Weakly Symmetric Stress Methods for Elasto-Plasticity

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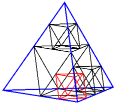

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Overview

Stress-Based Mixed Finite Elements for Linear Elasticity

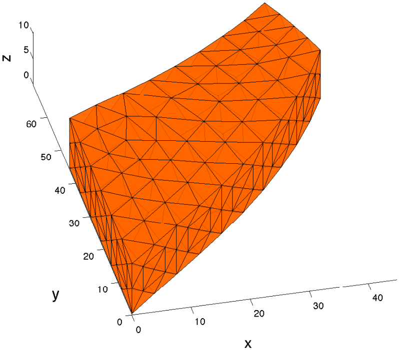
Stress-Based Methods for Elasto-Plasticity

Reconstruction-Based A Posteriori Error Estimation

Stress-Based Mixed Finite Elements for Linear Elasticity

The Linear Elasticity Model

Displacement field $\mathbf{u} : \Omega \to \mathbb{R}^3$
Stress tensor $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{3\times 3}$



$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\mathbf{u}^T}{2}$$
$$= \begin{bmatrix} \partial_1 u_1 & \frac{\partial_2 u_1 + \partial_1 u_2}{2} & \frac{\partial_3 u_1 + \partial_1 u_3}{2} \\ \frac{\partial_1 u_2 + \partial_2 u_1}{2} & \partial_2 u_2 & \frac{\partial_3 u_2 + \partial_2 u_3}{2} \\ \frac{\partial_1 u_3 + \partial_3 u_1}{2} & \frac{\partial_2 u_3 + \partial_3 u_2}{2} & \partial_3 u_3 \end{bmatrix}$$

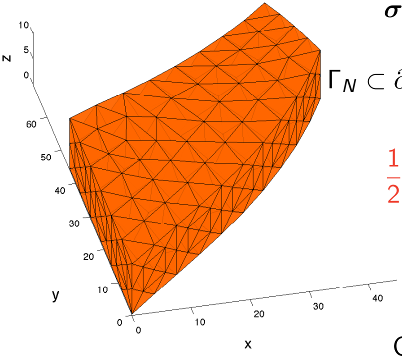
$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$
$$= \mathcal{C}\,\boldsymbol{\varepsilon}(\mathbf{u}) := 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda\,(\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}))\,\mathbf{I}$$

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \mathcal{A}\boldsymbol{\sigma} := \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{3\lambda + 2\mu}(\operatorname{tr}\boldsymbol{\sigma})\,\mathbf{I}\right) = \frac{1+\nu}{E}\left(\boldsymbol{\sigma} - \frac{\nu}{1+\nu}(\operatorname{tr}\boldsymbol{\sigma})\,\mathbf{I}\right)$$

Stress-Based Mixed Finite Elements for Linear Elasticity

The Linear Elasticity Model

Displacement field $\mathbf{u} : \Omega \to \mathbb{R}^3$
Stress tensor $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{3\times 3}$



$$\boldsymbol{\sigma} : \varepsilon = \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij}\varepsilon_{ij} \, , \; \mathbf{f} \cdot \mathbf{u} = \sum_{i=1}^{3} f_i u_i$$

$\Gamma_N \subset \partial\Omega$

$$\frac{1}{2} \int_\Omega \boldsymbol{\sigma} : \varepsilon(\mathbf{u}) \, dx$$

$$- \int_\Omega \mathbf{f} \cdot \mathbf{u} \, dx - \int_{\Gamma_N} \mathbf{t} \cdot \mathbf{u} \, dx \longrightarrow \min$$

Optimality condition:

