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Complete frequency band-gaps in the relaxed micromorphic model

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1 The isotropic relaxed micromorphic model

The micromorphic model is a generalized continuum model suitable for the effective multi-scale-description of heterogeneous media with strong contrast between microscopic and macroscopic properties through the introduction of a **characteristic length scale** L_c. It allows to incorporate new effects which extend the classical linear elastic description, e.g. **size-effects** and the **dispersion of waves**. This model couples the **macroscopic displacement** $u \in \mathbb{R}^3$ and an **affine substructure deformation** attached at each macroscopic point encoded by the **micro-distortion field** $P \in \mathbb{R}^{3\times 3}$.

The **relaxed micromorphic model** [3–6] has been introduced in 2013 in [6] and endows the standard Mindlin-Eringen's representation with more geometric structure by reducing the curvature energy term to depend **only** on the **second order dislocation density tensor** $\alpha = -\operatorname{Curl} P$:

$$W = \underbrace{\mu_e \,\|\operatorname{sym}(\nabla u - P)\|^2 + \frac{\lambda_e}{2}(\operatorname{tr}(\nabla u - P))^2}_{\text{isotropic elastic}-\text{energy}} + \underbrace{\mu_c \,\|\operatorname{skew}(\nabla u - P)\|^2}_{\text{rotational elastic coupling}} \tag{1}$$
$$+ \underbrace{\mu_{\text{micro}} \,\|\operatorname{sym} P\|^2 + \frac{\lambda_{\text{micro}}}{2}(\operatorname{tr} P)^2}_{\text{micro}-\text{self}-\text{energy}} + \underbrace{\frac{\mu L_c^2}{2}\|\operatorname{Curl} P\|^2}_{\text{isotropic curvature}} .$$

Here μ_e, μ_{micro}, λ_e and λ_{micro} are elasticity coefficients. The resulting elastic (relative) stress:

$$\sigma(\nabla u, P) = 2\,\mu_e \operatorname{sym}(\nabla u - P) + 2\,\mu_c \operatorname{skew}(\nabla u - P) + \lambda_e \operatorname{tr}(\nabla u - P)\,\mathbb{1}, \tag{2}$$

is related only to **elastic distortions** $e = \nabla u - P$. The **skew-symmetry** of σ is controlled by the **Cosserat couple modulus** $\mu_c \geq 0$ since

$$\operatorname{skew}\sigma = \mu_c \operatorname{skew}(\nabla u - P). \tag{3}$$

The model is well-posed in statics and dynamics including $\mu_c = 0$, see [2, 5].

2 Homogenization formula for the isotropic case

Comparing classical linear elasticity with our new relaxed model for $L_c \to 0$ we offer an **a priori relation** between μ_e, λ_e, μ_{micro} and λ_{micro} on the one side and λ_{macro} and μ_{macro} on the other side that we call **macroscopic consistency condition** (see [1] for the fully anisotropic case):

$$\mu_{\text{macro}} := \frac{\mu_{\text{micro}}\,\mu_e}{\mu_{\text{micro}} + \mu_e}, \qquad 2\mu_{\text{macro}} + 3\lambda_{\text{macro}} := \frac{(2\mu_{\text{micro}} + 3\lambda_{\text{micro}})(2\mu_e + 3\lambda_e)}{(2\mu_{\text{micro}} + 3\lambda_{\text{micro}}) + (2\mu_e + 3\lambda_e)}. \tag{4}$$

For $\mu_{\text{micro}} \to \infty$ we recover the **Cosserat model or micropolar model** which means that $P \in \mathfrak{so}(3)$ and for $L_c \to 0$ we obtain classical linear elasticity with μ_{macro}, λ_{macro} from (4).

For comparison, the standard isotropic Mindlin-Eringen model with $\mu_c > 0$ and curvature energy depending on $\|\nabla P\|^2$ tends to a **second gradient model** when $\mu_e, \mu_c \to \infty$.

