Reconsidering the linear isotropic indeterminate couple stress model (Koiter-Mindlin-Toupin-model)

Patrizio Neff *

September 1, 2008

1 The variational formulation

We present here a formulation which is consistent, as far as possible, with the Cosserat model. In fact, this model is formally obtained by setting $\mu_c = \infty$ in the linear isotropic Cosserat model, which enforces the constraint $\operatorname{curl} u = 2 \operatorname{axl} \overline{A} = \phi$ [3, 1, 6, 9, 5]. For the **displacement** $u : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ we consider therefore the **one-field** minimization problem

$$\begin{aligned} I(u) = \int_\Omega W_{\text{mp}}(\nabla u) + W_{\text{curv}}(\nabla \operatorname{curl} u) - \langle f, u\rangle - \frac{1}{2}\langle \operatorname{axl}(\overline{M}), \operatorname{curl} u\rangle \,\mathrm{dV} \qquad (1.1)\\ - \int_{\Gamma_S} \langle f_S, u\rangle - \frac{1}{2}\langle \operatorname{axl}(\overline{M}_S), \operatorname{curl} u\rangle \,\mathrm{dS} \mapsto \quad \min.\ \text{w.r.t. } u, \end{aligned}$$

under the constitutive requirements and boundary conditions

$$\begin{aligned} W_{\text{mp}}(\overline{\varepsilon}) &= \mu \,\|\operatorname{sym}\nabla u\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[\operatorname{sym}\nabla u\right]^2, \quad u_{|\Gamma} = u_{\text{d}}\,, \\ W_{\text{curv}}(\nabla \operatorname{curl} u) &= \frac{\gamma+\beta}{2}\|\operatorname{dev}\operatorname{sym}\nabla (1/2)\operatorname{curl} u\|^2 + \frac{\gamma-\beta}{2}\|\operatorname{skew}\nabla(1/2)\operatorname{curl} u\|^2 \\ &\quad + \frac{3\alpha+(\beta+\gamma)}{6}\operatorname{tr}\left[\nabla(1/2)\operatorname{curl} u\right]^2 \\ &= \frac{\gamma+\beta}{8}\|\operatorname{dev}\operatorname{sym}\nabla\operatorname{curl} u\|^2 + \frac{\gamma-\beta}{8}\|\operatorname{skew}\nabla\operatorname{curl} u\|^2 \\ &= \frac{\gamma+\beta}{8}\,\|\operatorname{sym}\nabla\operatorname{curl} u\|^2 + \frac{\gamma-\beta}{8}\|\operatorname{skew}\nabla\operatorname{curl} u\|^2 \\ &= \frac{\gamma+\beta}{8}\,\|\operatorname{sym}\nabla\operatorname{curl} u\|^2 + \frac{\gamma-\beta}{16}\|\operatorname{curl}\operatorname{curl} u\|^2_{\mathbb{R}^3}\,. \qquad (1.2) \end{aligned}$$

In this limit model, the curvature parameter α, related to the spherical part of the couple stress tensor m remains **indeterminate**, since $\operatorname{tr}[\nabla\phi] = \operatorname{Div}\operatorname{axl}\overline{A} = \operatorname{Div}\frac{1}{2}\operatorname{curl} u = 0$. We remark the intricate relation between $\mu_c \to \infty$[1] and the indeterminacy of α. The model has the advantage of easier physical interpretation, since the Cosserat rotation coincides with the continuum rotation [4]. It is previsaged in [2, p.30], where the rotations are then referred to "triedre cache", in contrast to the independent Cosserat rotations "triedre mobile".

2 The strong form of the equilibrium equations

For exposition, we assume no torque-loads, nor surface tractions or surface loads. Moreover, no higher order boundary conditions are imposed. The force stress part has the classical form:

$$\sigma = 2\mu\,\operatorname{sym}\nabla u + \lambda\operatorname{tr}\left[\operatorname{sym}\nabla u\right]\mathbb{1}\,. \qquad (2.1)$$

*Patrizio Neff, AG6, Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstrasse 7, 64289 Darmstadt, Germany, email: neff@mathematik.tu-darmstadt.de, Tel.: +49-6151-16-3495

[1] This formal limit $\mu_c \to \infty$ excludes that $\operatorname{axl}\overline{A}$ is an independent field.

The so-called hyperstress may be given in the format

$$\tau = -\left[\frac{\gamma+\beta}{4}\,\text{anti}(\text{Div}\,\text{sym}\,\nabla\,\text{curl}\,u) + \frac{\gamma-\beta}{4}\,\text{anti}(\text{Div}\,\text{skew}\,\nabla\,\text{curl}\,u)\right], \tag{2.2}$$

and the system reads

$$\text{Div}(\sigma+\tau) = f\,, \quad u_{|\Gamma} = u_{\text{d}}\,. \tag{2.3}$$

Since σ is symmetric, the antis-symmetric part of the total stress is entirely determined by τ. It represents a system of fourth order, which is numerically preventive. In this way, the Cosserat model can also be seen as a relaxed Koiter-Mindlin model where the Cosserat couple modulus μ_c is a penalty factor.

