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Numerical approximation of 3D stochastic Navier–Stokes equations

Stochastic Navier–Stokes
A priori error analysis
Computational fluid dynamics
Stochastic simulations

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The Mathematical Problem

We consider the stochastic Navier–Stokes equations, describing the flow of a viscous incompressible fluid in a bounded domain $\mathcal{O} \subset \mathbb{R}^3$:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \Delta \mathbf{u} - \nabla p + \dot{\eta}, \quad \mathrm{div}(\mathbf{u}) = 0, \tag{1}$$

where we put the physical constants density and viscosity to 1 for simplicity. The unknowns are the velocity field $\mathbf{u}$ and the pressure p. The noise term $\dot{\eta}$ can depend on the solution $\mathbf{u}$ itself. In the following we describe different scenarios concerning their role.

Deterministic case. In the deterministic case (that is $\dot{\eta} = 0$ or $\dot{\eta} = \mathbf{f}$ deterministic) it is well-known that unique smooth solutions exist locally in time. It is, however, still one of the biggest open problems in analysis if solutions can develop singularities in finite time.

Stochastic forcing. We speak about stochastic forcing if

$$\dot{\eta} = \Phi(\mathbf{u}) \frac{\mathrm{d}W}{\mathrm{d}t}$$

with a (possibly infinite-dimensional) Wiener process W and an operator Φ with appropriate growth assumptions. A simple example is given by

$$\dot{\eta} = \begin{pmatrix} u^1 \frac{\mathrm{d}W^1}{\mathrm{d}t} \\ u^2 \frac{\mathrm{d}W^2}{\mathrm{d}t} \\ u^3 \frac{\mathrm{d}W^3}{\mathrm{d}t} \end{pmatrix}, \quad W = \begin{pmatrix} W^1 \\ W^2 \\ W^3 \end{pmatrix},$$

with three independent Wiener processes W^1, W^2 and W^3. If Φ is independent of $\mathbf{u}$ we speak of additive noise. A particular instance of relevance is given by

$$\dot{\eta} = \sum_k \mathbf{e}_k \frac{\mathrm{d}W_k}{\mathrm{d}t}$$

with a possibly infinite sum, smooth solenoidal vector fields $\mathbf{e}_k$ and independent Wiener processes W_k. Explicit relations for the $\mathbf{e}_k'$s are given in [1], where the stochastic Navier–Stokes equation are used to derive the Kolmogorov–Obukhov Statistical Theory of Turbulence.

As in the deterministic case smooth solutions to (1) with stochastic forcing exist locally in time (here the existence time is a random variable which is almost surely positive).

Transport noise. We speak about transport noise if

$$\dot{\eta} = \sum_{k=1}^{K} (\boldsymbol{\sigma}_k \cdot \nabla)\mathbf{u} \circ \frac{\mathrm{d}W_k}{\mathrm{d}t}, \tag{2}$$

$$\text{or} \quad \dot{\eta} = \sum_{k=1}^{K} \mathrm{curl}^{-1}((\boldsymbol{\sigma}_k \cdot \nabla)\mathrm{curl}\, \mathbf{u}) \circ \frac{\mathrm{d}W_k}{\mathrm{d}t}, \tag{3}$$

with given vector fields $\boldsymbol{\sigma}_k$ and stochastic differentials with respect to independent Wiener process W_k in the Stratonovich sense (an infinite sum is possible too). Note that in the first case, the noise is energy conservative. Hence stochasticity does not enter as an external force but as an intrinsic property of the system.

Numerical approximation in 2D

it is shown in [2], [8] (space-periodic problem) and [5] (Dirichlet problem) that for any $\xi > 0$

$$\mathbb{P}\left[\frac{\max_m \|\mathbf{u}(t_m) - \mathbf{u}_{h,m}\|^2_{L^2_x} + \sum_m \tau \|\nabla \mathbf{u}(t_m) - \nabla \mathbf{u}_{h,m}\|^2_{L^2_x}}{h^{2\beta} + \tau^{2\alpha}} > \xi\right] \to 0 \tag{4}$$

as $h, \tau \to 0$ (where $\alpha < \frac{1}{2}$ and $\beta < 1$ are arbitrary). Here $\mathbf{u}$ is the solution to (1) with stochastic forcing and $\mathbf{u}_{h,m}$ the approximation of $\mathbf{u}(t_m)$ with discretisation parameters $\tau = T/M$ (in time, by semi-implicit Euler-Maruyama) and h (in space, by finite elements). The relation (4) tells us that the convergence in probability is of order (almost) 1/2 in time and 1 in space. It seems to be an intrinsic feature of stochastic partial differential equations (SPDEs) with general non-Lipschitz nonlinearities such as (1) that the more common concept of a pathwise error (an error measured in $L^2(\Omega)$) is too strong. Hence (4) is the best result we can hope for in the general case.

Note that the upper threshold $\alpha < \frac{1}{2}$ in (4) arises from the limited time regularity of the solution which is inherited from the driving Wiener process. If the stochastic forcing is additive and sufficiently smooth in space this can be improved to $\alpha < 1$, cf. [4]. This is based on a transformation to a random PDE which is not available for multiplicative noise.

The only available result on (1) with transport noise is the recent paper [6] concerning the temporal discretisation. Working with a structure preserving scheme, which is in line with the Stratonovich noise, it is shown that the pathwise error is of order $\frac{1}{2}$. The improvement compared to (4) is due to (higher order) pathwise energy estimates independent of the noise. The result is, however, limited to constant vector fields $\boldsymbol{\sigma}_k$ in (2) and periodic boundary conditions.

Numerical approximation in 3D

A counterpart of (4) is shown in [3] for the space-periodic problem. It holds locally in time, that is as long as t_M is below the hypothetical blow-up time of the solution (about which we only know that it is almost surely positive). The only other result concerning the 3D case is [7], where the consistency with a weak solution to (1) (which exists globally in time) is shown.

Objectives

The aim is to study the space-time discretisation of (1) (locally in time) with different types of noise, as explained above, and to prove optimal convergence rates (with respect to the error in probability, cf. (4)) and to confirm them by numerical simulations. In particular, we aim at the following work packages:

❶ Multiplicative stochastic forcing. Here we aim to extend the result from [3] (convergence of order 1 in space and 1/2 in time) in two directions: from periodic to Dirichlet boundary conditions and to consider further spatial discretisations in addition to finite element methods such as discontinuous Galerkin (DG) methods.

❷ Additive stochastic forcing. In this case we hope to improve the temporal convergence rate from the first working package to order 1. This would provide a counterpart of the two-dimensional result of [4]. Again, we plan to also include DG methods.

❸ Transport noise. We aim to analyse both cases (2) and (3) and to analyse the error with respect to space and time. So far, even in the 2D case only the temporal error has been studied. Moreover, we will include boundary effects as well as DG.

❹ In a final working package we are concerned with the implementation of the discretisation of (1). Here we plan to confirm the results from the previous work packages by simulations (this is a very expansive task and must be performed on a compute cluster). Also, it would be interesting to see if one can verify the regularisation effect observed in [9] if the noise from (3) is considered.

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