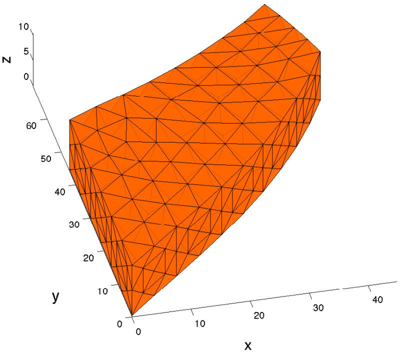
$$\int_\Omega \boldsymbol{\sigma} : \varepsilon(\mathbf{v}) \, dx - \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx - \int_{\Gamma_N} \mathbf{t} \cdot \mathbf{v} \, dx = 0$$

for all "variations" $\mathbf{v}$

Stress-Based Mixed Finite Elements for Linear Elasticity

The Linear Elasticity Model

Displacement field $\mathbf{u} : \Omega \to \mathbb{R}^3$
Stress tensor $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{3\times 3}$
Optimality condition: $\int_\Omega \boldsymbol{\sigma} : \varepsilon(\mathbf{v})\, dx - \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx - \int_{\Gamma_N} \mathbf{t} \cdot \mathbf{v}\, dx = 0$



$$\operatorname{div} \boldsymbol{\sigma} = \begin{bmatrix} \partial_1\sigma_{11} + \partial_2\sigma_{12} + \partial_3\sigma_{13} \\ \partial_1\sigma_{21} + \partial_2\sigma_{22} + \partial_3\sigma_{23} \\ \partial_1\sigma_{31} + \partial_2\sigma_{32} + \partial_3\sigma_{33} \end{bmatrix}$$

$$\begin{aligned} \operatorname{div} \boldsymbol{\sigma} + \mathbf{f} &= \mathbf{0} \text{ in } \Omega \\ \mathcal{A}\boldsymbol{\sigma} - \varepsilon(\mathbf{u}) &= \mathbf{0} \text{ in } \Omega \\ \mathbf{u} &= \mathbf{0} \text{ on } \Gamma_D \\ \boldsymbol{\sigma} \cdot \mathbf{n} &= \mathbf{t} \text{ on } \Gamma_N \end{aligned}$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \begin{bmatrix} n_1\sigma_{11} + n_2\sigma_{12} + n_3\sigma_{13} \\ n_1\sigma_{21} + n_2\sigma_{22} + n_3\sigma_{23} \\ n_1\sigma_{31} + n_2\sigma_{32} + n_3\sigma_{33} \end{bmatrix}$$

Stress-Based Mixed Finite Elements for Linear Elasticity

Hellinger-Reissner Principle

Formulation purely with respect to stress tensor $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{3\times 3}$

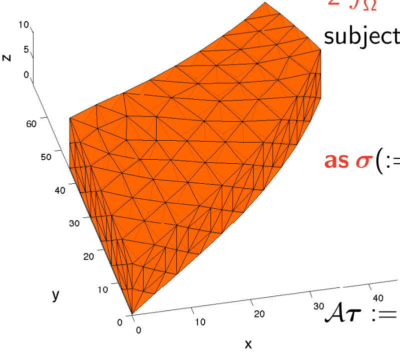
Minimize free energy $\displaystyle \frac{1}{2}\int_\Omega (\mathcal{A}\boldsymbol{\sigma}) : \boldsymbol{\sigma}\, dx$

subject to the constraints:

$$\operatorname{div} \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \text{ in } \Omega$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t} \text{ on } \Gamma_N$$

$$\mathbf{as}\, \boldsymbol{\sigma} (:= (\boldsymbol{\sigma} - \boldsymbol{\sigma}^T)/2) = \mathbf{0} \text{ in } \Omega$$



$$\mathcal{A}\boldsymbol{\tau} := \frac{1}{2\mu}\left(\boldsymbol{\tau} - \frac{\lambda}{3\lambda + 2\mu}(\operatorname{tr}\boldsymbol{\tau})\,\mathbf{I}\right)$$

$$\longrightarrow \frac{1}{2\mu}\left(\boldsymbol{\tau} - \frac{1}{3}(\operatorname{tr}\boldsymbol{\tau})\,\mathbf{I}\right) =: \frac{1}{2\mu}\,\mathbf{dev}\,\boldsymbol{\tau}$$

