SPP 2410 Hyperbolic Balance Laws in Fluid Mechanics: Complexity, Scales, Randomness (CoScaRa)

Compressible Euler equations with transport noise

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Motivation

The motivation to study transport noise in fluid mechanics is twofold:

- Holm et al. derived in [6] models with transport noise in fluid dynamics from a physical perspective with the aim of modelling turbulent effects.
- Transport noise can have regularisation effects as demonstrated in [7] for the transport equation and very recently in [8] for the 3D incompressible Navier–Stokes equations.

The Mathematical Problem

We consider the stochastic isentropic Euler equations, describing the flow of a compressible fluid in a bounded domain $\mathcal{O} \subset \mathbb{R}^n$ in $n = 1, 2, 3$ dimensions:

$$\begin{cases} \partial_t \mathbf{m} + \operatorname{div}\left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right) = -\nabla p + \dot{\boldsymbol{\eta}}, \\ \partial_t \varrho + \operatorname{div}(\mathbf{m}) = \dot{\xi}. \end{cases} \tag{1}$$

The unknowns are the momentum $\mathbf{m}$ and the density ϱ. We suppose the barotropic pressure law $p = p(\varrho) = \mathrm{Ma}^{-2}\varrho^{\gamma}$, where $\mathrm{Ma} > 0$ is the Mach-number and $\gamma \geq 1$ the adiabatic exponent.

We speak about transport noise if

$$\dot{\boldsymbol{\eta}} = \sum_{k=1}^{K} \operatorname{div}(\boldsymbol{\sigma}_k \otimes \mathbf{m}) \circ \frac{dW_k}{dt}, \quad \dot{\xi} = \sum_{k=1}^{K} \operatorname{div}(\varrho \boldsymbol{\sigma}_k) \circ \frac{dW_k}{dt}, \tag{2}$$

with given vector fields $\boldsymbol{\sigma}_k$ and stochastic differentials with respect to independent Wiener process W_k in the Stratonovich sense.

Preliminary Work

In [4] the applicant initiated a systematic study of the compressible Navier–Stokes system (the viscous counterpart of the Euler system (1)) subject to stochastic forcing, where

$$\dot{\boldsymbol{\eta}} = \Phi(\varrho, \mathbf{m}) \frac{dW}{dt}, \quad \dot{\xi} = 0,$$

with a (possibly infinite-dimensional) Wiener process W and an operator Φ with appropriate growth assumptions.

❶ This lead to the research monograph [3], which includes the existence of martingale solutions (see also [5] for a result concerning the system with transport noise) as well as stationary solutions.

❷ For problems where uniqueness is not at hand, stationary solutions are taken as a substitute for an invariant measure - a steady state of the system which is believed to fully characterize the behaviour of turbulent fluid flows.

❸ Eventually, we studied the existence of a Markov selection in [1]. This means obtaining a solution for which the probability distribution of the future only depends on the present state of the evolution and is independent of the past.

The results on the Navier–Stokes system serve as a basis to eventually study the Euler equations as inviscid limit.

❹ In [2] global-in-time weak solutions to (1) with stochastic forcing have been constructed by the method of convex integration: In fact, there exists infinitely many weak solutions with the same initial data. However, these solutions have yet two drawbacks: They only exists for very smooth initial data and only up to a (possibly large) stopping time .On the other hand, it is quite interesting to note that these solutions are strong in the probabilistic sense.

Well/ill-posedness

We aim at a rather complete picture concerning the well-posedness (and ill-posedness) of the compressible Euler equations (1) with transport noise (2). In particular, we aim at the following:

❶ Developing a robust existence theory for (1) with transport noise (2). To be more precise, we aim to prove the existence of measure-valued martingale solutions satisfying some form of energy inequality.

❷ Analysing further properties of the solutions constructed in 1. In particular, we will investigate the weak-strong uniqueness property as well as the existence of Markov selections and stationary solutions. This will heavily depend on the energy inequality derived in 1.

❸ Constructing pathwise weak solutions to (1) with transport noise (2) which exist until a given deterministic existence time $T > 0$. Here we intend to apply the method of convex integration and hence expect to obtain infinitely many weak solutions for any given initial datum belonging to the energy class.

❹ Analysing possible regularisation effects for (1) by considering some particular transport noise in the spirit of [8]. To be more precise, we plan to prove that, for any given deterministic time $T > 0$, the solution is regular until time T with high probability.

The Project's Research in the Context of SPP 2410

The isentropic Euler system (1) is an iconic example of a hyperbolic conservation law which is ubiquitous in many applications in physics and engineering. In turbulence theory one is often confronted with resolved large-scale, slow-varying and unresolved small-scale, fast varying components of the velocity field. This is typically modeled by random effects and stochastic differential equations.

Planned cooperations within SPP 2410

- **Project-<Barth>:** Exchange on the impact of random forcing in hyperbolic problems.
- **Project-<Giesselmann-Öffner>:** Exchange on generalized solution concepts.
- **Project-<Herty/Lukáčová>:** Synergy effects with the computational research: Discussion on regularizing effects and questions on the regularity of strong solutions.

References

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