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September 23, 2016

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The 6-parameter Cosserat-shell model

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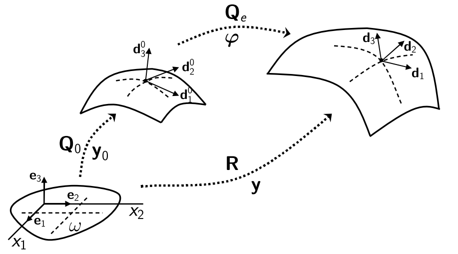
1 The Cosserat (6-parameter) model for shells

The deformed configuration of the shell is described by two independent fields:

$$\mathbf{y} = \mathbf{y}(x_1, x_2), \qquad \mathbf{R}(x_1, x_2) = \mathbf{Q}_e \mathbf{Q}_0 = \mathbf{d}_i(x_1, x_2) \otimes \mathbf{e}_i \in SO(3),$$

where $\mathbf{y}$ is the **position vector** of the midsurface and $\mathbf{R}$ is the **total microrotation** (or structure tensor). The orthonormal vectors $\mathbf{d}_i$ $(i = 1, 2, 3)$ form the triad of **directors**.



The curved reference configuration is characterized by the position vector $\mathbf{y}_0$ and the initial microrotation tensor $\mathbf{Q}_0$. We use the covariant base vectors $\mathbf{a}_\alpha = \frac{\partial \mathbf{y}_0}{\partial x_\alpha} = \mathbf{y}_{0,\alpha}$ of the reference midsurface, the contravariant base vectors $\mathbf{a}^\alpha$ $(\alpha = 1, 2)$ and the **first fundamental tensor** $\mathbf{a} = \mathbf{a}_\alpha \otimes \mathbf{a}^\alpha = a_{\alpha\beta}\, \mathbf{a}^\alpha \otimes \mathbf{a}^\beta$. Let $\mathbf{n}_0$ be the unit normal to the tangent plane of the reference midsurface. The **shell deformation gradient tensor** is

$$\mathbf{F} := \mathrm{Grad}_s\, \mathbf{y} = \mathbf{y}_{,\alpha} \otimes \mathbf{a}^\alpha .$$

The non-symmetric **elastic shell strain tensor** $\mathbf{E}_e$ and **elastic shell bending-curvature tensor** $\mathbf{K}_e$ are defined by

$$\mathbf{E}_e := \mathbf{Q}_e^T \mathbf{F} - \mathbf{a} = \mathbf{Q}_e^T\, \mathrm{Grad}_s \mathbf{y} - \mathbf{a},$$
$$\mathbf{K}_e := \mathrm{axl}(\mathbf{Q}_e^T \mathbf{Q}_{e,\alpha}) \otimes \mathbf{a}^\alpha = \mathbf{Q}_0 \big[\mathrm{axl}\big(\mathbf{R}^T \mathbf{R}_{,\alpha}\big) - \mathrm{axl}\big(\mathbf{Q}_0^T \mathbf{Q}_{0,\alpha}\big)\big].$$

2 The shell dislocation density tensor

We define first the **surface curl operator** curl_s for vector fields $\mathbf{v}$ and the operator Curl_s for tensor fields $\mathbf{T}$ by

$$(\mathrm{curl}_s\, \mathbf{v}) \cdot \mathbf{c} = \mathrm{Div}_s(\mathbf{v} \times \mathbf{c}) \qquad \text{for all constant vectors } \mathbf{c},$$
$$(\mathrm{Curl}_s\, \mathbf{T})^T \mathbf{c} = \mathrm{curl}_s(\mathbf{T}^T \mathbf{c}) \qquad \text{for all constant vectors } \mathbf{c}.$$

