Chair for Nonlinear Analysis and Modelling
Faculty of Mathematics
University of Duisburg-Essen
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Geometry of logarithmic strain measures

Patrizio Neff, Bernhard Eidel and Robert J. Martin

1 Introduction

The two logarithmic strain measures [11]

$$\omega_{\text{iso}} = \|\text{dev}_n \log U\| \quad \text{and} \quad \omega_{\text{vol}} = |\text{tr}(\log U)|\,,$$

which are isotropic invariants of the Hencky strain tensor $\log U = \log\sqrt{F^TF}$, can be uniquely characterized by purely geometric methods based on the geodesic distance on the general linear group $\text{GL}(n)$. Here, $F = \nabla\varphi$ is the deformation gradient, $U = \sqrt{F^TF}$ is the right Biot-stretch tensor, log denotes the principal matrix logarithm, $\|\,.\,\|$ is the Frobenius matrix norm, tr is the trace operator and $\text{dev}_n X = X - \frac{1}{n}\,\text{tr}(X)\cdot \mathbb{1}$ is the n-dimensional deviator of $X \in \mathbb{R}^{n\times n}$.

2 The Euclidean strain measure in linear and nonlinear elasticity

Let $\varphi(x) = x + u(x)$ with the displacement u. Then the **infinitesimal strain measure** may be obtained by taking the distance of the displacement gradient $\nabla u \in \mathbb{R}^{n\times n}$ to the set of *linearized rotations* $\mathfrak{so}(n) = \{A \in \mathbb{R}^{n\times n} : A^T = -A\}$, which is the vector space of skew symmetric matrices. An obvious choice for a distance measure on the linear space $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$ of $n\times n$-matrices is the *Euclidean distance* induced by the canonical Frobenius norm $\|\,.\,\|$. One can also use the more general weighted norm defined by

$$\|X\|^2_{\mu,\mu_c,\kappa} = \mu\,\|\text{dev}_n\,\text{sym}\,X\|^2 + \mu_c\,\|\text{skew}\,X\|^2 + \frac{\kappa}{2}\,[\text{tr}(X)]^2$$

for $\mu, \mu_c, \kappa > 0$, which separately weights the *deviatoric* (or *trace free*) *symmetric part* $\text{dev}_n\,\text{sym}\,X = \text{sym}\,X - \frac{1}{n}\text{tr}(\text{sym}\,X)\cdot\mathbb{1}$, the *spherical part* $\frac{1}{n}\text{tr}(X)\cdot\mathbb{1}$, and the *skew symmetric part* $\text{skew}\,X = \frac{1}{2}(X - X^T)$ of X.

Of course, the element of best approximation in $\mathfrak{so}(n)$ to ∇u with respect to the weighted Euclidean distance $\text{dist}_{\text{Euclid}}(X,Y) = \|X - Y\|_{\mu,\mu_c,\kappa}$ is given by the associated orthogonal projection of ∇u to $\mathfrak{so}(n)$. This projection is given by the *continuum rotation*, i.e. the skew symmetric part $\text{skew}\,\nabla u = \frac{1}{2}(\nabla u - (\nabla u)^T)$ of ∇u. Thus the distance is

$$\text{dist}_{\text{Euclid}}(\nabla u, \mathfrak{so}(n)) = \|\text{sym}\,\nabla u\|_{\mu,\mu_c,\kappa}\,.$$

We therefore find

$$\text{dist}^2_{\text{Euclid}}(\nabla u, \mathfrak{so}(n)) = \|\text{sym}\,\nabla u\|^2_{\mu,\mu_c,\kappa} = \mu\,\|\text{dev}_n\,\varepsilon\|^2 + \frac{\kappa}{2}\,[\text{tr}(\varepsilon)]^2$$

for the linear strain tensor $\varepsilon = \text{sym}\,\nabla u$, which is the quadratic isotropic energy for linear elasticity.

