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Shear in nonlinear elasticity

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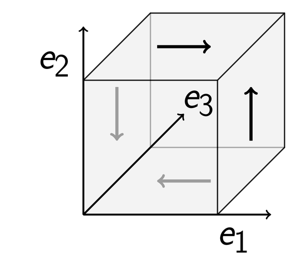
1 Introduction

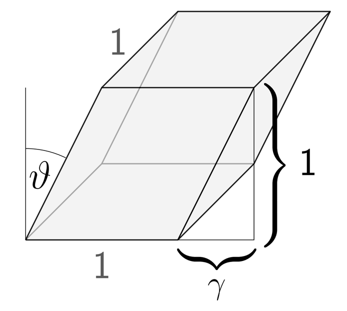
The term **shear** [8] describes a number of closely related but distinct concepts, including the **(pure) shear stress**

$$T^s = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

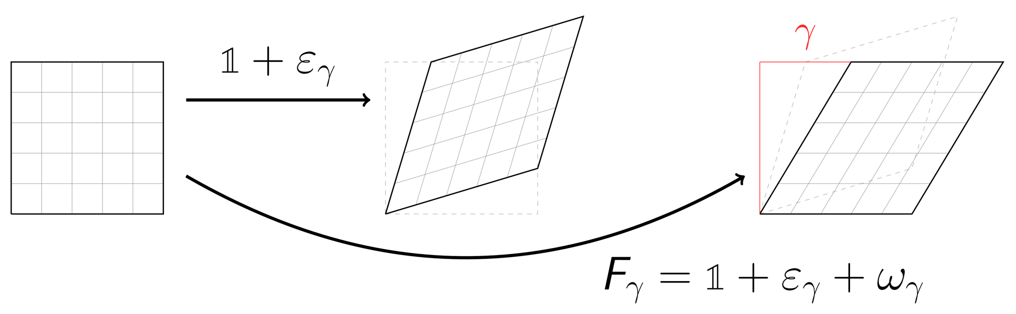


with $s \in \mathbb{R}$ and the **(simple) shear deformation**

$$F_\gamma = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



with the ammount of shear $\gamma \in \mathbb{R}$. It is well known that in isotropic linear elasticity, every simple shear deformation $F_\gamma = \mathbb{1} + \varepsilon_\gamma + \omega_\gamma$ corresponds to an infinitesimal pure shear stress tensor σ_{lin}.



Here, $\varepsilon_\gamma = \frac{\gamma}{2}(e_1 \otimes e_2 + e_2 \otimes e_1)$ donates the **infinitesimal shear strain**.

In nonlinear elasticity, on the other hand, a non-trivial Cauchy pure shear stress tensor $\sigma = T^s$ never corresponds to a simple shear deformation F_γ. Therefore, the finite generalization of infinitesimal shear must take another form. [1, 4, 3]

Given a Cauchy pure shear stress $\sigma = T^s$, the **guiding questions** are:

1. Independent of the particular elasticity law, which kind of deformations correspond to pure shear stress?
2. Which of these deformations are suitable to be called "shear"?
3. Which constitutive requirements ensure that only "shear" deformations correspond to pure shear Cauchy stress?

2 Pure shear stress

Starting with the **first question**, we utilize the fact that the left Cauchy-Green deformation tensor $B = FF^T$ and the corresponding Cauchy stress tensor $\widehat{\sigma}(B)$ commute for **any** isotropic stress response. Thus B and $\widehat{\sigma}(B)$ are simultaneously diagonalizable [7]. If $\widehat{\sigma}(B) = T^s$, then it can be shown that B commutes with $\widehat{\sigma}(B)$ if and only if B has the form

$$B = \begin{pmatrix} p & q & 0 \\ q & p & 0 \\ 0 & 0 & r \end{pmatrix} \tag{1}$$