From these definitions it follows $\mathrm{curl}_s\, \mathbf{v} = -\mathbf{v}_{,\alpha} \times \mathbf{a}^\alpha$ and $\mathrm{Curl}_s\, \mathbf{T} = -\mathbf{T}_{,\alpha} \times \mathbf{a}^\alpha$. We introduce the **shell dislocation density tensor** $\mathbf{D}_s$ by

$$\mathbf{D}_s := \mathbf{a}\, \mathbf{D}_e \qquad \text{with} \qquad \mathbf{D}_e := \mathbf{Q}_e^T\, \mathrm{Curl}_s\, \mathbf{Q}_e = -\big(\mathbf{Q}_e^T \mathbf{Q}_{e,\alpha}\big) \times \mathbf{a}^\alpha .$$

The shell dislocation density tensor $\mathbf{D}_s$ represents an alternative strain measure for orientation (curvature) change in Cosserat-type shells [1].

The **extended Nye's formula for shells** expresses the relationship between the shell dislocation density tensor $\mathbf{D}_s$ and the shell bending-curvature tensor $\mathbf{K}_e$:

$$\mathbf{D}_s = -\mathbf{K}_e^T + (\mathrm{tr}\, \mathbf{K}_e)\, \mathbf{a}, \qquad \text{or equivalently,} \qquad \mathbf{K}_e = -\mathbf{D}_s^T + (\mathrm{tr}\, \mathbf{D}_s)\, \mathbf{a}.$$

We remark the reciprocity of these two relations. We also have $\|\mathbf{D}_s\| = \|\mathbf{K}_e\|$ and

$$\mathrm{dev}_s\, \mathrm{sym}\, \mathbf{D}_s = -\mathrm{dev}_s\, \mathrm{sym}\, \mathbf{K}_e, \qquad \mathrm{skew}\, \mathbf{D}_s = \mathrm{skew}\, \mathbf{K}_e, \qquad \mathrm{tr}\, \mathbf{D}_s = \mathrm{tr}\, \mathbf{K}_e .$$

3 Existence of minimizers for the total energy functional

We formulate the following **two–field minimization problem**: find the pair $(\widehat{\mathbf{y}}, \widehat{\mathbf{R}})$ in the admissible set $\mathcal{A}$ which realizes the minimum of the total energy functional [3]

$$\mathcal{E}(\mathbf{y}, \mathbf{R}) = \int_\omega W(\mathbf{E}_e, \mathbf{D}_s) \sqrt{\det (a_{\alpha\beta})_{2\times 2}}\, \mathrm{d}x_1\, \mathrm{d}x_2 - \Lambda(\mathbf{y}, \mathbf{R}) \qquad \text{for} \quad (\mathbf{y}, \mathbf{R}) \in \mathcal{A},$$

where $\Lambda(\mathbf{y}, \mathbf{R})$ is the **potential of external surface loads** $\mathbf{f}$, $\boldsymbol{\ell}$ and **boundary loads** $\mathbf{n}^*$, $\mathbf{m}^*$, while the **admissible set** $\mathcal{A}$ is defined by

$$\mathcal{A} = \big\{(\mathbf{y}, \mathbf{R}) \in \mathbf{H}^1(\omega, \mathbb{R}^3) \times \mathbf{H}^1(\omega, SO(3)) \;\big|\; \mathbf{y}\big|_{\partial\omega_d} = \mathbf{y}^*, \; \mathbf{R}\big|_{\partial\omega_d} = \mathbf{R}^*\big\}.$$

Theorem. *Assume that the external loads and boundary data satisfy the conditions*

$$\mathbf{f} \in \mathbf{L}^2(\omega, \mathbb{R}^3), \quad \mathbf{n}^* \in \mathbf{L}^2(\partial\omega_f, \mathbb{R}^3), \quad \mathbf{y}^* \in \mathbf{H}^1(\omega, \mathbb{R}^3), \quad \mathbf{R}^* \in \mathbf{H}^1(\omega, SO(3))$$

and the curved reference configuration satisfies the regularity conditions

$$\mathbf{y}_0 \in \mathbf{H}^1(\omega, \mathbb{R}^3), \qquad \mathbf{Q}_0 \in \mathbf{H}^1(\omega, SO(3)),$$
$$\mathbf{a}_\alpha \in \mathbf{L}^\infty(\omega, \mathbb{R}^3), \qquad \det\big(a_{\alpha\beta}(x_1, x_2)\big) \geq a_0 > 0.$$

