varepsilon^{p,\eta}) + 2\mu_c(\mathrm{skew}(\nabla u^\eta) - A^\eta) + \lambda\mathrm{tr}(\varepsilon^\eta - \varepsilon^{p,\eta} + \eta\varepsilon^\eta)\mathbb{1}\,,\\
c\Delta p^\eta - \mathrm{div}u_t^\eta &= f\,,\\
-l_c\Delta\mathrm{axl}(A^\eta) &= \mu_c\mathrm{axl}(\mathrm{skew}(\nabla u^\eta) - A^\eta)\,,\\
\varepsilon_t^{p,\eta} &= \mathrm{w} - \lim_{\nu\to 0^+} G^{\eta,\nu}(T_E^{\eta,\nu}) = \chi\,.
\end{aligned}$$

From Lemma 4.5 we conclude that the sequence $\{T_E^{\eta,\nu}\}$ is a Cauchy sequence in the space $L^\infty(0,T;L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}}))$. From the definition of the Yosida approximation we get $g^{\eta,\nu}(T_E^{\eta,\nu}) = g^\eta(J_\nu(T_E^{\eta,\nu}))$. We know that the sequence $(J_\nu(T_E^{\eta,\nu}), g^{\eta,\nu}(T_E^{\eta,\nu}))$ is contained in the graph of g^η and converges strongly-weakly to (T_E^η, g^η) (note that J_ν is a globally Lipschitz operator). From the strong-weak closedness of the graph of the maximal monotone operator g^η we have $\mathrm{w} - \lim_{\nu\to 0^+} g^{\eta,\nu}(T_E^{\eta,\nu}) = g^\eta(T_E^\eta)$. The properties of G^η (G^η can be written as the sum of maximal monotone operator and a Lipschitz operator) implies that $\chi = G^\eta(T_E^\eta)$. This finishes the proof of Theorem 4.2. □

5 Energy estimates independent of η

We would like to obtain some estimates for the sequence $\{u^\eta, \varepsilon^{p,\eta}, p^\eta, A^\eta\}$ in order to pass to the weak limits in (4.1). This will be the main part in the proof of Theorem 2.2.

Again using the assumption $g_P \in W^{1,\infty}(0,T;H^{\frac{3}{2}}(\partial\Omega;\mathbb{R}))$ we rewrite the system (4.1) in the form

$$\begin{aligned}
\mathrm{div}\sigma^\eta - \nabla(p^\eta - p^*) &= -F + \nabla p^*\,,\\
c\Delta(p^\eta - p^*) - \mathrm{div}u_t^\eta &= f - c\Delta p^*\,,\\
-l_c\Delta\mathrm{axl}(A^\eta) &= \mu_c\mathrm{axl}(\mathrm{skew}(\nabla u^\eta) - A^\eta)\,,\\
\varepsilon_t^{p,\eta} &= G^\eta(T_E^\eta)\,.
\end{aligned} \tag{5.1}$$

Theorem 5.1 (Energy estimate)
Let us suppose that all hypotheses of Definition 2.1 *hold. We also assume that the number* β *satisfies* $\beta > r' > r = \alpha \deg(Y)(\deg(Y) - 1) > 1$. *Then for all* $T > 0$ *the following estimate*

$$\mathcal{E}^\eta(u^\eta, \varepsilon^\eta, \varepsilon^{p,\eta}, A^\eta)(t) + \eta \int_0^t \int_\Omega |T_E^\eta|^{\beta+1}\, dx d\tau + \int_0^t \int_\Omega \mathcal{F}(Y(T_E^\eta))\, \frac{\partial P}{\partial T}(T_E^\eta) T_E^\eta \, dx d\tau$$
$$+ c \int_0^t \int_\Omega |\nabla(p^\eta - p^*)|^2 \, dx d\tau \le C(T)$$

holds and $C(T)$ *does not depend on* $\eta > 0$ *(it depends only on the domain and the data).*