Determine $\boldsymbol{\sigma} \in \boldsymbol{\sigma}^N + \boldsymbol{\Sigma}$, $\boldsymbol{\Sigma} \subseteq H_{\Gamma_N}(\mathrm{div},\Omega)^3$ (with $\boldsymbol{\sigma}^N \cdot \mathbf{n} = \mathbf{t}$) and Lagrange multipliers $\mathbf{u} \in \mathbf{V} \subseteq L^2(\Omega)^3$, $\boldsymbol{\rho} \in \mathbf{R} \subseteq L^2(\Omega)^{3\times 3,\mathbf{as}}$ such that

$$\begin{aligned}
(\mathcal{A}\boldsymbol{\sigma},\boldsymbol{\tau})_{L^2(\Omega)} + (\mathbf{u},\mathrm{div}\,\boldsymbol{\tau})_{L^2(\Omega)} + (\boldsymbol{\rho},\mathbf{as}\,\boldsymbol{\tau})_{L^2(\Omega)} &= 0 \\
(\mathrm{div}\,\boldsymbol{\sigma} + \mathbf{f},\mathbf{v})_{L^2(\Omega)} &= 0 \\
(\mathbf{as}\,\boldsymbol{\sigma},\boldsymbol{\vartheta})_{L^2(\Omega)} &= 0
\end{aligned}$$

for all $\boldsymbol{\tau} \in \boldsymbol{\Sigma}$, $\mathbf{v} \in \mathbf{V}$ and $\boldsymbol{\vartheta} \in \mathbf{R}$

$L^2(\Omega)^{3\times 3,\mathbf{as}} := \{\boldsymbol{\tau} \in L^2(\Omega)^{3\times 3} : \boldsymbol{\tau} + \boldsymbol{\tau}^T = \mathbf{0}\}$

Why use a stress-based variational formulation?

- ▷ More accurate stress approximation (in $H(\mathrm{div})$)!
- ▷ Local momentum conservation satisfied to higher accuracy!
- ▷ More straightforward handling of stress-based constraints like for plasticity or (frictional) contact

Stress-Based Mixed Finite Elements for Linear Elasticity

Optimality (KKT) Conditions

Determine $\boldsymbol{\sigma} \in \boldsymbol{\sigma}^N + \boldsymbol{\Sigma}$, $\mathbf{u} \in \mathbf{V}$ and $\boldsymbol{\rho} \in \mathbf{R}$ such that

$$\begin{aligned} (\mathcal{A}\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathbf{u}, \operatorname{div} \boldsymbol{\tau}) + (\boldsymbol{\rho}, \mathbf{as}\, \boldsymbol{\tau}) &= 0 \text{ for all } \boldsymbol{\tau} \in \boldsymbol{\Sigma} \\ (\operatorname{div} \boldsymbol{\sigma} + \mathbf{f}, \mathbf{v}) &= 0 \text{ for all } \mathbf{v} \in \mathbf{V} \\ (\mathbf{as}\, \boldsymbol{\sigma}, \boldsymbol{\vartheta}) &= 0 \text{ for all } \boldsymbol{\vartheta} \in \mathbf{R} \end{aligned}$$

Admissible combinations of spaces need to satisfy the inf-sup condition

$$\inf_{\mathbf{v} \in \mathbf{V}, \boldsymbol{\vartheta} \in \mathbf{R}} \sup_{\boldsymbol{\tau} \in \boldsymbol{\Sigma}} \frac{(\operatorname{div} \boldsymbol{\tau}, \mathbf{v})_{L^2(\Omega)} + (\mathbf{as}\, \boldsymbol{\tau}, \boldsymbol{\vartheta})_{L^2(\Omega)}}{\|\boldsymbol{\tau}\|_{H(\operatorname{div},\Omega)} \left(\|\mathbf{v}\|_{L^2(\Omega)} + \|\boldsymbol{\vartheta}\|_{L^2(\Omega)} \right)} \geq \beta > 0$$

One possibility (Boffi/Brezzi/Fortin, 2009):

$$\begin{aligned} \operatorname{div} \boldsymbol{\Sigma} &= \mathbf{V} \\ \boldsymbol{\Sigma} &\supseteq \mathbf{curl}\, \boldsymbol{\Xi} \text{ with a Stokes-stable pair } (\boldsymbol{\Xi}, \mathbf{R}) \end{aligned}$$

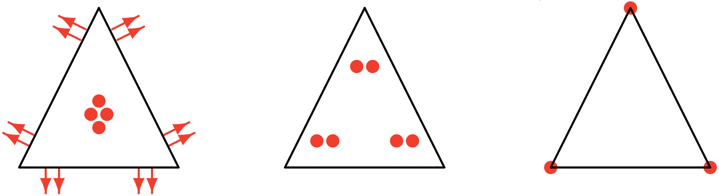
Stress-Based Mixed Finite Elements for Linear Elasticity