The strain energy density $W(\mathbf{E}_e, \mathbf{D}_s)$ is assumed to be a quadratic, convex and coercive function of $(\mathbf{E}_e, \mathbf{D}_s)$, in the sense that $W(\mathbf{E}_e, \mathbf{D}_s) \geq C_0\big(\|\mathbf{E}_e\|^2 + \|\mathbf{D}_s\|^2\big)$. Then, ***the minimization problem admits at least one minimizing solution*** *pair $(\widehat{\mathbf{y}}, \widehat{\mathbf{R}}) \in \mathcal{A}$.*

We consider the following form of the strain energy density for **isotropic Cosserat shells**:

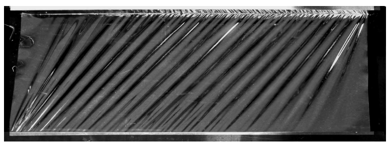
$$2\, W(\mathbf{E}_e, \mathbf{D}_s) = \alpha_1 \big[\mathrm{tr}(\mathbf{a}\mathbf{E}_e)\big]^2 + \alpha_2\, \mathrm{tr}(\mathbf{a}\mathbf{E}_e)^2 + \alpha_3\, \|\mathbf{a}\mathbf{E}_e\|^2 + \alpha_4\, \|\mathbf{n}_0 \mathbf{E}_e\|^2$$
$$+ \beta_1 \big[\mathrm{tr}(\mathbf{D}_s\, \mathbf{a})\big]^2 + \beta_2\, \mathrm{tr}(\mathbf{D}_s\, \mathbf{a})^2 + \beta_3\, \|\mathbf{D}_s\, \mathbf{a}\|^2 + \beta_4\, \|\mathbf{D}_s\, \mathbf{n}_0\|^2 \qquad \text{with}$$

$$\alpha_1 = h \frac{2\lambda\mu}{\lambda + 2\mu}, \quad \alpha_2 = h\,(\mu - \mu_c), \quad \alpha_3 = h\,(\mu + \mu_c), \quad \alpha_4 = \kappa h\,(\mu + \mu_c),$$
$$\beta_1 = -\frac{h^3}{12}(\mu - \mu_c), \qquad \beta_2 = -2\mu\Big(\frac{h^3}{12}\,\frac{\lambda}{\lambda + 2\mu} + \frac{2\,h\,L_c^2}{3}\Big),$$
$$\beta_3 = 4\mu\Big(\frac{h^3}{12}\,\frac{\lambda + \mu}{\lambda + 2\mu} + \frac{4\,h\,L_c^2}{3}\Big), \qquad \beta_4 = \frac{16\,h\,L_c^2}{3},$$

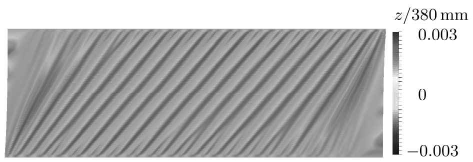
where h is the thickness, λ, μ are the Lamé coefficients, $\mu_c \geq 0$ is the **Cosserat couple modulus** and L_c an internal characteristic length of the Cosserat material [4].

4 Wrinkling of a sheared rectangular sheet

A thin elastic sheet under shear exhibits **wrinkles**. This problem has been investigated **experimentally** in [5] for a rectangular polyimide sheet of dimension 380 × 128 mm:



Using the Cosserat shell model and **geodesic finite elements** [2] we determine the numerical solution corresponding to the dimensions $h = 25\,\mu$m, $L_c = 0.025\,\mu$m and the boundary data $\delta_h = 3$ mm (horizontal shearing), $\delta_v = 0.4$ mm (vertical displacement):



We remark a **very good quantitative match**, including the same number of wrinkles, the secondary wrinkles near the horizontal sides and the wrinkles near the vertical sides.

References

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