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Compressible fluid-structure interactions

Compressible Navier–Stokes
Linearised Koiter-type shells
Heat-conducting fluids
Regularity and uniqueness

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Plates versus shells

Almost all results so far are concerned with a simplified geometrical set-up, where the domain Ω is given by a rectangle and the flexible part of the boundary is flat (see Figure 1); this is the case of elastic plates. We aim to study general geometries (see Figure 2) including cylinders or sphere; this is the case of elastic shells.

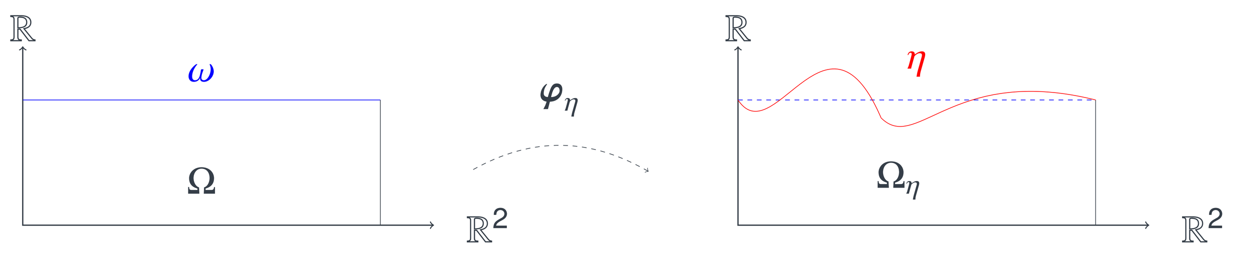

Figure 1: Domain transformation in the simplified set-up.

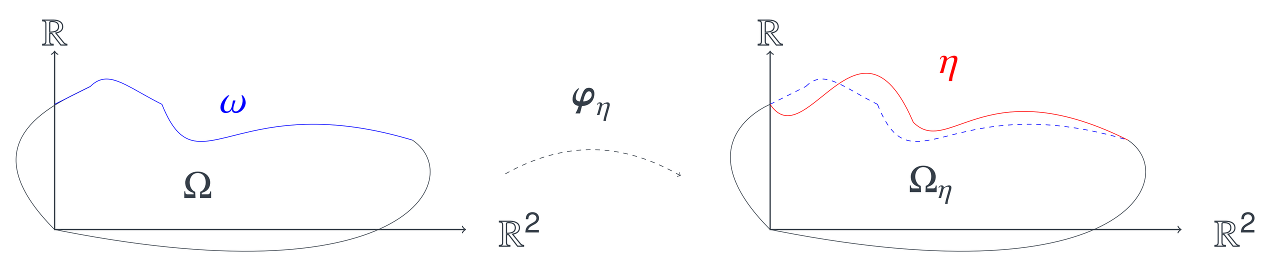

Figure 2: Domain transformation in the general set-up.

The Mathematical Problem

We are interested in the case, where a viscous fluid interacts with a flexible shell which is located at a part of the boundary (or even describes the complete boundary) of the underlying domain $\Omega \subset \mathbb{R}^3$ denoted by ω. The shell, described by a function $\eta : (0,T)\times\omega \to \mathbb{R}$ for some $T>0$, reacts to the surface forces induced by the fluid and deforms the domain Ω to $\Omega_{\eta(t)}$, where the function $\varphi_{\eta(t)}$ describes the coordinate transform (see Figures 1 and 2 above) and $\mathbf{n}_{\eta(t)}$ is the normal at the deformed boundary.
The motion of the fluid is governed by the Navier–Stokes equations

$$\varrho\big(\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\big) = \mu\Delta\mathbf{u} - \nabla p + \varrho\mathbf{f}, \quad \partial_t\varrho + \mathrm{Div}(\varrho\mathbf{u}) = 0, \tag{1}$$

in the moving domain Ω_η where $\mathbf{u} : (0,T)\times\Omega_\eta \to \mathbb{R}^3$ is the velocity field and $p : (0,T)\times\Omega_\eta \to \mathbb{R}$ is the pressure function. The equations are supplemented with initial conditions and the boundary condition $\mathbf{u}\circ\varphi_\eta = \partial_t\eta\mathbf{n}$ at the flexible part of the boundary with normal $\mathbf{n}$. There exist various models in the literature to model the behaviour of the shell and a typical example is given by

$$\partial_t^2\eta - \gamma\partial_t\Delta_y\eta + \alpha\Delta_y^2\eta = g - \mathbf{n}\boldsymbol{\tau}\circ\varphi_\eta\mathbf{n}_\eta\det(\nabla\varphi_\eta) \tag{2}$$

on ω supplemented with initial and boundary conditions. Here $\boldsymbol{\tau}$ denotes the Cauchy stress of the fluid given by Newton's rheological law, that is

$$\boldsymbol{\tau} = \mu\big(\nabla\mathbf{u} + \nabla\mathbf{u}^\top\big) + \nu\,\mathrm{Div}\,\mathbf{u}\,\mathbb{I}_{3\times3} - p\mathbb{I}_{3\times3}.$$