P r o o f. Compute the time derivative

$$\begin{aligned}\frac{d}{dt}\Big(\mathcal{E}^\eta(u^\eta, \varepsilon^\eta, \varepsilon^{p,\eta}, A^\eta)(t)\Big) &= \int_\Omega \Big(2\mu(\varepsilon^\eta - \varepsilon^{p,\eta})(\varepsilon_t^\eta - \varepsilon_t^{p,\eta}) + \lambda \mathrm{tr}(\varepsilon^\eta - \varepsilon^{p,\eta})\mathrm{tr}(\varepsilon_t^\eta - \varepsilon_t^{p,\eta}) \\ &\quad + \lambda\eta \mathrm{tr}(\varepsilon^\eta)\mathrm{tr}(\varepsilon_t^\eta) + 2\mu_c(\mathrm{skew}(\nabla u^\eta) - A^\eta)(\mathrm{skew}(\nabla u_t^\eta) - A_t^\eta)\Big)\, dx \\ &\quad + 4 l_c \int_\Omega \nabla \mathrm{axl}(A^\eta) \nabla \mathrm{axl}(A_t^\eta)\, dx \\ &= \int_\Omega \sigma^\eta \nabla u_t^\eta \, dx - \int_\Omega T_E^\eta \varepsilon_t^{p,\eta}\, dx - 2\int_\Omega \mu_c(\mathrm{skew}(\nabla u^\eta) - A^\eta) A_t^\eta \, dx \\ &\quad + 4 l_c \int_\Omega \nabla \mathrm{axl}(A^\eta) \nabla \mathrm{axl}(A_t^\eta)\, dx. \end{aligned} \tag{5.2}$$

Integrating by parts in the first and last term on the right hand side of (5.2), using the first and third equation in the system (5.1) we have (notice that $\|A\|^2 = 2\|\mathrm{axl} A\|^2$)

$$\begin{aligned}\frac{d}{dt}\Big(\mathcal{E}^\eta(u^\eta, \varepsilon^\eta, \varepsilon^{p,\eta}, A^\eta)(t)\Big) &= \int_\Omega (F - \nabla p^*) u_t^\eta \, dx - \int_\Omega (\nabla p^\eta - \nabla p^*) u_t^\eta \, dx \\ &\quad + \int_{\partial\Omega} \sigma^\eta n \, u_t^\eta \, dS - \int_\Omega T_E^\eta \varepsilon_t^{p,\eta}\, dx + 4 l_c \int_{\partial\Omega} \nabla \mathrm{axl}(A^\eta) n \, \mathrm{axl}(A_t^\eta)\, dS.\end{aligned} \tag{5.3}$$

Again, integrating by parts in the second term on the right hand side of (5.3), using the second equation of the system (5.1) and the boundary conditions we finally obtain

$$\begin{aligned}\frac{d}{dt}(\mathcal{E}^\eta(u^\eta, \varepsilon^\eta, \varepsilon^{p,\eta}, A^\eta)(t)) &= \int_\Omega (F - \nabla p^*) u_t^\eta \, dx - \int_\Omega (f - c\Delta p^*)(p^\eta - p^*) dx \\ &\quad - c\int_\Omega |\nabla p^\eta - \nabla p^*|^2 \, dx + \int_{\partial\Omega} \Big(\sigma^\eta - c \mathbb{1} \cdot (p^\eta - p^*)\Big) n \, g_{D,t}\, dS \\ &\quad - \int_\Omega T_E^\eta \varepsilon_t^{p,\eta}\, dx + 4 l_c \int_{\partial\Omega} (\nabla \mathrm{axl}(A^\eta)) n \, \mathrm{axl}(A_{D,t})\, dS.\end{aligned} \tag{5.4}$$

Now we integrate (5.4) with respect to time and have

$$\mathcal{E}^\eta(u^\eta,\varepsilon^\eta,\varepsilon^{p,\eta},A^\eta)(t)+\eta\int_0^t\int_\Omega |T_E^\eta|^{\beta+1}\,dxd\tau+\int_0^t\int_\Omega \mathcal{F}(Y(T_E^\eta))\,\frac{\partial P}{\partial T}(T_E^\eta)T_E^\eta\,dxd\tau$$
$$+\,c\int_0^t\int_\Omega |\nabla p^\eta-\nabla p^*|^2\,dxd\tau=\mathcal{E}^\eta(u^\eta,\varepsilon^\eta,\varepsilon^{p,\eta},A^\eta)(0)+\int_0^t\int_\Omega (F-\nabla p^*)u_t^\eta\,dxd\tau$$
$$+\int_0^t\int_{\partial\Omega}\Big(\sigma^\eta-c\mathbb{1}\cdot(p^\eta-p^*)\Big)n\,g_{D,t}\,dSd\tau-\int_0^t\int_\Omega (f-c\Delta p^*)(p^\eta-p^*)\,dxd\tau$$
$$+\,4l_c\int_0^t\int_{\partial\Omega}(\nabla\mathrm{axl}(A^\eta))n\,\mathrm{axl}(A_{D,t})\,dSd\tau. \tag{5.5}$$