3 Koiter's remark on couple stresses

It seems appropriate to cite also Koiter [5]: "The predictions of the (classical elasticity) theory are usually in satisfactory agreement with careful experiments, if the stresses remain within the elastic limit of the material. Unfortunately the theory of elasticity apparently fails, however, to give an adequate description of the behaviour in fatigue of machine parts or other structural elements involving high stress concentrations. This failure can hardly be ascribed completely to inelastic behaviour of the material, because the endurance limit in fatigue is usually well below the macroscopic elastic limit of the material. A more likely explanation is that the classical (finite elasticity) theory is not adequate in the presence of large stress gradients. The latter explanation is entirely plausible in view of the discontinuous polycrystalline structure of actual engineering materials. It is also supported by evidence that the discrepancy between the theoretical predictions and fatigue test results is more marked for materials with a coarse grain structure. It would seem therefore that the idealized model of an elastic continuum is not quite appropriate for the analysis of stress and strain in an actual discontinuous polycrystalline material involving large stress gradients. It need hardly be argued, however, that a detailed analysis of the transmission of loads between the individual grains in a polycrystalline material would pose a formidable problem. Some idealisation, preferably in the form of a continuous model, is highly desirable in order to make the problem amenable to analysis. At first sight it might seem that this return to a continuous model would also imply a complete return to the classical theory of elasticity. It should be remembered, however, that we have already alluded to additional assumptions made in the classical theory, apart from the model of a continuum. The assumption in question is that the transmission of loads between the material on both sides of an infinitesimal surface element dS is described completely by a force vector $p\,$dS acting in the center of gravity of the surface element. We emphasize that this is an assumption which can neither be proved directly, nor disproved. It can only be tested by a confrontation of its predictions for measurable quantities with experiments. For most purposes it has indeed proved to be an appropriate assumption, resulting in satisfactory agreement between theoretical predictions and experimental evidence. The lack of agreement between theory and experiment on the effect of stress gradients, however, makes this assumption questionable at least in cases of large stress gradients. A quite natural generalisation of the classical theory of the elastic continuum is thus obtained, if we drop the additional assumption. (...) It turned out, however, that the magnitude of the effect of couple-stresses, required to explain quantitatively the effect of stress gradients in fatigue tests, was such that it could not easily have escaped attention in other careful experiments." and he continues [5, p.41]: "We venture to conjecture that the stress gradient effect in fatigue cannot be described satisfactorily by allowing the presence of couple-stresses in an isotropic elastic medium." Brackets my addition.

It must be noted that Koiter came to reject the significant presence of couple stresses because he based his investigations on the so called indeterminate couple stress theory (**??**), which tends to maximize the influence of length scale effects. His arguments only show that this special constrained gradient theory cannot be based on experimental evidence. However, the main thrust of his comments remains valid. I have not been aware of Koiters contribution during the preparation of my main arguments, but it squares with my development

4 Parameter ranges

The parameter-range for μ, λ is the same as for linear elasticity. For the curvature term Koiter concludes [5] that $\gamma+\beta > 0$, $\gamma-\beta > 0$ is the correct choice for this model. However, by Korn's second inequality it is easy to see that $\gamma = \beta$ still leads to a well-posed problem. Inspection of analytical solutions [7] shows that $\gamma = \beta$ must be chosen to avoid unbounded stiffness for small samples (extrapolated from the corresponding statement for the Cosserat model).

5 Criticism

Compared to the classical Cosserat model with uniform positive curvature, the Koiter-Mindlin model with $\gamma > \beta$ exacerbates the stiffness increase for small samples of the material and leads thus to certain singularities for which no physical basis can be found. Experimental methods based on trying to identify $\gamma > \beta$ have failed [8]. On this basis, the Koiter-Mindlin model has been consistently neglected in the literature.

However, since $\gamma = \beta$ is also allowed, this criticism has lost its main thrust. The Koiter-Mindlin model is thus waiting for a complete rediscovery.

References

[1] E.L. Aero and E.V. Kuvshinskii. Fundamental equations of the theory of elastic media with rotationally interacting particles. *Soviet Physics-Solid State*, 2:1272–1281, 1961.

[2] E. Cosserat and F. Cosserat. *Théorie des corps déformables.* Librairie Scientifique A. Hermann et Fils (Translation: Theory of deformable bodies, NASA TT F-11 561, 1968), Paris, 1909.

[3] G. Grioli. Elasticitá asimmetrica. *Ann. Mat. Pura Appl., Ser. IV*, 50:389–417, 1960.

[4] G. Grioli. Microstructure as a refinement of Cauchy theory. Problems of physical concreteness. *Cont. Mech. Thermod.*, 15:441–450, 2003.