3 Dynamic formulation

The dynamical formulation is obtained defining a joint Hamiltonian and assuming stationary action. For this, we introduce a micro-inertia density contribution $\frac{\eta}{2}\|P_{,t}\|^2$, where η is the scalar **micro-inertia density**. The dynamical equilibrium equations are:

$$\begin{aligned} \rho\, u_{,tt} &= \operatorname{Div}\sigma = \operatorname{Div}[2\,\mu_e \operatorname{sym}(\nabla u - P) + 2\,\mu_c \operatorname{skew}(\nabla u - P) + \lambda_e \operatorname{tr}(\nabla u - P)\,\mathbb{1}], \\ \eta\, P_{,tt} &= 2\,\mu_e \operatorname{sym}(\nabla u - P) + 2\,\mu_c \operatorname{skew}(\nabla u - P) + \lambda_e \operatorname{tr}(\nabla u - P)\,\mathbb{1} \\ &\quad - [2\mu_{\text{micro}} \operatorname{sym} P + \lambda_{\text{micro}} \operatorname{tr}(P)\mathbb{1}] - \mu L_c^2 \operatorname{Curl}\operatorname{Curl} P. \end{aligned} \tag{5}$$

This system is a **generalized tensorial Maxwell-problem** for the micro-distortion P coupled to balance of linear momentum.

In our study of wave propagation in micromorphic media we limit ourselves to the case of **plane waves** traveling in an **infinite domain**. We suppose that the space dependence of all introduced kinematic fields are limited to the component x_1 of x which is the direction of propagation of the wave. Therefore we look for solutions of (5) in the form:

$$u(x,t) = \alpha\, e^{i(k\,x_1 - \omega\, t)},\ \alpha \in \mathbb{R}^3, \qquad P(x,t) = \beta\, e^{i(k\,x_1 - \omega\, t)},\ \beta \in \mathbb{R}^{3\times 3}. \tag{6}$$

4 Band-gaps for generalized continuum models

We present the **dispersion relations** obtained with different generalized continuum models. In the figures we consider uncoupled waves (a), longitudinal waves (b) and transverse waves (c). TRO: transverse rotational optic, TSO: transverse shear optic, TCVO: transverse constant-volume optic, LA: longitudinal acoustic, LO_1-LO_2: 1st and 2nd longitudinal optic, TA: transverse acoustic, TO_1-TO_2: 1st and 2nd transverse optic.

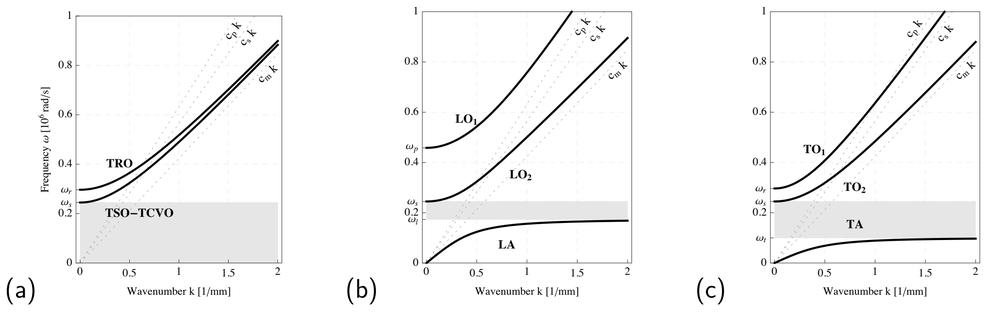

Figure 1: Dispersion relations $\omega = \omega(k)$ for the **relaxed micromorphic model** with non-vanishing Cosserat couple modulus $\mu_c > 0$. **Complete frequency band gap** is the shaded intersected domain. The width of the band gap is related to $\mu_c > 0$.