The energy of the shell is given by

$$\tfrac12\int_\omega|\partial_t\eta|^2\mathrm{d}y + \tfrac{\alpha}{2}\int_\omega|\Delta_y\eta|^2\mathrm{d}y.$$

We are interested in the well-posedness of the system (1)–(2), that is, existence and uniqueness of strong solutions (at least locally in time) as well as conditional regularity of weak solutions (under which assumptions is a weak solution a strong one?).

Incompressible fluids

In the case of a homogeneous incompressible fluid the density ϱ is a positive constant. The second equation of (1) reduces to $\mathrm{Div}\,\mathbf{u} = 0$ and the pressure function p is an unknown.

- There exists several results concerning the existence of local-in-time strong solutions. In the 2D case these solutions exist globally in time, cf. [6]. The study of well-posedness for fluid-structure interactions with general reference geometries as in Figure 2 (the case of elastic shells) has already been started very recently. The existence of strong solutions to (1)–(2) has been shown in [1] (in 2D, globally in time) and [4] (in 3D, locally in time).
- Recently we proved in [4] a version of the classical Ladyzhenskaya-Prodi-Serrin condition for (1)–(2): under additional integrability conditions on the velocity field, the weak solution must be regular as well as unique in the class of weak solutions. This is a consequence of an acceleration estimate and a weak-strong uniqueness result for (1)–(2).

Isentropic compressible fluids

In the isentropic compressible Navier–Stokes equations the density $\varrho : (0,T)\times\Omega_\eta \to [0,\infty)$ is an unknown function and the pressure relates to it via the adiabatic law

$$p = p(\varrho) = \tfrac{1}{\mathrm{Ma}^2}\varrho^\gamma,$$

where $\mathrm{Ma} > 0$ is the Mach-number and $\gamma > 1$ is the adiabatic exponent.

- The existence of weak solutions to (1)–(2) in this case was shown in [2] (in the case of a general reference geometry as in Figure 2). These solutions satisfy an energy inequality, where the energy of the fluid system (1) is given by
$$\tfrac12\int_{\Omega_\eta}\varrho|\mathbf{u}|^2\mathrm{d}x + \tfrac{1}{(\gamma-1)\mathrm{Ma}^2}\int_{\Omega_\eta}\varrho^\gamma\mathrm{d}x.$$
The result from [2] gives a counterpart to the celebrated theory by Lions and Feireisl.
- The local-well-posedness in the case of elastic plates is studied in [8] for the flat reference geometry and a weak-strong uniqueness theorem in that case can be found in [9].
- Well-posedness results for the interaction of compressible fluids with elastic shells are completely missing.

Heat-conducting compressible fluids

The motion of a general compressible and heat-conducting fluid is described by the Navier–Stokes–Fourier equations. In addition to the velocity field $\mathbf{u}$ and density ϱ, the absolute temperature $\vartheta : (0,T)\times\Omega_\eta \to [0,\infty)$ is an unknown. In the case of an ideal gas the pressure law is given by

$$p = p(\varrho,\vartheta) = \varrho\vartheta.$$

The internal energy balance is given by

$$c_v\big(\partial_t(\varrho\vartheta) + \mathrm{Div}(\varrho\vartheta\mathbf{u})\big) + \mathrm{Div}(\kappa\nabla\vartheta) = \boldsymbol{\tau} : \nabla\mathbf{u}, \tag{3}$$

where $c_v = \frac{1}{\gamma-1}$ with the adiabatic exponent $\gamma > 1$ and $\kappa = \kappa(\vartheta)$ is the heat-conductivity.

- The existence of weak solutions to (1)–(3) is shown in [3] (in the case of a general reference geometry as in Figure 2). These solutions satisfy an energy equality, where the energy of the fluid system (1), (3) is given by
$$\tfrac12\int_{\Omega_\eta}\varrho|\mathbf{u}|^2\mathrm{d}x + \int_{\Omega_\eta}\varrho c_v\vartheta\,\mathrm{d}x.$$
- Further results can be found in [7] and [10], where the possibility of heat-transfer through the shell is included.
- The local well-posedness of (1)–(3) in the case of elastic plates is studied in [8].
- Results regarding conditional regularity and weak-strong uniqueness seem to be missing completely.

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