In order to obtain a (geometrically) **nonlinear strain measure**, we must compute the distance

$$\text{dist}(\nabla\varphi, \text{SO}(n)) = \text{dist}(F, \text{SO}(n)) = \inf_{Q\in\text{SO}(n)} \text{dist}(F, Q)$$

of the deformation gradient $F = \nabla\varphi \in \text{GL}^+(n)$ to the actual set of pure rotations $\text{SO}(n) \subset \text{GL}^+(n)$. It is therefore necessary to choose a distance function on $\text{GL}^+(n)$; an obvious choice is the restriction of the Euclidean distance on $\mathbb{R}^{n\times n}$ to $\text{GL}^+(n)$. For the canonical Frobenius norm $\|\,.\,\|$, the Euclidean distance between $F, P \in \text{GL}^+(n)$ is

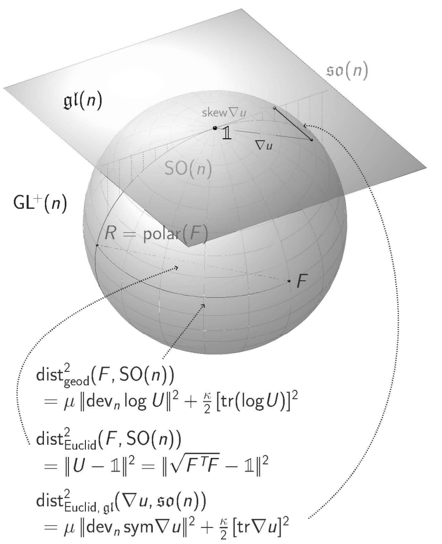
$$\text{dist}_{\text{Euclid}}(F,P) = \|F - P\| = \sqrt{\text{tr}[(F-P)^T(F-P)]}\,.$$

Thus the computation of the strain measure induced by the Euclidean distance on $\text{GL}^+(n)$ reduces to the *matrix nearness problem* [5]

$$\text{dist}_{\text{Euclid}}(F, \text{SO}(n)) = \inf_{Q\in\text{SO}(n)} \|F - Q\|\,.$$

By a well-known optimality result discovered by Giuseppe Grioli [3] (cf. [10, 4, 9, 1]), also called "Grioli's Theorem" by Truesdell and Toupin [12, p. 290], this minimum is attained for the orthogonal polar factor R.

However, we observe that the Euclidean distance is not an *intrinsic* distance measure on $\text{GL}^+(n)$: for example, $A - B \notin \text{GL}^+(n)$ for $A, B \in \text{GL}^+(n)$ in general, hence the term $\|A - B\|$ depends on the underlying linear structure of $\mathbb{R}^{n\times n}$. Furthermore, because $\text{GL}^+(n)$ is not convex, the straight line $\{A + t\,(B-A)\,|\,t\in[0,1]\}$ connecting A and B is not necessarily contained in $\text{GL}^+(n)$, which shows that the characterization of the Euclidean distance as the length of a shortest connecting curve is also not possible in a way intrinsic to $\text{GL}^+(n)$.



3 $\text{GL}^+(n)$ as a Riemannian manifold

In order to find an intrinsic distance function on $\text{GL}^+(n)$ that alleviates the drawbacks of the Euclidean distance, we endow $\text{GL}(n)$ with a *Riemannian metric*. Such a metric g is defined by an inner product $g_A\colon T_A\,\text{GL}(n)\times T_A\,\text{GL}(n) \to \mathbb{R}$ on each tangent space $T_A\,\text{GL}(n)$, $A \in \text{GL}(n)$. Then the **geodesic distance** between $A, B \in \text{GL}^+(n)$ is defined as the infimum over the lengths of all (twice continuously differentiable) curves connecting A to B. Mechanical considerations suggest a **left-GL(n)-invariant and right-O(n)-invariant metric** g of the form

$$g_A(X,Y) = \langle A^{-1}X, A^{-1}Y\rangle_{\mu,\mu_c,\kappa}\,,$$

where $\langle\cdot,\cdot\rangle_{\mu,\mu_c,\kappa}$ is the fixed inner product on the tangent space $\mathfrak{gl}(n) = T_{\mathbb{1}}\,\text{GL}(n) = \mathbb{R}^{n\times n}$ at the identity with

$$\langle X,Y\rangle_{\mu,\mu_c,\kappa} = \mu\,\langle\text{dev}_n\,\text{sym}\,X, \text{dev}_n\,\text{sym}\,Y\rangle + \mu_c\langle\text{skew}\,X, \text{skew}\,Y\rangle + \tfrac{\kappa}{2}\,\text{tr}(X)\,\text{tr}(Y)\,.$$