The initial values $u^\eta(0)$, $\varepsilon^\eta(0)$, $A^\eta(0)$ solve the following elliptic boundary-value problem

$$\begin{aligned}
\mathrm{div}_x\sigma^\eta(0)-\nabla_x p^\eta(0)&=-F(0),\\
\mathrm{div}_x u^\eta(0)&=\mathrm{div}u^0,\\
-l_c\Delta_x\mathrm{axl}(A^\eta(0))&=\mu_c\mathrm{axl}(\mathrm{skew}(\nabla_x u^\eta(0))-A^\eta(0)),\\
u^\eta(0)_{|\partial\Omega}=g_D(0),p^\eta(0)_{|\partial\Omega}&=g_P(0),\quad A^\eta(0)_{|\partial\Omega}=A_D(0).
\end{aligned}$$

Notice that the system above is the Stokes problem coupled with a linear elliptic system for the microrotation. From the Lax-Milgram theorem and the regularity on the initial data we obtain a unique solution $u^\eta(0)\in H^1(\Omega;\mathbb{R}^3)$, $\varepsilon^\eta(0)\in L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})$, $A^\eta(0)\in H^2(\Omega;\mathfrak{so}(3))$ and this solution satisfies the following inequality

$$\|u^\eta(0)\|_{H^1(\Omega;\mathbb{R}^3)}+\eta\|\mathrm{div}u^\eta(0)\|_{L^2(\Omega;\mathbb{R})}+\|A^\eta(0)\|_{H^1(\Omega;\mathfrak{so}(3))}\le C(\Omega)\Big(\|F(0)\|_{L^2(\Omega;\mathbb{R}^3)}$$
$$+\|\mathrm{div}\mathcal{D}(\varepsilon^{p,0})\|_{L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})}+\|g_D(0)\|_{H^{\frac12}(\partial\Omega;\mathbb{R}^3)}+\|A_D(0)\|_{H^{\frac12}(\partial\Omega;\mathfrak{so}(3))}+\|\mathrm{div}u^0\|_{H^1(\Omega;\mathbb{R})}\Big).$$

Thus, the energy $\mathcal{E}^\eta(u^\eta,\varepsilon^\eta,\varepsilon^{p,\eta},A^\eta)(0)$ is bounded by a constant independent of η.

Next, we estimate all integral terms from the right hand side of (5.5).

$$\int_0^t\int_\Omega (F-\nabla p^*)\,u_t^\eta\,dxd\tau=-\int_0^t\int_\Omega (F-\nabla p^*)_t\,u^\eta\,dxd\tau+\int_\Omega (F-\nabla p^*)u^\eta\,dx$$
$$-\int_\Omega (F-\nabla p^*)(0)u^\eta(0)\,dx. \tag{5.6}$$

From the regularity of the functions $(F-\nabla p^*)$ and $u^\eta(0)$ we conclude that the last integral of (5.6) is bounded. From Hölder inequality we get

$$\int_0^t\int_\Omega (F-\nabla p^*)_t\,u^\eta\,dxd\tau\le\int_0^t\|(F-\nabla p^*)_t\|_{L^3(\Omega;\mathbb{R}^3)}\|u^\eta\|_{L^{\frac32}(\Omega;\mathbb{R}^3)}d\tau. \tag{5.7}$$

Let us define the space $LD(\Omega;\mathbb{R}^3)=\{u\in L^1(\Omega;\mathbb{R}^3):\ \varepsilon(u)\in L^1(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})\}$. It is well known that $LD(\Omega;\mathbb{R}^3)$ is continuously embedded in $L^{\frac32}(\Omega;\mathbb{R}^3)$ (see for example [26]) and

$$\int_0^t\int_\Omega F_t u^\eta\,dxd\tau\le C(\Omega)\int_0^t\|F_t\|_{L^3(\Omega;\mathbb{R}^3)}\Big(\|\varepsilon^\eta\|_{L^1(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})}+\int_{\partial\Omega}|g_D|\,dS\Big)\,d\tau$$
$$\le C(\Omega,\beta)\Big(1+\int_0^t\|F_t\|_{L^3(\Omega;\mathbb{R}^3)}\|\varepsilon^\eta\|_{L^{1+\frac1\beta}(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})}\,d\tau\Big)$$

