The linear isotropic Cosserat (micropolar) model

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August 12, 2008

Introduction

General continuum models involving **independent rotations** have been introduced by the Cosserat brothers [9] at the beginning of the last century. Their originally nonlinear, geometrically exact development has been largely forgotten for decades only to be rediscovered in a restricted linearized setting in the early sixties [15, 13, 45, 46, 21, 33, 42, 47]. Since then, the original Cosserat concept has been generalized in various directions, notably by Eringen and his coworkers who extended the Cosserat concept to include also microinertia effects and to rename it subsequently into **micropolar theory**. For an overview of these so called **microcontinuum** theories we refer to [14, 12, 8, 7, 31, 39].

The Cosserat model includes in a natural way **size effects**, i.e., small samples behave comparatively stiffer than large samples. In classical, size-independent models this would lead to an apparent increase of elastic moduli for smaller samples of the same material. The micropolar theory can explain and analyze more efficiently the diagonal fracture plane under compressive loading for heterogeneous materials, e.g., sand, soil and high porous rock, as compared to the classical continuum theory [48, 44, 1, 2, 23, 17]. Of course, other theories can also provide the micro-rotations of particles and their localizations on the shear bands, e.g., via 2D numerical methods [3, 4]. Unfortunately, the direct measurement of micro-rotation of particles is not achievable with high accuracy but one can measure the displacements in the diagonal fracture plane by means of stereo-photometric methods [10].

The micropolar theory can also be viewed as a generalized continuum theory in which microstructure details are averaged out by a "characteristic internal length scale" L_c [6, 5, 41, 16]. This last parameter can be considered as the size of a representative volume element (RVE) in heterogeneous media and it is frequently used to model damage phenomenon in concrete [40, 32]. a dislocated single crystal is another example of a Cosserat continuum for which lattice curvature is due to geometrically necessary dislocations [38].

The mathematical analysis establishing well-posedness for the infinitesimal strain, Cosserat elastic solid is presented in [24, 11, 22, 19, 20] and extended in [27, 25, 26] for so called linear microstretch models. This analysis has always been based on the **uniform positivity** of the free quadratic energy of the Cosserat solid. The first author has extended the existence results for both the Cosserat model and the more general micromorphic models to the geometrically exact, finite-strain case, see e.g. [37, 34, 36]. More on the mathematical analysis for the nonlinear case can be found in [30, 43]

The important problem of the determination of Cosserat material parameters for **continuous solids with random microstructure** is still a major unsolved problem both analytically and practically. In the linear, isotropic case there are the classical linear elastic Lamé moduli μ and λ

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whose determination is simple and the possibility of four additional constants, one coupling constant $\mu_c \geq 0$ with dimension [MPa] and three curvature length scales. One of the major problems of the micropolar theory is therefore to relate these parameters in an experimental setting, which is not easy to achieve. Lakes [29] proposed an experimental procedure to determine the four supplementary material moduli $(\mu_c, \alpha, \beta, \gamma)$ but the setup is difficult and it is not always achievable for all heterogeneous materials in reality. Usually, a series of experiments with specimens of different slenderness is performed in order to determine the additional four Cosserat parameters [18, 29]. By using the traditional curvature energy complying with pointwise definiteness, one observes, however, an **unphysical unbounded stiffening** behavior for slender specimens which seems to make it impossible to arrive at consistent values for the Cosserat parameters: the values for the parameters will depend strongly on the smallest investigated specimen size. Thus a size-independent determination of the material parameters (which must be the ultimate goal) is impossible. This inconsistency may be in part responsible for the fact that 1. (linear) Cosserat parameters for continuous solids have never gained general acceptance even in the "Cosserat community" and 2. that the linear elastic Cosserat model has never been really accepted by a majority of applied scientists as a useful model to describe size effects in continuous solids.