[5] W.T. Koiter. Couple stresses in the theory of elasticity I,II. *Proc. Kon. Ned. Akad. Wetenschap*, B 67:17–44, 1964.

[6] R.D. Mindlin and H.F. Tiersten. Effects of couple stresses in linear elasticity. *Arch. Rat. Mech. Anal.*, 11:415–447, 1962.

[7] P. Neff. The Cosserat couple modulus for continuous solids is zero viz the linearized Cauchy-stress tensor is symmetric. *Preprint 2409, http://www3.mathematik.tu-darmstadt.de/fb/mathe/bibliothek/preprints.html, Zeitschrift f. Angewandte Mathematik Mechanik (ZAMM)*, 86(DOI 10.1002/zamm.200510281):892–912, 2006.

[8] J. Schijve. Note on couple stresses. *J. Mech. Phys. Solids*, 14:113–120, 1966.

[9] R.A. Toupin. Elastic materials with couple stresses. *Arch. Rat. Mech. Anal.*, 11:385–413, 1962.

Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 second order tensors, written with capital letters and Sym denotes symmetric second orders tensors. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times 3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times 3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3\times 3}$. The identity tensor on $\mathbb{M}^{3\times 3}$ will be denoted by $1\!\!1$, so that $\operatorname{tr}[X] = \langle X, 1\!\!1\rangle$. We set $\operatorname{sym}(X) = \frac{1}{2}(X^T + X)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \operatorname{sym}(X) + \operatorname{skew}(X)$. For $X \in \mathbb{M}^{3\times 3}$ we set for the deviatoric part $\operatorname{dev} X = X - \frac{1}{3}\operatorname{tr}[X]\, 1\!\!1 \in \mathfrak{sl}(3)$ where $\mathfrak{sl}(3)$ is the Lie-algebra of traceless matrices. The set $\operatorname{Sym}(n)$ denotes all symmetric $n \times n$-matrices. The Lie-algebra of $\operatorname{SO}(3) := \{X \in \operatorname{GL}(3)\ |X^T X = 1\!\!1,\ \det[X] = 1\}$ is given by the set $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times 3}\ |X^T = -X\}$ of all skew symmetric tensors. The canonical identification of $\mathfrak{so}(3)$ and $\mathbb{R}^3$ is denoted by $\operatorname{axl}\overline{A} \in \mathbb{R}^3$ for $\overline{A} \in \mathfrak{so}(3)$. The Curl operator on the three by three matrices acts row-wise, i.e.

$$\operatorname{Curl}\begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33}\end{pmatrix} = \begin{pmatrix}\operatorname{curl}(X_{11}, X_{12}, X_{13})^T \\ \operatorname{curl}(X_{21}, X_{22}, X_{23})^T \\ \operatorname{curl}(X_{31}, X_{32}, X_{33})^T\end{pmatrix}. \tag{5.1}$$

Moreover, we have

$$\forall\, A \in \mathbb{C}^1(\mathbb{R}^3, \mathfrak{so}(3)) : \quad \operatorname{Div} A(x) = -\operatorname{curl}\operatorname{axl}(A(x)). \tag{5.2}$$

Note that $(\operatorname{axl}\overline{A}) \times \xi = \overline{A}.\xi$ for all $\xi \in \mathbb{R}^3$, such that

$$\operatorname{axl}\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0\end{pmatrix} := \begin{pmatrix}-\gamma \\ \beta \\ -\alpha\end{pmatrix}, \quad \overline{A}_{ij} = \sum_{k=1}^{3} -\varepsilon_{ijk}\cdot \operatorname{axl}\overline{A}_k\,,$$
$$\|\overline{A}\|^2_{\mathbb{M}^{3\times 3}} = 2\,\|\operatorname{axl}\overline{A}\|^2_{\mathbb{R}^3}\,, \qquad \langle\overline{A}, \overline{B}\rangle_{\mathbb{M}^{3\times 3}} = 2\langle\operatorname{axl}\overline{A}, \operatorname{axl}\overline{B}\rangle_{\mathbb{R}^3}\,, \tag{5.3}$$

where ε_{ijk} is the totally antisymmetric permutation tensor. Here, $\overline{A}.\xi$ denotes the application of the matrix $\overline{A}$ to the vector ξ and $a \times b$ is the usual cross-product. Moreover, the inverse of axl is denoted by anti and defined by

$$\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} := \operatorname{anti} \begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix}, \quad \operatorname{axl}(\operatorname{skew}(a \otimes b)) = -\frac{1}{2}\, a \times b\,, \tag{5.4}$$

and

$$2\operatorname{skew}(b \otimes a) = \operatorname{anti}(a \times b) = \operatorname{anti}(\operatorname{anti}(a).b)\,. \tag{5.5}$$

Moreover,

$$\operatorname{curl} u = 2\operatorname{axl}(\operatorname{skew} \nabla u)\,. \tag{5.6}$$