$$\leq C(\Omega,\beta)\Big(1+\int_0^t \|F_t\|_{L^3(\Omega;\mathbb{R}^3)}\Big(\|\varepsilon^\eta-\varepsilon^{p,\eta}\|_{L^2(\Omega;\mathbb{R}^{3\times3}_{\rm sym})}+\|\varepsilon^{p,\eta}\|_{L^{1+\frac{1}{\beta}}(\Omega;\mathbb{R}^{3\times3}_{\rm sym})}\Big)\,d\tau\Big). \tag{5.8}$$

Applying Cauchy and Young inequalities we obtain

$$\int_0^t\int_\Omega F_t u^\eta\,dxd\tau \leq C(\Omega,\beta,\alpha)+C(\Omega,\beta)\Big(\frac{1}{2}\int_0^t \|T_E^\eta\|^2_{L^2(\Omega;\mathbb{R}^{3\times3}_{\rm sym})}\,d\tau+\alpha\int_0^t \|\varepsilon^{p,\eta}\|^{1+\frac{1}{\beta}}_{L^{1+\frac{1}{\beta}}(\Omega;\mathbb{R}^{3\times3}_{\rm sym})}\,d\tau\Big), \tag{5.9}$$

where $\alpha>0$ is a positive number and $C(\Omega,\beta,\alpha)>0$ does not depend on η. From the above calculations we also have the following inequality

$$\int_\Omega (F-\nabla p^*)\,u^\eta\,dx \leq C(\Omega,\beta,\alpha)+C(\Omega,\beta)\Big(\alpha\|T_E^\eta\|^2_{L^2(\Omega;\mathbb{R}^{3\times3}_{\rm sym})}+\alpha\int_0^t \|\varepsilon_t^{p,\eta}\|^{1+\frac{1}{\beta}}_{L^{1+\frac{1}{\beta}}(\Omega;\mathbb{R}^{3\times3}_{\rm sym})}\,d\tau\Big)\,. \tag{5.10}$$

The inelastic constitutive equation gives us additionally the following information
(notice that $1+\frac{1}{\beta}<1+\frac{1}{r}$)

$$\int_0^t \|\varepsilon_t^{p,\eta}\|^{1+\frac{1}{\beta}}_{L^{1+\frac{1}{\beta}}(\Omega;\mathbb{R}^{3\times3}_{\rm sym})}\,d\tau \tag{5.11}$$
$$\leq C(\beta)\eta\int_0^t \|T_E^\eta\|^{\beta+1}_{L^{\beta+1}(\Omega;\mathbb{R}^{3\times3}_{\rm sym})}\,d\tau + C(\Omega,\beta,r)\int_0^t \Big\|\mathcal{F}(Y(T_E^\eta))\,\frac{\partial P}{\partial T}(T_E^\eta)\Big\|^{1+\frac{1}{r}}_{L^{1+\frac{1}{r}}(\Omega;\mathbb{R}^{3\times3}_{\rm sym})}\,d\tau.$$

Now we estimate the appearing boundary integrals

$$\int_0^t\int_{\partial\Omega}\Big(\sigma^\eta-c\mathbb{1}\cdot(p^\eta-p^*)\Big)n\,g_{D,t}\,dSd\tau$$
$$\leq \int_0^t \Big\|\Big(\sigma^\eta-c\mathbb{1}\cdot(p^\eta-p^*)\Big)n\Big\|_{H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^3)}\|g_{D,t}\|_{H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^3)}\,d\tau$$
$$\leq \quad \text{(from the trace theorem in the space } L^2_{\rm div}(\Omega) \text{ see for example in [26])} \quad \leq$$
$$\leq C\int_0^t \Big(\|\sigma^\eta-c\mathbb{1}\cdot(p^\eta-p^*)\|^2_{L^2(\Omega;\mathbb{R}^{3\times3}_{\rm sym})}$$
$$+\Big\|{\rm div}\Big(\sigma^\eta-c\mathbb{1}\cdot(p^\eta-p^*)\Big)\Big\|^2_{L^2(\Omega;\mathbb{R}^3)}\Big)\|g_{D,t}\|_{H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^3)}\,d\tau$$
$$\leq C\int_0^t \|F-\nabla p^*\|^2_{L^2(\Omega;\mathbb{R}^{3\times3}_{\rm sym})}\|g_{D,t}\|_{H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^3)}\,d\tau+\alpha C\int_0^t (\|\sigma^\eta-c\mathbb{1}\cdot(p^\eta-p^*)\|^2_{L^2(\Omega;\mathbb{R}^{3\times3}_{\rm sym})}\,d\tau$$
$$+C(\alpha)\int_0^t \|g_{D,t}\|_{H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^3)}\,d\tau, \tag{5.12}$$