$\mathcal{T} = \{T : T \subset \mathbb{R}^d\}$: Triangulation (member of a non-degenerate family)

Boffi/Brezzi/Fortin (2009): $k \geq 1$ ($d = 2$ or 3)

$$\begin{aligned} \mathbf{\Sigma} \subset H(\mathrm{div}, \Omega)^d &: \boldsymbol{\sigma}|_T \in RT_k(T) = P_k^{d\times d}(T) + P_k^d(T)\,\mathbf{x} \\ \mathbf{V} \subset L^2(\Omega)^d &: \mathbf{u}|_T \in P_k^d(T) \\ \mathbf{R} \subset H^1(\Omega)^{d\times d} &: \boldsymbol{\rho}|_T \in P_k^{d\times d,\mathrm{as}}(T) \end{aligned}$$

$k = 1$: $\boldsymbol{\sigma}$ $\mathbf{u}$ $\boldsymbol{\rho}$



$RT_1(\mathcal{T})^2$ $DP_1(\mathcal{T})^2$ $P_1(\mathcal{T})$

Here: $\mathbf{\Sigma} \supseteq \mathbf{curl}\,\mathbf{\Xi}$ with Stokes-stable pair $(\mathbf{\Xi}, \mathbf{R})$
$(\mathbf{\Xi}, \mathbf{R})$ ($P_2(\mathcal{T})/P_1(\mathcal{T})$: Taylor-Hood elements)

Stress-Based Mixed Finite Elements for Linear Elasticity

(Incomplete) list of weakly symmetric mixed finite element combinations:

Arnold/Brezzi/Douglas (1984): PEERS

the elimination of the stresses from the equilibrium and constitutive equations of a material exhibiting plastic behaviour is difficult; consequently, only a mixed formulation is feasible.

. . .

Brezzi/Douglas/Marini (1986)

Stenberg (1988)

. . .

Stein/Rolfes (1990)

Klaas/Schröder/Stein/Miehe (1995)

. . .

Lonsing/Verfürth (2004): $PEERS_k$ and $BDMS_k$

. . .

Arnold/Falk/Winther (2007)

. . .

Boffi/Brezzi/Fortin (2009)

Gopalakrishnan/Guzmán (2011)

Pechstein/Schöberl (2011)

Stress-Based Methods for Elasto-Plasticity

Von Mises Yield Criterion (Perfect Plasticity)

Admissible set for the stresses:

$$\mathcal{K}(t) = \{\boldsymbol{\sigma} \in H(\mathrm{div},\Omega)^d : \mathrm{div}\,\boldsymbol{\sigma} + \mathbf{f} = \mathbf{0}\,,\ \mathbf{as}\,\boldsymbol{\sigma} = \mathbf{0} \text{ in } \Omega\,,\\ \boldsymbol{\sigma}\cdot\mathbf{n} = \ell(t) \text{ on } \Gamma_N\,,\ |\mathbf{dev}\,\boldsymbol{\sigma}| \le \kappa \text{ in } \Omega\}$$

not a subspace any more!

Stress formulation of elastoplasticity:

Find $\boldsymbol{\sigma}(t) \in \mathcal{K}(t)$ such that $(\mathcal{A}\,\dot{\boldsymbol{\sigma}}(t), \boldsymbol{\tau} - \boldsymbol{\sigma}(t)) \ge 0$ holds for all $\boldsymbol{\tau} \in \mathcal{K}$

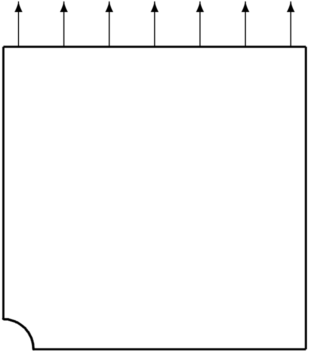
Incremental formulation of elastoplasticity (quasi-static problem):