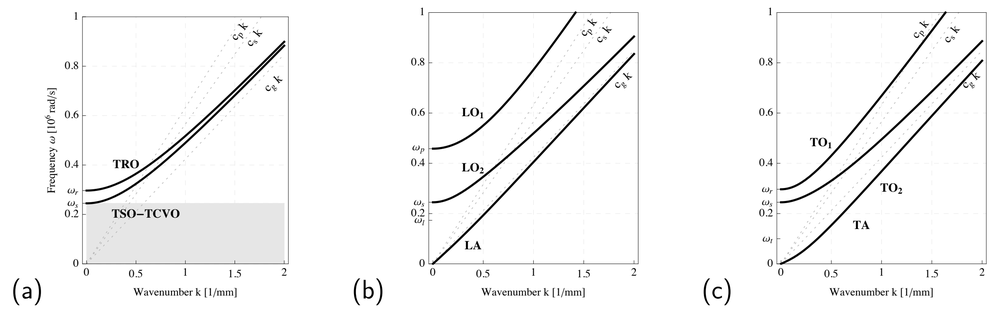

Figure 2: Dispersion relations $\omega = \omega(k)$ for the **standard micromorphic model** with $\|\nabla P\|^2$: only a **partial band gap** can be modeled.

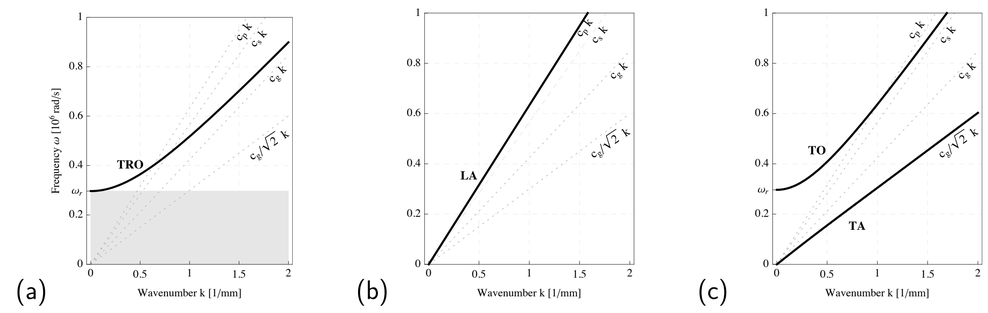

Figure 3: Dispersion relations $\omega = \omega(k)$ for the **Cosserat model** obtained by letting $\mu_{\text{micro}} \to \infty$ in the relaxed micromorphic model: only a **partial band gap** can be modeled.

Dispersion relations $\omega = \omega(k)$ for the **second gradient model** obtained as limit case of the standard micromorphic model by letting $\mu_e \to \infty$ and $\mu_c \to \infty$ show **no band gap** at all and for the **linear elastic model** obtained as limit case of the relaxed micromorphic model by letting $L_c \to 0$ there is **no dispersion** of waves.

5 Conclusion

Metamaterials are artifacts composed by **microstructural elements** in periodic or quasi-periodic patterns, giving rise to materials with **unorthodox properties**. For some of these metamaterials, the presence of a microstructure allows for **local resonances** at the micro-level which globally result in **macroscopic wave-inhibition**: the energy of the incident wave remains trapped at the level of the microstructure.

The presence of band gaps can be observed even in natural materials such as **perovskites**. Indeed, these materials are characterized by **microscopic rotational** and **stretch** motions which can been observed using **Raman Spectroscopy**. The respective micro-vibrational modes, with frequencies much higher than the acoustic modes, give rise to some local resonances and thus to the onset of **band gaps**.

The **relaxed micromorphic model** is the **only linear, isotropic, reversibly elastic, nonlocal generalized continuum model** known to date able to **predict complete frequency band gaps**. It is decisive to use $\operatorname{Curl} P$ instead of the full micro-distortion gradient ∇P and to take a positive Cosserat couple modulus $\mu_c > 0$. A material not showing band gaps must be modeled with $\mu_c \equiv 0$.

6 References

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