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Rank-one convexity implies polyconvexity for isotropic energies on SL(2)

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1 Rank-one convexity and polyconvexity on SL(2)

The notion of polyconvexity was introduced into the context of nonlinear elasticity theory by Sir John Ball in his seminal paper [1]. An exhaustive self-contained study giving necessary and sufficient conditions for polyconvexity in arbitrary spatial dimension was given by Alexander Mielke [7]. It is well known that the implications

$$\text{polyconvexity} \implies \text{quasiconvexity} \implies \text{rank-one convexity}$$

hold for functions on $\mathbb{R}^{n\times n}$ (as well as on $\mathrm{SL}(n)=\{X\in\mathbb{R}^{n\times n}\mid \det X=1\}$, see [2, Theorem 1.1]) for arbitrary dimension n. The reverse implications do not hold in general: rank-one convexity does not imply polyconvexity for dimension $n\geq 2$, and rank-one convexity does not imply quasiconvexity [3] for $n>2$. Whether this latter implication holds for $n=2$ is still an open question: the conjecture that **rank-one convexity** and **quasiconvexity** are **not equivalent** for $n=2$ is also called **Morrey's conjecture** [8]. For certain classes of functions on $\mathbb{R}^{2\times 2}$, however, it has been demonstrated that the two convexity properties are equivalent [12, 11, 10, 9]. In this spirit, it has been shown in [6] that any energy function $W\colon \mathrm{GL}^+(2)\to\mathbb{R}$ which is isotropic and objective (i.e. bi-SO(2)-invariant) as well as isochoric is rank-one convex if and only if it is polyconvex. Here, we consider the case of incompressible materials, i.e., we consider objective-isotropic energies $W:\mathrm{SL}(2)\to\mathbb{R}$ and **we negatively answer Morrey's conjecture for isotropic and objective energies defined on SL(2).**

The restrictions imposed by rank-one convexity are less strict in this case:

Definition. (Rank-one convexity) A function $W\colon\mathrm{SL}(2)\to\mathbb{R}$ is called *rank-one convex* if the mapping $t\mapsto W(F+t\,\xi\otimes\eta)$, $t\in\mathbb{R}$ is convex for all $F\in\mathrm{SL}(2)$ and all $\xi,\eta\in\mathbb{R}^2$ such that $\langle\xi\otimes\eta,F^{-T}\rangle=0$.

We recall some definitions regarding the polyconvexity of an energy W on $\mathbb{R}^{2\times2}$ and $\mathrm{SL}^+(2)$, respectively:

Definition. (Polyconvexity)

i) (Ball [1]) A function $\widetilde{W}:\mathbb{R}^{2\times2}\to\mathbb{R}\cup\{\infty\}$ is called *polyconvex* if there exists a convex function $P:\mathbb{R}^5\to\mathbb{R}\cup\{\infty\}$ such that

$$\widetilde{W}(F)=P(F,\det F)\qquad\text{for all }\ F\in\mathbb{R}^{2\times2}.$$

ii) (Mielke [7]) A function $W_{\mathrm{inc}}:\mathrm{SL}(2)\to\mathbb{R}$ is called *polyconvex* if the function

$$\widetilde{W}:\mathbb{R}^{2\times2}\to\mathbb{R}\cup\{\infty\},\quad \widetilde{W}(F)=\begin{cases}W_{\mathrm{inc}}(F), & F\in\mathrm{SL}(2),\\ \infty, & F\notin\mathrm{SL}(2)\end{cases}$$

is polyconvex on $\mathbb{R}^{2\times2}$.

The following proposition is due to Alexander Mielke and assumes no regularity of the energy [7, Theorem 5.1].

Proposition. (Mielke [7]) *Let $W\colon\mathrm{SL}(2)\to\mathbb{R}$ be an objective and isotropic function, and $\phi:[0,\infty)\to\mathbb{R}$ the unique function with $W(F)=\phi\Big(\lambda_{\max}(F)-\frac{1}{\lambda_{\max}(F)}\Big)$ for all $F\in\mathrm{SL}(2)$, where $\lambda_{\max}(F)$ is the largest singular value of F. Then the following are equivalent:*

i) ϕ is nondecreasing and convex on $[0,\infty)$,

ii) W is polyconvex (in the sense of Mielke's polyconvexity definition).

For objective and isotropic differentiable energies, a combination of the above proposition and an adaptation of the three dimensional result concerning rank-one convexity of differentiable functions due to Dunn, Fosdick and Zhang [4] leads to the following proposition [5]:

Proposition. *Let $W\colon\mathrm{SL}(2)\to\mathbb{R}$ be an objective and isotropic* ***differentiable*** *function. Then, the energy W is rank-one convex if and only if W is polyconvex.*

2 The main result

Our result [5] shows that rank-one convexity implies polyconvexity of an objective and isotropic energy **without assuming any regularity** of the energy. As an intermediate step in our approach we use the fact that any $F\in\mathrm{SL}(2)$ can be viewed locally as a simple shear in a suitable direction with local amount of shear γ, followed or preceded by a suitable rotation.