where $\alpha>0$ is a positive number and $C>0$ does not depend on η. We use the H^2 regularity of the microrotations to estimate the following integral

$$\int_0^t \int_{\partial\Omega} (\nabla \operatorname{axl}(A^\eta))n \operatorname{axl}(A_{D,t})\, dS d\tau \le \int_0^t \|\nabla \operatorname{axl}(A^\eta))n\|_{H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^3)} \|\operatorname{axl}(A_{D,t})\|_{H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^3)}\, d\tau$$

$$\le C \int_0^t \Big(\|\nabla \operatorname{axl}(A^\eta)\|^2_{L^2(\Omega;\mathbb{R}^9)} + \|\Delta \operatorname{axl}(A^\eta)\|^2_{L^2(\Omega;\mathbb{R}^3)} \Big) \|\operatorname{axl}(A_{D,t})\|_{H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^3)}\, d\tau$$

$$= C \int_0^t \Big(\|\nabla \operatorname{axl}(A^\eta)\|^2_{L^2(\Omega;\mathbb{R}^9)} + \frac{\mu_c}{l_c} \|\operatorname{skew}(\nabla u^\eta) - A^\eta\|^2_{L^2(\Omega;\mathbb{R}^9)} \Big) \|\operatorname{axl}(A_{D,t})\|_{H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^3)}\, d\tau$$

$$\le C\alpha \int_0^t \Big(\|\nabla \operatorname{axl}(A^\eta)\|^2_{L^2(\Omega;\mathbb{R}^9)} + \|\operatorname{skew}(\nabla u^\eta) - A^\eta\|^2_{L^2(\Omega;\mathbb{R}^9)} \Big)\, d\tau$$

$$+ C(\alpha) \int_0^t \|\operatorname{axl}(A_{D,t})\|_{H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^3)}\, d\tau. \tag{5.13}$$

Let us denote by L the left hand side of the inequality in Theorem 5.1, which we are going to prove. Using (5.7)–(5.13) we obtain the inequality

$$L \stackrel{\text{def}}{=} \mathcal{E}^\eta(u^\eta, \varepsilon^\eta, \varepsilon^{p,\eta}, A^\eta)(t) + \eta \int_0^t \int_\Omega |T_E^\eta|^{\beta+1}\, dx d\tau + \int_0^t \int_\Omega \mathcal{F}(Y(T_E^\eta))\, \frac{\partial P}{\partial T}(T_E^\eta) T_E^\eta\, dx d\tau$$

$$+ c \int_0^t \int_\Omega |\nabla(p^\eta - p^*)|^2\, dx d\tau$$

$$\le C(\Omega,\beta,T,\alpha) + C(\Omega,\beta) \int_0^t \|T_E^\eta\|^2_{L^2(\Omega;\mathbb{R}^{3\times3}_{\text{sym}})}\, d\tau + \alpha C(\Omega) c \int_0^t \int_\Omega |\nabla(p^\eta - p^*)|^2\, dx d\tau$$

$$+ \alpha C(\Omega,\beta) \|T_E^\eta\|^2_{L^2(\Omega;\mathbb{R}^{3\times3}_{\text{sym}})} + \alpha C(\beta)\eta \int_0^t \|T_E^\eta\|^{\beta+1}_{L^{\beta+1}(\Omega;\mathbb{R}^{3\times3}_{\text{sym}})}\, d\tau$$