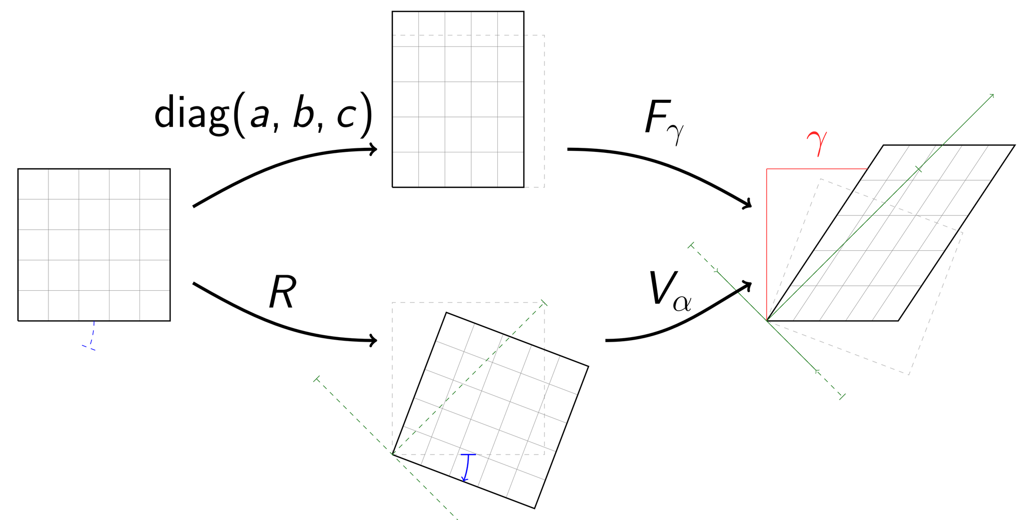
with $p = \frac{1}{2}(\mu_1 + \mu_2)$, $q = \frac{1}{2}(\mu_1 - \mu_2)$ and $r = \mu_3$, where $\mu_1, \mu_2, \mu_3 \in \mathbb{R}_+$ are the eigenvalues of B. This determines the form of the deformation gradient F itself:

If $B = FF^T$ commutes with a Cauchy pure shear stress tensor, then F is uniquely determined by

$$F = F_\gamma \operatorname{diag}(a, b, c)\, Q = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} Q\,, \tag{2}$$

up to an arbitrary $Q \in \mathrm{SO}(3)$.

3 Finite simple shear deformation



In order to answer the **second question**, we introduce the notion of an idealized shear deformation which translates the characteristic infinitesimal properties of the simple shear into the setting of finite elasticity:

We call $F = VR \in \mathrm{GL}^+(3)$ with $V \in \mathrm{Sym}^+(3)$ and $R \in \mathrm{SO}(3)$ an (idealized) **finite shear deformation** if the following requirements are satisfied:

i) The stretch V (or, equivalently, the deformation F) is **volume preserving**, i.e. $\det V = 1$.

ii) The stretch V is **planar**, i.e. V has the eigenvalue 1 to the eigenvector e_3.

iii) The rotation R is such that the deformation F is **ground parallel**, i.e. e_1, e_3 are eigenvectors of F.

In terms of the singular values $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}_+$ of F, i.e. the eigenvalues of V, the first two conditions can be stated as $\lambda_1\lambda_2\lambda_3 = 1$ and $\lambda_3 = 1$, respectively. In particular, i) and ii) are satisfied if and only if there exists $\lambda \in \mathbb{R}_+$ with $\lambda_1 = \lambda$, $\lambda_2 = \frac{1}{\lambda}$ and $\lambda_3 = 1$.

These considerations lead to the concept of the (idealized) **left finite simple shear deformation** as well as the (idealized) **finite pure shear stretch** as the class of deformations which exhibit the general form (2) and are suitable to be called "shear":

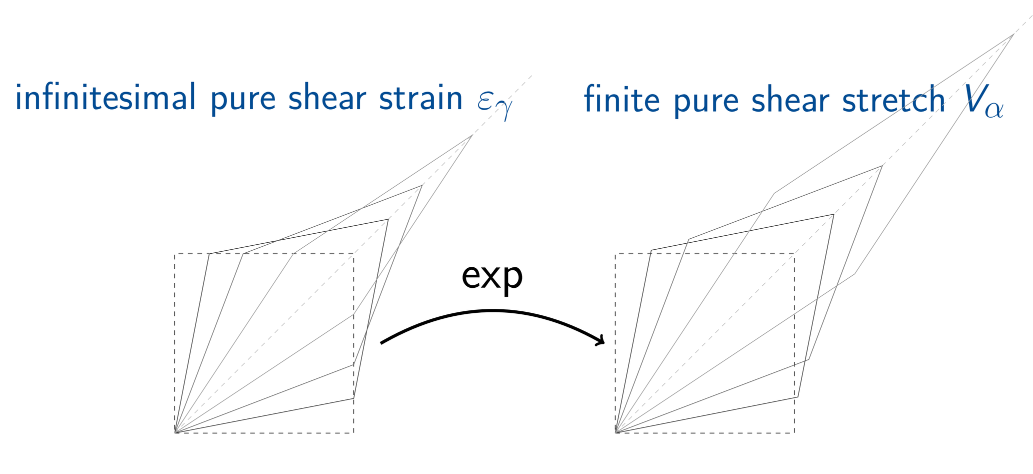
For $\alpha \in \mathbb{R}$, we call $F \in \mathrm{GL}^+(3)$ an (idealized) **left finite simple shear deformation gradient** if $F = F_\alpha$ has the form

$$F_\alpha = \frac{1}{\sqrt{\cosh(2\alpha)}} \begin{pmatrix} 1 & \sinh(2\alpha) & 0 \\ 0 & \cosh(2\alpha) & 0 \\ 0 & 0 & \sqrt{\cosh(2\alpha)} \end{pmatrix} \tag{3}$$

and $V \in \mathrm{Sym}^+(3)$ a **finite pure shear stretch** if $V = V_\alpha$ has the form

$$V_\alpha = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) & 0 \\ \sinh(\alpha) & \cosh(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp \underbrace{\begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{infinitesimal pure shear strain } \varepsilon_\gamma}. \tag{4}$$

