

D. Breit, R. Mensah & P. M. Ngougoue Ngougoue,  
University of Duisburg-Essen,  
Faculty of Mathematics

## Plates versus shells

Almost all results so far are concerned with a simplified geometrical set-up, where the domain  $\Omega$  is given by a rectangle and the flexible part of the boundary is flat (see Figure 1); this is the case of elastic plates. We aim to study general geometries (see Figure 2) including cylinders or sphere; this is the case of elastic shells.

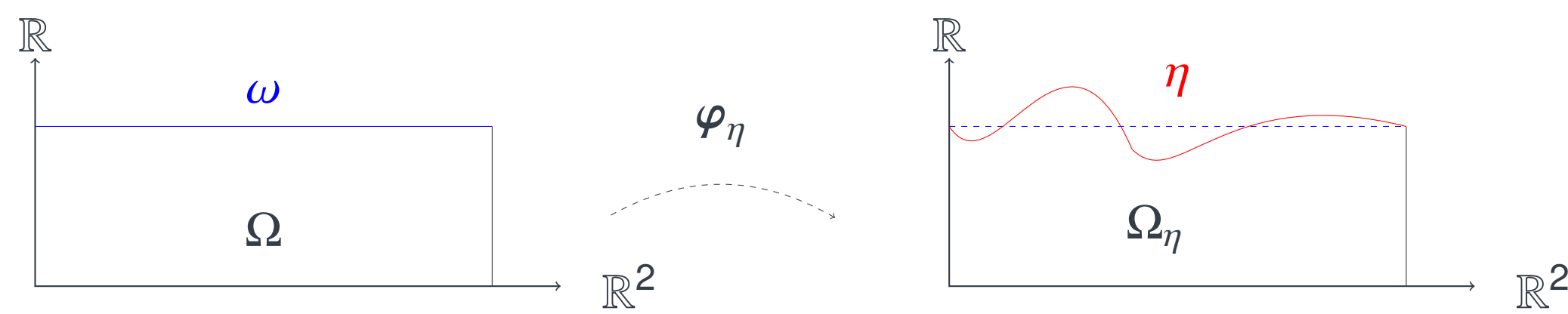


Figure 1: Domain transformation in the simplified set-up.

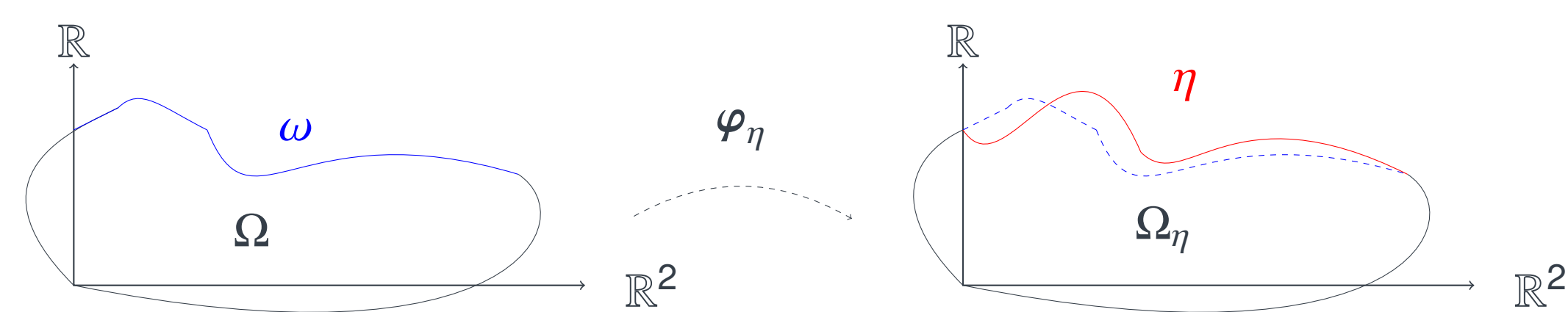


Figure 2: Domain transformation in the general set-up.

## The Mathematical Problem

We are interested in the case, where a viscous fluid interacts with a flexible shell which is located at a part of the boundary (or even describes the complete boundary) of the underlying domain  $\Omega \subset \mathbb{R}^3$  denoted by  $\omega$ . The shell, described by a function  $\eta : (0, T) \times \omega \rightarrow \mathbb{R}$  for some  $T > 0$ , reacts to the surface forces induced by the fluid and deforms the domain  $\Omega$  to  $\Omega_{\eta(t)}$ , where the function  $\varphi_{\eta(t)}$  describes the coordinate transform (see Figures 1 and 2 above) and  $\mathbf{n}_{\eta(t)}$  is the normal at the deformed boundary. The motion of the fluid is governed by the Navier–Stokes equations

$$\varrho(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}) = \mu \Delta \mathbf{u} - \nabla p + \varrho \mathbf{f}, \quad \partial_t \varrho + \text{Div}(\varrho \mathbf{u}) = 0, \quad (1)$$