The linear elastic Cosserat model in variational form

For the **displacement** $u : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ and the **skew-symmetric infinitesimal microrotation** $\overline{A} : \Omega \subset \mathbb{R}^3 \mapsto \mathfrak{so}(3)$ we consider the **two-field** minimization problem

$$I(u, \overline{A}) = \int_\Omega W_{\mathrm{mp}}(\overline{\varepsilon}) + W_{\mathrm{curv}}(\nabla \operatorname{axl} \overline{A}) - \langle f, u \rangle \, \mathrm{dx} \mapsto \quad \min . \text{ w.r.t. } (u, \overline{A}), \tag{1}$$

under the following constitutive requirements and boundary conditions

$$\begin{aligned}
\overline{\varepsilon} &= \nabla u - \overline{A}, \quad \textbf{first Cosserat stretch tensor} \\
u_{|_\Gamma} &= u_{\mathrm{d}} \,, \quad \textbf{essential displacement boundary conditions} \\
W_{\mathrm{mp}}(\overline{\varepsilon}) &= \mu \, \| \operatorname{sym} \overline{\varepsilon} \|^2 + \mu_c \, \| \operatorname{skew} \overline{\varepsilon} \|^2 + \frac{\lambda}{2} \operatorname{tr} [\operatorname{sym} \overline{\varepsilon}]^2 \qquad \textbf{strain energy} \\
&= \mu \, \| \operatorname{sym} \nabla u \|^2 + \mu_c \, \| \operatorname{skew}(\nabla u - \overline{A}) \|^2 + \frac{\lambda}{2} \operatorname{tr} [\operatorname{sym} \nabla u]^2 \qquad (2) \\
&= \mu \, \| \operatorname{dev} \operatorname{sym} \nabla u \|^2 + \mu_c \, \| \operatorname{skew}(\nabla u - \overline{A}) \|^2 + \frac{2\mu + 3\lambda}{6} \operatorname{tr} [\operatorname{sym} \nabla u]^2 \\
&= \mu \, \| \operatorname{sym} \nabla u \|^2 + \frac{\mu_c}{2} \, \| \operatorname{curl} u - 2 \operatorname{axl} \overline{A} \|^2_{\mathbb{R}^3} + \frac{\lambda}{2} \, (\operatorname{Div} u)^2 \,, \\
\phi &:= \operatorname{axl} \overline{A} \in \mathbb{R}^3, \quad \overline{\mathfrak{k}} = \nabla \phi \,, \quad \| \operatorname{curl} \phi \|^2_{\mathbb{R}^3} = 4 \| \operatorname{axl} \operatorname{skew} \nabla \phi \|^2_{\mathbb{R}^3} = 2 \| \operatorname{skew} \nabla \phi \|^2_{\mathbb{M}^{3\times 3}} \,, \\
W_{\mathrm{curv}}(\nabla \phi) &= \frac{\gamma + \beta}{2} \| \operatorname{sym} \nabla \phi \|^2 + \frac{\gamma - \beta}{2} \| \operatorname{skew} \nabla \phi \|^2 + \frac{\alpha}{2} \operatorname{tr} [\nabla \phi]^2 \qquad \textbf{curvature energy} \\
&= \frac{\gamma + \beta}{2} \| \operatorname{dev} \operatorname{sym} \nabla \phi \|^2 + \frac{\gamma - \beta}{2} \| \operatorname{skew} \nabla \phi \|^2 + \frac{3\alpha + (\beta + \gamma)}{6} \operatorname{tr} [\nabla \phi]^2 \\
&= \frac{\gamma}{2} \| \nabla \phi \|^2 + \frac{\beta}{2} \langle \nabla \phi, \nabla \phi^T \rangle + \frac{\alpha}{2} \operatorname{tr} [\nabla \phi]^2 \\
&= \frac{\gamma + \beta}{2} \| \operatorname{sym} \nabla \phi \|^2 + \frac{\gamma - \beta}{4} \| \operatorname{curl} \phi \|^2_{\mathbb{R}^3} + \frac{\alpha}{2} \, (\operatorname{Div} \phi)^2 \,.
\end{aligned}$$

Here, f are given volume forces while u_{d} are Dirichlet boundary conditions[1] for the displacement at $\Gamma \subset \partial\Omega$. Surface tractions, volume couples and surface couples can be included in the stan-

[1]Note that it is always possible to prescribe essential boundary values for the microrotations $\overline{A}$ but we abstain from such a prescription because the physical meaning of it is doubtful.

dard way. The strain energy W_{mp} and the curvature energy W_{curv} are the most general isotropic quadratic forms in the **infinitesimal non-symmetric first Cosserat strain tensor** $\overline{\varepsilon} = \nabla u - \overline{A}$ and the **micropolar curvature tensor** $\overline{\mathfrak{k}} = \nabla \operatorname{axl} \overline{A} = \nabla \phi$ (curvature-twist tensor). The parameters μ, λ[MPa] are the classical Lamé moduli and α, β, γ are additional micropolar moduli with dimension $[\mathrm{Pa} \cdot \mathrm{m}^2] = [\mathrm{N}]$ of a force.