Find $\boldsymbol{\sigma}(t) \in \mathcal{K}(t)$ such that $(\mathcal{A}\,(\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}^{\mathrm{old}}), \boldsymbol{\tau} - \boldsymbol{\sigma}(t)) \ge 0\ \ \forall\ \ \boldsymbol{\tau} \in \mathcal{K}$

Han/Reddy (1999/2013)
Wieners (1999/2007)
Bartels (2015)

Stress-Based Mixed Finite Elements for Elasto-Plasticity

Benchmark Test (Perfect Plasticity)



Poisson ratio: $\nu = 0.29$

Boundary conditions:
$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0}$ at right bdy and circle
$\boldsymbol{\sigma} \cdot \mathbf{n} = (0, \gamma)$ at upper bdy

Symmetry conditions:
$(\sigma_{11}, \sigma_{12}) \cdot \mathbf{n} = 0, u_2 = 0$ at bottom
$u_1 = 0, (\sigma_{21}, \sigma_{22}) \cdot \mathbf{n} = 0$ at left bdy

Load cycle: γ from 0 to 4.5 and then back
Load step size: $\delta\gamma = 0.025$

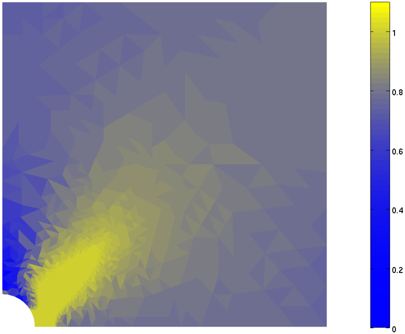
Plane strain

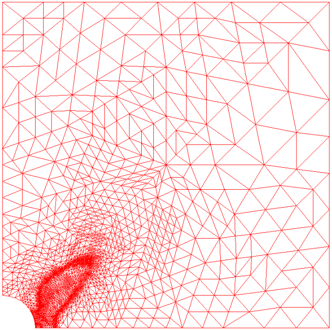
Collection of benchmark problems for elastoplasticity from
Stein/Wriggers/Rieger/Schmidt and Lang/Wieners/Wittum
in E. Stein (editor): Error-controlled Adaptive Finite Elements in Solid Mechanics. Wiley, 2002

Stress-Based Mixed Finite Elements for Elasto-Plasticity

Benchmark Test (Perfect Plasticity)

$\gamma = 4.0$: $|\mathbf{dev}\,\boldsymbol{\sigma}|$ triangulation

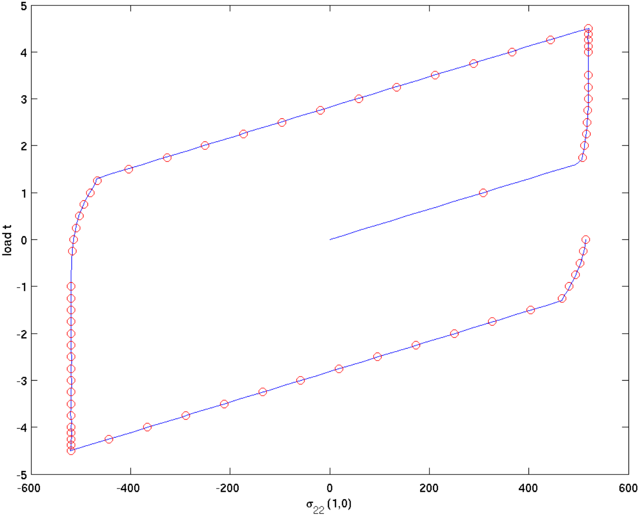




Stress-Based Mixed Finite Elements for Elasto-Plasticity

Benchmark Test (Perfect Plasticity)

Stress values at lower left corner for one complete load cycle



Stress-Based Mixed Finite Elements for Elasto-Plasticity

Optimality (KKT) Conditions (Perfect Plasticity)

Determine $\boldsymbol{\sigma} \in \boldsymbol{\sigma}^N + \boldsymbol{\Sigma}$, Lagrange multipliers $\mathbf{u} \in \mathbf{V}$, $\boldsymbol{\rho} \in \mathbf{R}$, $\chi \in X^+$ s.t.