Then, combining an explicit representation of the geodesic curves [8] with a novel **logarithmic minimization property** [7], the geodesic distance of $F \in \text{GL}^+(n)$ to the special orthogonal group $\text{SO}(n)$ can be computed explicitly [11] (cf. [6]):

Theorem. *Let g be the left-$\text{GL}(n)$-invariant, right-$\text{O}(n)$-invariant Riemannian metric on $\text{GL}(n)$ defined by*

$$g_A(X,Y) = \langle A^{-1}X, A^{-1}Y\rangle_{\mu,\mu_c,\kappa}\,, \qquad \mu,\mu_c,\kappa > 0\,.$$

Then for all $F \in \text{GL}^+(n)$, the geodesic distance of F to the special orthogonal group $\text{SO}(n)$ induced by g is given by

$$\text{dist}^2_{\text{geod}}(F, \text{SO}(n)) = \mu\,\|\text{dev}_n \log U\|^2 + \frac{\kappa}{2}\,[\text{tr}(\log U)]^2\,.$$

The orthogonal factor $R \in \text{SO}(n)$ of the polar decomposition $F = R\,U$ is the unique element of best approximation in $\text{SO}(n)$, i.e.

$$\text{dist}_{\text{geod}}(F, \text{SO}(n)) = \text{dist}_{\text{geod}}(F, R)\,.$$

Similarly, the **partial strain measures** $\|\text{dev}_n \log U\|$ and $|\text{tr}(\log U)|$ can also be characterized separately.

Theorem (Partial strain measures). *Let*

$$\omega_{\text{iso}}(F) = \|\text{dev}_n \log\sqrt{F^TF}\|\,,$$
$$\omega_{\text{vol}}(F) = |\text{tr}(\log\sqrt{F^TF})|\,.$$

Then

$$\omega_{\text{iso}}(F) = \text{dist}_{\text{geod, SL}(n)}\left(\frac{F}{\det F^{1/n}},\ \text{SO}(n)\right)$$
$$\omega_{\text{vol}}(F) = \sqrt{n}\cdot\text{dist}_{\text{geod},\,\mathbb{R}^+\cdot\mathbb{1}}\left((\det F)^{1/n}\cdot\mathbb{1},\ \mathbb{1}\right)\,,$$

where the geodesic distances $\text{dist}_{\text{geod, SL}(n)}$ and $\text{dist}_{\text{geod},\,\mathbb{R}^+\cdot\mathbb{1}}$ on the Lie groups $\text{SL}(n)$ and $\mathbb{R}^+\cdot\mathbb{1}$ are induced by the canonical left-invariant metric

$$\bar{g}_A(X,Y)1 = \langle A^{-1}X, A^{-1}Y\rangle = \text{tr}(X^TA^{-T}A^{-1}Y)\,.$$

This theorem states that ω_{iso} and ω_{vol} appear as natural measures of the *isochoric* and *volumetric* strain, respectively: if $F = F_{\text{iso}}\,F_{\text{vol}}$ is decomposed [2] into an isochoric part $F_{\text{iso}} = (\det F)^{-1/n}\cdot F$ and a volumetric part $F_{\text{vol}} = (\det F)^{1/n}\cdot\mathbb{1}$, then $\omega_{\text{iso}}(F)$ measures the $\text{SL}(n)$-geodesic distance of F_{iso} to $\text{SO}(n)$, whereas $\frac{1}{\sqrt{n}}\,\omega_{\text{vol}}(F)$ gives the geodesic distance of F_{vol} to the identity $\mathbb{1}$ in the group $\mathbb{R}^+\cdot\mathbb{1}$ of purely volumetric deformations.

4 References

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Prof. Dr. rer. nat. Patrizio Neff
Dr.-Ing. Bernhard Eidel
Dipl.-Math. Robert J. Martin

Faculty of Mathematics
University of Duisburg-Essen
Thea-Leymann-Straße 9
45127 Essen