The additional parameter $\mu_c \geq 0$[MPa] in the strain energy is the **Cosserat couple modulus**. For $\mu_c = 0$ the two fields of displacement and microrotations decouple and one is left formally with classical linear elasticity for the displacement u.

Non-negativity of the energy

From the representation of the energy in (2) we can read off immediately the necessary and sufficient conditions for the non-negativity of the free energy. Using the irreducible Lie-Algebra decomposition of a second order tensor into antisymmetric and symmetric-trace free and volumetric parts we must have

$$\begin{aligned} \mu \geq 0\,, \quad \mu_c \geq 0\,, \quad 2\mu + 3\lambda \geq 0\,, \\ \gamma + \beta \geq 0\,, \quad \gamma - \beta \geq 0\,, \quad 3\alpha + (\beta + \gamma) \geq 0\,. \end{aligned} \tag{3}$$

Certain of these inequalities need to be strict in order for the well-posedness of the model. However, the uniform pointwise positivity (strict inequalities everywhere) is not necessary, although it is assumed most often in treatments of linear Cosserat elasticity.

Bounded stiffness for small samples

For every physical material, it is essential that small samples still show bounded rigidity. However, this may or may not be true for Cosserat models, depending on the values of Cosserat parameters. Based on analytic solution formulas for simple three-dimensional Cosserat boundary value problems it has been shown in [35] that for **bounded stiffness for arbitrary thin cylindrical samples we must have**

1. in torsion of a slender cylinder: $\beta + \gamma = 0$ or $\Psi = \frac{\beta+\gamma}{\alpha+\beta+\gamma} = \frac{3}{2}$.
2. in bending of a slender cylinder: $(\beta + \gamma)\,(\gamma - \beta) = 0$.

Foams and bones have been identified by Lakes as prototype Cosserat solids. In order to identify the material parameters, however, Lakes had to leave the traditionally admitted parameter range motivated by strict pointwise positivity. In [28, 29] the value $\Psi = \frac{3}{2}$ has been chosen in order to accommodate bounded stiffness with experimental findings. For a syntactic foam [28] $\beta = \gamma$ has been taken for a best fit. In this case, the curvature energy looks like $W_{\mathrm{curv}}(\nabla\phi) = \gamma \,\|\operatorname{dev}\operatorname{sym}\nabla\phi\|^2$ with $\gamma > 0$. For a polyurethane foam [28] the same procedure has led to a curvature energy that looks like $W_{\mathrm{curv}}(\nabla\phi) = \frac{\beta+\gamma}{2}\,\|\operatorname{dev}\operatorname{sym}\nabla\phi\|^2 + \frac{\gamma-\beta}{4}\|\operatorname{curl}\phi\|^2$.

The linear elastic Cosserat balance equations: strong form

Taking variations of the energy in (1) w.r.t. both displacement $u \in \mathbb{R}^3$ and infinitesimal microrotation $\overline{A} \in \mathfrak{so}(3)$, one arrives at the equilibrium system (the Euler-Lagrange equations of (1))

$$
\begin{aligned}
\operatorname{Div} \sigma &= f\,, \qquad \textbf{balance of linear momentum}\\
-\operatorname{Div} m &= 4\,\mu_c \cdot \operatorname{axl} \operatorname{skew} \overline{\varepsilon}\,, \quad \textbf{balance of angular momentum}\\
\sigma &= 2\mu \cdot \operatorname{sym} \overline{\varepsilon} + 2\mu_c \cdot \operatorname{skew} \overline{\varepsilon} + \lambda \cdot \operatorname{tr}[\overline{\varepsilon}] \cdot 1\!\!1 = (\mu+\mu_c)\cdot \overline{\varepsilon} + (\mu-\mu_c)\cdot \overline{\varepsilon}^T + \lambda \cdot \operatorname{tr}[\overline{\varepsilon}] \cdot 1\!\!1\\
&= 2\mu \cdot \operatorname{dev} \operatorname{sym} \overline{\varepsilon} + 2\mu_c \cdot \operatorname{skew} \overline{\varepsilon} + K \cdot \operatorname{tr}[\overline{\varepsilon}] \cdot 1\!\!1\,,\\
m &= \gamma\, \nabla\phi + \beta\, \nabla\phi^T + \alpha \operatorname{tr}[\nabla\phi] \cdot 1\!\!1\\
&= (\gamma+\beta)\, \operatorname{dev} \operatorname{sym} \nabla\phi + (\gamma-\beta)\, \operatorname{skew} \nabla\phi + \frac{3\alpha + (\gamma+\beta)}{2} \operatorname{tr}[\nabla\phi]\, 1\!\!1\,,\\
\phi &= \operatorname{axl} \overline{A}\,, \quad u_{|_\Gamma} = u_{\mathrm{d}}\,.
\end{aligned} \tag{4}
$$

