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Rank-one convexity vs. polyconvexity for isochoric energies on $\mathrm{GL}^+(2)$

Robert J. Martin, Ionel-Dumitrel Ghiba and Patrizio Neff

1 Morrey's conjecture

Different notions of convexity play a fundamental role in nonlinear elasticity theory, especially the concepts of **polyconvexity** (i.e. convexity in terms of minors) of an energy function $W\colon \mathrm{GL}^+(n) \to \mathbb{R}$, **quasiconvexity** (which is tantamount to the weak lower semicontinuity of the energy functional $\varphi \mapsto \int W(\nabla\varphi)\,\mathrm{d}x$ on appropriate Sobolev spaces) and **rank-one convexity** (or *Legendre-Hadamard ellipticity)*. Polyconvexity, in particular, has been an important notion in continuum mechanics and a cornerstone of the direct methods of the calculus of variations since its introduction to elasticity theory in John Ball's seminal paper [1, 2]. It is well known that the implications

polyconvexity $\Rightarrow$ **quasiconvexity** $\Rightarrow$ **rank-one convexity**

hold for arbitrary dimension n. However, it is also known that rank-one convexity does not imply polyconvexity in general, and that for $n > 2$ rank-one convexity does not imply quasiconvexity. The question whether rank-one convexity implies quasiconvexity in the two-dimensional case is considered to be one of the major open problems in the calculus of variations. Charles B. Morrey conjectured in 1952 that the two are not equivalent [5], i.e. that there exists a function $W\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ which is rank-one convex but not quasiconvex.

2 Isochoric energy functions

In [4], we present a condition under which rank-one convexity implies polyconvexity (and thus quasiconvexity), thereby further complicating the search for a counterexample: any function $W\colon\ \mathrm{GL}^+(2) \to \mathbb{R}$ which is ***isotropic*** and ***objective*** (i.e. bi-SO(2)-invariant) as well as ***isochoric*** is **rank-one convex if and only if it is polyconvex**. A function $W\colon\ \mathrm{GL}^+(2) \to \mathbb{R}$ is called isochoric if

$$W(\alpha\, F) = W(F) \quad \text{for all} \quad \alpha \in \mathbb{R}_+ := (0,\infty)\,.$$

Such energy functions play an important role in nonlinear elasticity theory [6, 7], where an additive **volumetric-isochoric split**

$$W(F) = W_{\mathrm{iso}}(F) + W_{\mathrm{vol}}(\det F) = W_{\mathrm{iso}}\left(\frac{F}{(\det F)^{1/n}}\right) + W_{\mathrm{vol}}(\det F)$$

of the elastic energy potential W into an isochoric part W_{iso} and a volumetric part W_{vol} is oftentimes assumed. This constitutive requirement is equivalent [8] to the existence of a function $p\colon \mathbb{R}_+ \to \mathbb{R}$ with

$$\underbrace{\frac{1}{n}\,\mathrm{tr}(\sigma(V))}_{\text{mean pressure}} = p(\det V) = p(\det F)\,,$$

where $\sigma(V)$ denotes the Cauchy-stress tensor corresponding to the left Biot stretch tensor $V = \sqrt{FF^T}$.

Our approach is based on certain representation formulae for isochoric energy functions: it is well known that any objective, isotropic function can be **expressed in terms of the singular values** of F, i.e. that there exists a uniquely determined function $g\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$W(F) = g(\lambda_1,\, \lambda_2) \tag{1}$$

for all $F \in \mathrm{GL}^+(2)$ with singular values λ_1, λ_2. If, in addition, W is isochoric, then there exists a unique function $h\colon \mathbb{R}_+ \to \mathbb{R}$ with $h(t) = h(\frac{1}{t})$ such that

$$W(F) = h\left(\frac{\lambda_1}{\lambda_2}\right) \tag{2}$$

for all $F \in \mathrm{GL}^+(2)$ with singular values $\lambda_1, \lambda_2 \in \mathbb{R}_+$.

The main result is the following:

Theorem. *Let* $W\colon\ \mathrm{GL}^+(2) \to \mathbb{R}, \quad F \mapsto W(F)$ *be an objective, isotropic and isochoric function, and let* $h\colon \mathbb{R}_+ \to \mathbb{R},\ g\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *denote the functions uniquely determined by* (1), (2). *Then* ***the following are equivalent****:*

i) W *is* ***polyconvex****,*

ii) W *is* ***rank-one convex****,*

iii) g *is separately convex,*

iv) h *is convex on* $\mathbb{R}_+$*,*

v) h *is convex and non-decreasing on* $[1,\infty)$.

In particular, rank-one convexity (and thus quasiconvexity) implies polyconvexity for such functions. The implication v)$\Rightarrow$ i) follows from an earlier observation [3] that the function $Z\colon\ \mathrm{GL}^+(2) \to [1,\infty)$ with $Z(F) = \frac{\lambda_1}{\lambda_2}$ for $F \in \mathrm{GL}^+(2)$ with singular values $\lambda_1 \geq \lambda_2$ is polyconvex on $\mathrm{GL}^+(2)$.

Every isochoric, isotropic and objective function W on $\mathrm{GL}^+(2)$ can also be expressed in terms of the **deviatoric quadratic Hencky strain energy**

$$\|\mathrm{dev}_2 \log U\|^2 \;=\; \left\|\log \frac{U}{(\det U)^{1/2}}\right\|^2,$$

i.e. there exists a unique function $f\colon [0,\infty) \to \mathbb{R}$ such that $W(F) = f(\|\mathrm{dev}_2 \log U\|^2)$. If W is additionally a C^2-function, then the following are equivalent:

i) W *is polyconvex* (*i.e.* $W(F) = P(F, \det F)$, P *convex),*

ii) W *is rank-one convex* (*i.e.* $D^2W(F).(\xi\otimes\eta, \xi\otimes\eta) \geq 0$*),*

iii) $2\,\eta\, f''(\eta) \,+\, (1 - \sqrt{2\,\eta}\,)\, f'(\eta) \;\geq\; 0$ *for all* $\eta \in (0,\infty)$.

3 References

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Dipl. Math. Robert J. Martin
Dr. Ionel-Dumitrel Ghiba
Prof. Dr. Patrizio Neff

Faculty of Mathematics
University of Duisburg-Essen
Thea-Leymann-Straße 9
45127 Essen