$$
\begin{aligned}
&(\mathcal{A}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{old}}), \boldsymbol{\tau})_{L^2(\Omega)} + (\mathbf{u}, \operatorname{div} \boldsymbol{\tau})_{L^2(\Omega)} + (\boldsymbol{\rho}, \mathbf{as}\,\boldsymbol{\tau})_{L^2(\Omega)} \\
&\qquad\qquad + \left(\chi, \frac{\mathbf{dev}\boldsymbol{\sigma}}{|\mathbf{dev}\boldsymbol{\sigma}|} : \mathbf{dev}\,\boldsymbol{\tau}\right)_\star = 0 \\
&(\operatorname{div} \boldsymbol{\sigma} + \mathbf{f}, \mathbf{v})_{L^2(\Omega)} = 0 \\
&(\mathbf{as}\,\boldsymbol{\sigma}, \boldsymbol{\vartheta})_{L^2(\Omega)} = 0 \\
&(|\mathbf{dev}\,\boldsymbol{\sigma}| - \kappa, \omega)_\star \leq 0
\end{aligned}
$$

for all $\boldsymbol{\tau} \in \boldsymbol{\Sigma}$, $\mathbf{v} \in \mathbf{V}$, $\boldsymbol{\vartheta} \in \mathbf{R}$ and $\omega \in X^+$

$$
D_{\boldsymbol{\tau}}(|\mathbf{dev}\,\boldsymbol{\sigma}|) = \frac{\mathbf{dev}\,\boldsymbol{\sigma}}{|\mathbf{dev}\boldsymbol{\sigma}|} : \mathbf{dev}\,\boldsymbol{\tau} \text{ for } \mathbf{dev}\,\boldsymbol{\sigma} \neq \mathbf{0}
$$

Complementarity condition: $(|\mathbf{dev}\,\boldsymbol{\sigma}| - \kappa, \chi)_{L^\infty(\Omega) \times L^\infty(\Omega)'} = 0$

What is the meaning of Lagrange multiplier $\chi \in X^+$?

Stress-Based Mixed Finite Elements for Elasto-Plasticity

Optimality (KKT) Conditions (Perfect Plasticity)

$$(\mathcal{A}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{old}}), \boldsymbol{\tau}) + (\mathbf{u}, \operatorname{div} \boldsymbol{\tau}) + (\boldsymbol{\rho}, \mathbf{as}\, \boldsymbol{\tau}) + (\chi, \frac{\mathbf{dev}\boldsymbol{\sigma}}{|\mathbf{dev}\boldsymbol{\sigma}| : \mathbf{dev}\, \boldsymbol{\tau}}) = 0$$

$\Longleftrightarrow$ (with $\boldsymbol{\rho} = \mathbf{as}(\boldsymbol{\nabla}\mathbf{u})$)

$$(\mathcal{A}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{old}}) - \boldsymbol{\varepsilon}(\mathbf{u}) + \chi \frac{\mathbf{dev}\, \boldsymbol{\sigma}}{|\mathbf{dev}\, \boldsymbol{\sigma}|}, \boldsymbol{\tau}) = 0$$

Stress-Based Mixed Finite Elements for Elasto-Plasticity

Optimality (KKT) Conditions (Perfect Plasticity)

$$(\mathcal{A}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\text{old}}), \boldsymbol{\tau}) + (\mathbf{u}, \mathrm{div}\, \boldsymbol{\tau}) + (\boldsymbol{\rho}, \mathbf{as}\, \boldsymbol{\tau}) + (\chi, \frac{\mathbf{dev}\boldsymbol{\sigma}}{|\mathbf{dev}\boldsymbol{\sigma}|} : \mathbf{dev}\, \boldsymbol{\tau}) = 0$$