Here, m is the (second order) **couple stress tensor** which is given as a linear function of the curvature $\nabla\phi = \nabla \operatorname{axl} \overline{A}$.

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Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 second order tensors, written with capital letters and Sym denotes symmetric second orders tensors. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3\times3}$. The identity tensor on $\mathbb{M}^{3\times3}$ will be denoted by $1\!\!1$, so that $\operatorname{tr}[X] = \langle X, 1\!\!1\rangle$. We set $\operatorname{sym}(X) = \frac{1}{2}(X^T + X)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \operatorname{sym}(X) + \operatorname{skew}(X)$. For $X \in \mathbb{M}^{3\times3}$ we set for the deviatoric part $\operatorname{dev} X = X - \frac{1}{3}\operatorname{tr}[X]\,1\!\!1 \in \mathfrak{sl}(3)$ where $\mathfrak{sl}(3)$ is the Lie-algebra of traceless matrices. The set $\operatorname{Sym}(n)$ denotes all symmetric $n \times n$-matrices. The Lie-algebra of $\mathrm{SO}(3) := \{X \in \mathrm{GL}(3)\,|X^T X = 1\!\!1,\ \det[X] = 1\}$ is given by the set $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times3}\,|\,X^T = -X\}$ of all skew symmetric tensors. The canonical identification of $\mathfrak{so}(3)$ and $\mathbb{R}^3$ is denoted by $\operatorname{axl}\overline{A} \in \mathbb{R}^3$ for $\overline{A} \in \mathfrak{so}(3)$. Moreover, we have

$$\forall\ A \in \mathbb{C}^1(\mathbb{R}^3, \mathfrak{so}(3)): \quad \operatorname{Div} A(x) = -\operatorname{curl}\operatorname{axl}(A(x))\,. \tag{1}$$

Note that $(\operatorname{axl}\overline{A}) \times \xi = \overline{A}.\xi$ for all $\xi \in \mathbb{R}^3$, such that

$$\begin{gathered}\operatorname{axl}\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0\end{pmatrix} := \begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix}, \quad \overline{A}_{ij} = \sum_{k=1}^{3} -\varepsilon_{ijk}\cdot\operatorname{axl}\overline{A}_k\,, \\ \|\overline{A}\|^2_{\mathbb{M}^{3\times3}} = 2\,\|\operatorname{axl}\overline{A}\|^2_{\mathbb{R}^3}\,, \qquad \langle \overline{A}, \overline{B}\rangle_{\mathbb{M}^{3\times3}} = 2\langle \operatorname{axl}\overline{A}, \operatorname{axl}\overline{B}\rangle_{\mathbb{R}^3}\,,\end{gathered} \tag{2}$$

where ε_{ijk} is the totally antisymmetric permutation tensor. Here, $\overline{A}.\xi$ denotes the application of the matrix $\overline{A}$ to the vector ξ and $a \times b$ is the usual cross-product. Moreover, the inverse of axl is denoted by anti and defined by

$$\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0\end{pmatrix} := \operatorname{anti}\begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix}, \quad \operatorname{axl}(\operatorname{skew}(a \otimes b)) = -\frac{1}{2}\,a \times b\,, \tag{3}$$

and

$$2\operatorname{skew}(b \otimes a) = \operatorname{anti}(a \times b) = \operatorname{anti}(\operatorname{anti}(a).b)\,. \tag{4}$$

Moreover,

$$\operatorname{curl} u = 2\operatorname{axl}(\operatorname{skew}\nabla u)\,. \tag{5}$$