$$+ C(\beta,r)\alpha \int_0^t \|\mathcal{F}(Y(T_E^\eta))\, \frac{\partial P}{\partial T}(T_E^\eta)\|^{1+\frac{1}{r}}_{L^{1+\frac{1}{r}}(\Omega;\mathbb{R}^{3\times3}_{\text{sym}})}\, d\tau + \alpha C \int_0^t (\|\sigma^\eta - c\mathbb{1}\cdot(p^\eta - p^*)\|^2_{L^2(\Omega;\mathbb{R}^{3\times3}_{\text{sym}})}\, d\tau$$

$$+ C\alpha \int_0^t (\|\nabla \operatorname{axl}(A^\eta)\|^2_{L^2(\Omega;\mathbb{R}^9)} + \|\operatorname{skew}(\nabla u^\eta) - A^\eta\|^2_{L^2(\Omega;\mathbb{R}^3)})\, d\tau, \tag{5.14}$$

which is satisfied for all numbers $\alpha > 0$ and the constant $C(\Omega,\beta,T,\alpha) > 0$ does not depend on $\eta > 0$. Now we choose $\alpha > 0$ so small that the following inequality

$$L \le C(\Omega,T,\beta) + C(\Omega,\beta) \int_0^t \|T_E^\eta\|^2_{L^2(\Omega;\mathbb{R}^{3\times3}_{\text{sym}})}\, d\tau \tag{5.15}$$

holds for $T_E^\eta \in \{|T| > D\}$ (this follows from the fact that the functions P and Y are comparable). For $T_E^\eta \in \{|T| \le D\}$ the sixth term on the right hand side of (5.14) is bounded independently of $\eta > 0$. Finally we thus arrive at the inequality

$$L \le C(\Omega,T,\beta) + C(\Omega,\beta) \int_0^t \|T_E^\eta\|^2_{L^2(\Omega;\mathbb{R}^{3\times3}_{\text{sym}})}\, d\tau \le C(\Omega,T,\beta) + C(\Omega,\beta) \int_0^t L(\tau)\, d\tau\,. \tag{5.16}$$

The Gronwall inequality completes the proof. □

6 Proof of the Theorem 2.2: the limit $\eta \to 0^+$

The reasoning from the last section gives us that the sequence:

- $\{\sigma^\eta, A^\eta, p^\eta\}_{\eta>0}$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}}) \times H^1(\Omega;\mathfrak{so}(3))) \times L^2(0,T;H^1(\Omega;\mathbb{R}))$.
- $\{\sqrt{\eta}\,\mathrm{tr}(\varepsilon(u^\eta))\}_{\eta>0}$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}}))$.
- $\eta|T^\eta_E|^{\beta+1}$ is bounded in $L^1(0,T;L^1(\Omega;\mathbb{R}))$.
- $\{\varepsilon^{p,\eta}_t, \varepsilon^\eta\}$ is bounded in $L^{1+\frac{1}{\beta}}(0,T;L^{1+\frac{1}{\beta}}(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}}) \times L^{1+\frac{1}{\beta}}(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}}))$.

The first three statements follow directly from Theorem 5.1 and the last statement is a consequence of the structure of the inelastic constitutive equation. Therefore, the sequence $\{\mathrm{div}u^\eta = \mathrm{tr}(\varepsilon(u^\eta))\}_{\eta>0}$ is bounded in $L^{1+\frac{1}{\beta}}(0,T;L^{1+\frac{1}{\beta}}(\Omega;\mathbb{R}))$.

Using higher integrability of the sequence $\{\varepsilon^{p,\eta}_t\}_{\eta>0}$ we conclude that $\{\varepsilon^{p,\eta}_t\}_{\eta>0}$ is weakly precompact in $L^1(0,T;L^1(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}}))$. Hence, using that the leading term in $(3.1)_5$ converges to zero in $L^1(0,T;L^1(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}}))$ the sequence

$$\{\mathcal{F}(Y(T^\eta_E))\,\frac{\partial P}{\partial T}(T^\eta_E)\}_{\eta>0}$$

is weakly precompact in $L^1(0,T;L^1(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}}))$. Therefore, there exists a family of Young measures $\nu_{(x,t)}$ (see [2]) generated by the sequence $\{T^\eta_E\}_{\eta>0}$ such that $w-\lim_{\eta\to0^+}\varepsilon^{p,\eta}_t = \hat{\chi}$ is of the form

$$\hat{\chi}(x,t) = \int_{\mathbb{R}^{3\times3}_{\mathrm{sym}}} \mathcal{F}(Y(S))\,\frac{\partial P}{\partial T}(S)d\nu_{(x,t)}(S).$$

Passing to the limit in the system (4.1) with $\eta \to 0^+$ completes the proof. □

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Book Review

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Mathematik! Welch eine Anspruch für ein Lehrwerk. Ist es nicht vermessen, in einem einzigen schwergewichtigen Buch – überdimensioniert für jedes Regalbrett – die gesamte Mathematik darzustellen?