$\Longleftrightarrow$ (with $\boldsymbol{\rho} = \mathbf{as}(\boldsymbol{\nabla}\mathbf{u})$)

$$(\mathcal{A}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\text{old}}) - \boldsymbol{\varepsilon}(\mathbf{u}) + \chi \frac{\mathbf{dev}\, \boldsymbol{\sigma}}{|\mathbf{dev}\, \boldsymbol{\sigma}|}, \boldsymbol{\tau}) = 0$$

$$\Longleftrightarrow \qquad \begin{aligned} \frac{1}{d\lambda + 2\mu}\mathrm{tr}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\text{old}}) &= \mathrm{tr}\, \boldsymbol{\varepsilon}(\mathbf{u}) \\ \frac{1}{2\mu}\mathbf{dev}\, \boldsymbol{\sigma} + \chi \frac{\mathbf{dev}\, \boldsymbol{\sigma}}{|\mathbf{dev}\, \boldsymbol{\sigma}|} &= \mathbf{dev}\, (\frac{1}{2\mu}\boldsymbol{\sigma}^{\text{old}} + \boldsymbol{\varepsilon}(\mathbf{u})) \end{aligned}$$

Stress-Based Mixed Finite Elements for Elasto-Plasticity

Optimality (KKT) Conditions (Perfect Plasticity)

$$(\mathcal{A}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\rm old}), \boldsymbol{\tau}) + (\mathbf{u}, \operatorname{div} \boldsymbol{\tau}) + (\boldsymbol{\rho}, \mathbf{as}\, \boldsymbol{\tau}) + (\chi, \frac{\mathbf{dev}\boldsymbol{\sigma}}{|\mathbf{dev}\boldsymbol{\sigma}| : \mathbf{dev}\, \boldsymbol{\tau}}) = 0$$

$\Longleftrightarrow$ (with $\boldsymbol{\rho} = \mathbf{as}(\boldsymbol{\nabla}\mathbf{u})$)

$$(\mathcal{A}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\rm old}) - \boldsymbol{\varepsilon}(\mathbf{u}) + \chi \frac{\mathbf{dev}\, \boldsymbol{\sigma}}{|\mathbf{dev}\, \boldsymbol{\sigma}|}, \boldsymbol{\tau}) = 0$$

$$\Longleftrightarrow \qquad \begin{aligned} \frac{1}{d\lambda + 2\mu} \operatorname{tr}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\rm old}) &= \operatorname{tr} \boldsymbol{\varepsilon}(\mathbf{u}) \\ \frac{1}{2\mu} \mathbf{dev}\, \boldsymbol{\sigma} + \chi \frac{\mathbf{dev}\, \boldsymbol{\sigma}}{|\mathbf{dev}\, \boldsymbol{\sigma}|} &= \mathbf{dev}\, (\frac{1}{2\mu} \boldsymbol{\sigma}^{\rm old} + \boldsymbol{\varepsilon}(\mathbf{u})) \end{aligned}$$

(2nd eqn) $\qquad \Longrightarrow \qquad \dfrac{1}{2\mu} |\mathbf{dev}\, \boldsymbol{\sigma}| + \chi = |\mathbf{dev}\, (\dfrac{1}{2\mu} \boldsymbol{\sigma}^{\rm old} + \boldsymbol{\varepsilon}(\mathbf{u}))|$

χ: Difference between elastic and admissible stress deviator
a.k.a. return mapping parameter

Stress-Based Mixed Finite Elements for Elasto-Plasticity

Optimality (KKT) Conditions (Perfect Plasticity)

Determine $\boldsymbol{\sigma} \in \boldsymbol{\sigma}^N + \boldsymbol{\Sigma}$, Lagrange multipliers $\mathbf{u} \in \mathbf{V}$, $\boldsymbol{\rho} \in \mathbf{R}$, $\chi \in X^+$ s.t.

$$\begin{aligned}
(\mathcal{A}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\text{old}}), \boldsymbol{\tau})_{L^2(\Omega)} + (\mathbf{u}, \operatorname{div} \boldsymbol{\tau})_{L^2(\Omega)} + (\boldsymbol{\rho}, \mathbf{as}\, \boldsymbol{\tau})_{L^2(\Omega)} & \\
+ \left(\chi, \frac{\mathbf{dev}\boldsymbol{\sigma}}{|\mathbf{dev}\boldsymbol{\sigma}|} : \mathbf{dev}\, \boldsymbol{\tau}\right)_\star &= 0 \\
(\operatorname{div} \boldsymbol{\sigma} + \mathbf{f}, \mathbf{v})_{L^2(\Omega)} &= 0 \\
(\mathbf{as}\, \boldsymbol{\sigma}, \boldsymbol{\vartheta})_{L^2(\Omega)} &= 0 \\
(|\mathbf{dev}\, \boldsymbol{\sigma}| - \kappa, \omega)_\star &\le 0
\end{aligned}$$

for all $\boldsymbol{\tau} \in \boldsymbol{\Sigma}$, $\mathbf{v} \in \mathbf{V}$, $\boldsymbol{\vartheta} \in \mathbf{R}$ and $\omega \in X^+$