in the moving domain  $\Omega_{\eta}$  where  $\mathbf{u} : (0, T) \times \Omega_{\eta} \rightarrow \mathbb{R}^3$  is the velocity field and  $p : (0, T) \times \Omega_{\eta} \rightarrow \mathbb{R}$  is the pressure function. The equations are supplemented with initial conditions and the boundary condition  $\mathbf{u} \circ \varphi_{\eta} = \partial_t \eta \mathbf{n}$  at the flexible part of the boundary with normal  $\mathbf{n}$ . There exist various models in the literature to model the behaviour of the shell and a typical example is given by

$$\partial_t^2 \eta - \gamma \partial_t \Delta_y \eta + \alpha \Delta_y^2 \eta = g - \mathbf{n} \tau \circ \varphi_{\eta} \mathbf{n}_{\eta} \det(\nabla \varphi_{\eta}) \quad (2)$$

on  $\omega$  supplemented with initial and boundary conditions. Here  $\tau$  denotes the Cauchy stress of the fluid given by Newton's rheological law, that is

$$\tau = \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) + \nu \text{Div} \mathbf{u} \mathbb{I}_{3 \times 3} - p \mathbb{I}_{3 \times 3}.$$

The energy of the shell is given by

$$\frac{1}{2} \int_{\omega} |\partial_t \eta|^2 dy + \frac{\alpha}{2} \int_{\omega} |\Delta_y \eta|^2 dy.$$

We are interested in the well-posedness of the system (1)–(2), that is, existence and uniqueness of strong solutions (at least locally in time) as well as conditional regularity of weak solutions (under which assumptions is a weak solution a strong one?).

## Incompressible fluids

In the case of a homogeneous incompressible fluid the density  $\varrho$  is a positive constant. The second equation of (1) reduces to  $\text{Div} \mathbf{u} = 0$  and the pressure function  $p$  is an unknown.

- There exists several results concerning the existence of local-in-time strong solutions. In the 2D case these solutions exist globally in time, cf. [6]. The study of well-posedness for fluid-structure interactions with general reference geometries as in Figure 2 (the case of elastic shells) has already been started very recently. The existence of strong solutions to (1)–(2) has been shown in [1] (in 2D, globally in time) and [4] (in 3D, locally in time).
- Recently we proved in [4] a version of the classical Ladyzhenskaya-Prodi-Serrin condition for (1)–(2): under additional integrability conditions on the velocity field, the weak solution must be regular as well as unique in the class of weak solutions. This is a consequence of an acceleration estimate and a weak-strong uniqueness result for (1)–(2).

## Isentropic compressible fluids

In the isentropic compressible Navier–Stokes equations the density  $\varrho : (0, T) \times \Omega_{\eta} \rightarrow [0, \infty)$  is an unknown function and the pressure relates to it via the adiabatic law

$$p = p(\varrho) = \frac{1}{\text{Ma}^2} \varrho^{\gamma},$$

where  $\text{Ma} > 0$  is the Mach-number and  $\gamma > 1$  is the adiabatic exponent.

- The existence of weak solutions to (1)–(2) in this case was shown in [2] (in the case of a general reference geometry as in Figure 2). These solutions satisfy an energy inequality, where the energy of the fluid system (1) is given by

$$\frac{1}{2} \int_{\Omega_{\eta}} \varrho |\mathbf{u}|^2 dx + \frac{1}{(\gamma-1)\text{Ma}^2} \int_{\Omega_{\eta}} \varrho^{\gamma} dx.$$

The result from [2] gives a counterpart to the celebrated theory by Lions and Feireisl.

- The local-well-posedness in the case of elastic plates is studied in [8] for the flat reference geometry and a weak-strong uniqueness theorem in that case can be found in [9].
- Well-posedness results for the interaction of compressible fluids with elastic shells are completely missing.

## Heat-conducting compressible fluids

The motion of a general compressible and heat-conducting fluid is described by the Navier–Stokes–Fourier equations. In addition to the velocity field  $\mathbf{u}$  and density  $\varrho$ , the absolute temperature  $\vartheta : (0, T) \times \Omega_{\eta} \rightarrow [0, \infty)$  is an unknown. In the case of an ideal gas the pressure law is given by

$$p = p(\varrho, \vartheta) = \varrho \vartheta.$$

The internal energy balance is given by

$$c_v(\partial_t(\varrho \vartheta) + \text{Div}(\varrho \vartheta \mathbf{u})) + \text{Div}(\kappa \nabla \vartheta) = \tau : \nabla \mathbf{u}, \quad (3)$$

where  $c_v = \frac{1}{\gamma-1}$  with the adiabatic exponent  $\gamma > 1$  and  $\kappa = \kappa(\vartheta)$  is the heat-conductivity.

- The existence of weak solutions to (1)–(3) is shown in [3] (in the case of a general reference geometry as in Figure 2). These solutions satisfy an energy equality, where the energy of the fluid system (1), (3) is given by

$$\frac{1}{2} \int_{\Omega_{\eta}} \varrho |\mathbf{u}|^2 dx + \int_{\Omega_{\eta}} \varrho c_v \vartheta dx.$$

- Further results can be found in [7] and [10], where the possibility of heat-transfer through the shell is included.
- The local well-posedness of (1)–(3) in the case of elastic plates is studied in [8].
- Results regarding conditional regularity and weak-strong uniqueness seem to be missing completely.

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