Ja und Nein.

Ja – natürlich ist die moderne Mathematik so umfangreich und zu facettenreich, dass jeder Versuch, sie in ein umfassendes Lehrwerk zu pressen, zum Scheitern verurteilt ist.

Nein – natürlich braucht nicht jeder die gesamte moderne Mathematik, eine breite und intelligent ausgewählte Zusammenstellung vermittelt womöglich einen besseren Überblick als ein enzyklopädisches Werk.

Dieses Werk, verfasst von 6 Kollegen mit langjähriger Erfahrung in der universitären Mathematikausbildung, ist genau so eine Zusammenstellung. Sie hat den Anspruch, alle Grundlagen abzudecken, die Standardkurse der höheren Mathematik an technischen Universitäten umfassen. Ganz gezielt richtet sie sich dabei an Studierende, für die Mathematik ein wichtiges Nebenfach ist, und die Mathematik als unverzichtbares Werkzeug in ihrem wissenschaftlichen Hauptfach benötigen. Dieses Spannungsfeld, Mathematik sowohl als eigenständige wissenschaftliche Disziplin wie auch als wissenschaftliches Werkzeug zu betrachten, wird in einem Eingangskapitel diskutiert.

Es folgt ein einführender Mathematikkurs in rund vierzig übersichtlichen Kapiteln, aufgeteilt in sechs Teile. Die Kapitel werden neben dem klassischen Begriff auch mit einer mehr beschreibenden Wendung überschrieben, etwa „Reihen – Summieren bis zum Letzten“. Nicht alle dieser Titel sind wirklich überzeugend, sie unterstreichen aber das Bemühen den formalen Aufbau möglichst anschaulich zu gestalten. Alle Aussagen werden sauber mathematisch formuliert, und einige wichtige Schlüsselresultate werden vollständig bewiesen. Kompliziertere Sachverhalte, wie der Hauptsatz über implizite Funktionen, werden stattdessen an einer Reihe von klug ausgewählten Beispielen motiviert und erläutert.

Der Text arbeitet auf verschiedenen Ebenen. Der eigentliche Lernstoff enthält alle Begriffe und Definitionen sowie Beispiele und Sätze. Das Zusatzmaterial in farblich abgesetzten Kästen enthält Übersichten zu einem Themenkomplex, und in jedem Kapitel auch Anwendungen. Ergänzungen im Lernstoff werden als Vertiefungen angefügt, zusätzliches Bonusmaterial wird auf einer Webseite bereitgestellt. Jedes Kapitel endet mit einer Zusammenfassung und Übungsaufgaben, die in Verständnisaufgaben, Rechenaufgaben und Anwendungen untergliedert sind. In den Text eingestreut sind Selbstfragen (mit Lösungen am Ende der Kapitel), die zur Kontrolle bei eigenständigem Studium verwendet werden können.

Der letzte Teil behandelt die Grundlagen der Wahrscheinlichkeit und Statistik und umfasst auch eine Einführung in die Schätz- und Testtheorie und die lineare Regression. Damit wird ein extrem wichtiges Anwendungskapitel in diesen Grundkurs integriert, der in keiner Mathematikausbildung fehlen sollte. Leider fehlt ein entsprechender Teil über elementare numerische Methoden, wahrscheinlich weil ansonsten das dicke Buch zu dick würde und in sieben einzelne handliche Bände aufgeteilt werden müsste.

Dieses Lehrwerk, nun in der zweiten korrigierten Auflage erhältlich, ist die ideale Ergänzung zu den einführenden Mathematikkursen für Ingenieure und Naturwissenschaftler. Auch als Nachschlagewerk für Dozenten enthält es eine Reihe von nützlichen Zusammenstellungen, Beispielen und Aufgaben. Die hervorragende farbliche Gestaltung mit einer Fülle von Illustrationen macht das Gesamtwerk zu einer anregenden und lesenswerten Lektüre.

Karlsruhe **Christian Wieners**