Note that the definition of finite shear deformation is a direct generalization of the infinitesimal behavior: the infinitesimal (classcial) simple shear F_γ is planar, ground parallel and infinitesimally volume preserving ($\operatorname{tr}(\varepsilon_\gamma) = 0$). The transition mechanism of inifinitesimal stretch to finite stretch is the **matrix exponential** via the identification $\gamma = 2\alpha$.



4 Constitutive conditions

For the **third question**, it is important to note that whether or not a deformation gradient F corresponding to a Cauchy pure shear stress is a finite shear deformation depends on the particular stress response function. Similarly, not every constitutive law ensures that every idealized finite shear of the form (3) induces a Cauchy pure shear stress tensor.

In particular, for a given stress response, a finite pure shear stretch **always** induces a pure shear stress **if and only if** for all $\lambda \in \mathbb{R}_+$, there exists $s \in \mathbb{R}$ such that

$$\lambda_1 = \lambda\,,\ \lambda_2 = \frac{1}{\lambda}\,,\ \lambda_3 = 1 \qquad \Longrightarrow \qquad \sigma_1 = s\,,\ \sigma_2 = -s\,,\ \sigma_3 = 0\,,$$

where σ_i denotes the i–th eigenvalue of $\widehat{\sigma}(FF^T)$ for $F = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$. It follows:

Let $W\colon \mathrm{GL}^+(3) \to \mathbb{R}$ be an elastic energy of the form

$$W(F) = W_{\text{tc}}(F) + f(\det F)\,, \tag{5}$$

where $W_{\text{tc}}\colon \mathrm{GL}^+(3) \to \mathbb{R}$ is a sufficiently smooth **tension-compression symmetric** function, i.e. $W_{\text{tc}}(F^{-1}) = W_{\text{tc}}(F)$ for all $F \in \mathrm{GL}^+(3)$, and $f\colon \mathbb{R}_+ \to \mathbb{R}$ is differentiable with $f'(1) = 0$. Then $\widehat{\sigma}(B) = \widehat{\sigma}(V^2)$ is a pure shear stress for every finite pure shear stretch $V = V_\alpha$.

The most important classes of energy functions that satisfy the above conditions are the **Hencky-type** isotropic elastic energy functions [6, 5]

$$W(F) = \mathcal{W}(\|\operatorname{dev} \log V\|^2,\ |\operatorname{tr} \log V|^2) \tag{6}$$

and energy functions which exhibit an **additive isochoric-volumetric split** of the form

$$W(F) = W_{\text{iso}}\left(\frac{F}{(\det F)^{1/3}}\right) + f(\det F) \tag{7}$$

with a tension-compression symmetric isochoric part W_{iso}.

While the above conditions ensure that every pure shear stretch V_α induces a Cauchy pure shear stress tensor, additional assumptions on the energy function are required to ensure the reverse implication (i.e. that Cauchy pure shear stress induces pure shear stretch V_α) since the Cauchy stress response is in general not invertible.

Let W be a sufficiently smooth isotropic elastic energy satisfying the conditions (5). Furthermore, assume that W is p-coercive for some $p \geq 1$, i.e. $W(F) \geq c \cdot \|F\|^p + d$ for some $c > 0$ and $d \in \mathbb{R}$, and that W satisfies Hill's (strict) inequality [2]

$$\langle \tau(V_1) - \tau(V_2),\ \log(V_1) - \log(V_2) \rangle > 0 \quad \text{for all } V_1, V_2 \in \mathrm{Sym}^+(n)\,,\ V_1 \neq V_2\,, \tag{8}$$

Then $\sigma(V) = \widehat{\sigma}(B)$ is a pure shear stress tensor **if and only if** $V = V_\alpha$ is a pure shear stretch.

Here, $\tau(V) = \det(V) \cdot \sigma(V)$ denotes the **Kirchhoff stress** corresponding to the stretch V. For hyperelastic materials, inequality (8) is equivalent to the strict convexity of the mapping $X \mapsto W(\exp(X))$ on $\mathrm{Sym}(n)$.

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