Theorem. *Let $W\colon\mathrm{SL}(2)\to\mathbb{R}$ be an objective and isotropic function. Then the following are* ***equivalent****:*

i) W is ***rank-one convex****;*

ii) the mapping $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$, $\widetilde{\phi}(\gamma)=W\big(\big(\begin{smallmatrix}1&\gamma\\0&1\end{smallmatrix}\big)\big)$ is convex;

iii) W is ***polyconvex****;*

iv) the function $\phi:[0,\infty)\to\mathbb{R}$ satisfying $W(F)=\phi\Big(\lambda_{\max}(F)-\dfrac{1}{\lambda_{\max}(F)}\Big)$ is nondecreasing and convex.

3 Isochoric energies and functions on SL(2)

Any objective, isotropic and isochoric function can be written as

$$W(F)=W_{\mathrm{inc}}\left(\frac{F}{(\det F)^{1/2}}\right),$$

where $W_{\mathrm{inc}}=W\big|_{\mathrm{SL}(2)}$ is the restriction of W to the special linear group SL(2).

Proposition. *Let $W\colon\mathrm{GL}^+(2)\to\mathbb{R}$ be an objective, isotropic and isochoric function. If W is rank-one convex (equivalently polyconvex) on $\mathrm{GL}^+(2)$, then $W_{\mathrm{inc}}\colon\mathrm{SL}(2)\to\mathbb{R}$ is rank-one convex (equivalently polyconvex) on SL(2).*

The reverse of the above Proposition does not hold true, in general. Consider the function $W_{\mathrm{iso}}:\mathrm{GL}^+(2)\to\mathbb{R}$ with

$$W_{\mathrm{iso}}(F)=\left|\sqrt{\frac{\lambda_1}{\lambda_2}}-\sqrt{\frac{\lambda_2}{\lambda_1}}\,\right|$$

for all $F\in\mathrm{GL}^+(2)$ with singular values $\lambda_1,\lambda_2\in\mathbb{R}^+$. Then

i) W_{iso} is objective, isotropic and isochoric on $\mathrm{GL}^+(2)$,

ii) W_{iso} is *not* rank-one convex on $\mathrm{GL}^+(2)$,

iii) the restriction $W_{\mathrm{inc}}=W_{\mathrm{iso}}\big|_{\mathrm{SL}(2)}$ of W_{iso} to SL(2) is polyconvex on SL(2).

References

[1] J. M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rat. Mech. Anal.*, 63:337–403, 1977.

[2] S. Conti. Quasiconvex functions incorporating volumetric constraints are rank-one convex. *J. Math. Pures Appl.*, 90(1):15–30, 2008.

[3] B. Dacorogna. *Direct Methods in the Calculus of Variations.*, volume 78 of *Applied Mathematical Sciences.* Springer, Berlin, 2. edition, 2008.

[4] J. E. Dunn, R. Fosdick, and Y. Zhang. Rank 1 convexity for a class of incompressible elastic materials. In *Rational Continua, Classical and New*, pages 89–96. Springer, 2003.

[5] I. D. Ghiba, R. J. Martin, and P. Neff. Rank-one convexity implies polyconvexity in isotropic planar incompressible elasticity. *submitted*, 2016.

[6] R. J. Martin, I. D. Ghiba, and P. Neff. Rank-one convexity implies polyconvexity for all isotropic, objective and isochoric elastic energies defined in two-dimensions. *Proc. Roy. Soc. Edinburgh Sect. A*, 2016.

[7] A. Mielke. Necessary and suffcient conditions for polyconvexity of isotropic functions. *J. Conv. Anal.*, 12(2):291–314, 2005.

[8] C. B. Morrey. Quasi-convexity and the lower semicontinuity of multiple integrals. *Pacific J. Math*, 2(1):25–53, 1952.

[9] S. Müller. Rank-one convexity implies quasiconvexity on diagonal matrices. *Int. Math. Res. Not.*, 1999(20):1087–1095, 1999.

[10] P. Rosakis and H. Simpson. On the relation between polyconvexity and rank-one convexity in nonlinear elasticity. *J. Elasticity*, 37:113–137, 1995.

[11] D. Serre. Formes quadratiques et calcul des variations. *J. Math. Pures Appl.*, 62(9):177–196, 1983.

[12] F. J. Terpstra. Die Darstellung biquadratischer Formen als Summen von Quadraten mit Anwendung auf die Variationsrechnung. *Math. Ann.*, 116(1):166–180, 1939.

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