Complementarity condition: $(|\mathbf{dev}\, \boldsymbol{\sigma}| - \kappa, \chi)_{L^\infty(\Omega) \times L^\infty(\Omega)'} = 0$
Combine inequality constraints and complementarity condition into

$$(\chi, \omega) - \max\{0, (\chi, \omega) + \delta(|\mathbf{dev}\, \boldsymbol{\sigma}| - \kappa, \omega)\} = 0 \text{ for all } \omega \in X$$

Mixed Finite Element Discretization of Elasto-Plasticity

Semi-smooth Newton method ($\widehat{\boldsymbol{\sigma}}, \widehat{\chi}$ denote previous iterates):

$X^{\mathrm{a}} := \mathrm{span}\{\phi \in X : (\widehat{\chi}, \phi) + \delta(|\mathbf{dev}\,\widehat{\boldsymbol{\sigma}}| - \kappa, \phi) > 0\}$

$$\widehat{\mathbf{N}} := \frac{\mathbf{dev}\,\widehat{\boldsymbol{\sigma}}}{|\mathbf{dev}\,\widehat{\boldsymbol{\sigma}}|}$$

Determine $\boldsymbol{\sigma} \in \boldsymbol{\sigma}^N + \boldsymbol{\Sigma}$, $\mathbf{u} \in \mathbf{V}$, $\boldsymbol{\rho} \in \mathbf{R}$ and $\chi \in X_h$ s.t.

$$\begin{aligned}
(\mathcal{A}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{old}}), \boldsymbol{\tau}) + (\mathbf{u}, \mathrm{div}\,\boldsymbol{\tau}) + (\boldsymbol{\rho}, \mathbf{as}\,\boldsymbol{\tau}) + (\chi, \widehat{\mathbf{N}} : \mathbf{dev}\,\boldsymbol{\tau}) &= 0 \\
(\mathrm{div}\,\boldsymbol{\sigma} + \mathbf{f}, \mathbf{v}) &= 0 \\
(\mathbf{as}\,\boldsymbol{\sigma}, \boldsymbol{\vartheta}) &= 0 \\
\omega \in X^{\mathrm{a}} : \quad (\widehat{\mathbf{N}} : \mathbf{dev}\,\boldsymbol{\sigma} - \kappa, \omega) &= 0 \\
\omega \notin X^{\mathrm{a}} : \quad (\chi, \omega) &= 0
\end{aligned}$$

for all $\boldsymbol{\tau} \in \boldsymbol{\Sigma}$, $\mathbf{v} \in \mathbf{V}$, $\boldsymbol{\vartheta} \in \mathbf{R}$ and $\omega \in X$

Interpretation as primal-dual active set strategy

Hintermüller/Ito/Kunisch (2003)

Possible choices of combinations of finite element spaces?

Verification of the Inf-Sup Condition on Vertex Patches

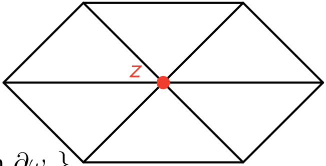
Vertex Patch Decomposition

$$\inf_{\mathbf{v},\boldsymbol{\vartheta},\chi} \sup_{\boldsymbol{\tau}\in\boldsymbol{\Sigma}} \frac{(\operatorname{div}\boldsymbol{\tau},\mathbf{v}) + (\mathbf{as}\,\boldsymbol{\tau},\boldsymbol{\vartheta}) + (\widehat{\mathbf{N}} : \mathbf{dev}\,\boldsymbol{\tau},\chi)}{\|\boldsymbol{\tau}\|_{H(\operatorname{div},\Omega)}\left(\|\mathbf{v}\| + \|\boldsymbol{\vartheta}\| + \|\chi\|\right)} \geq \beta > 0$$

for all $\widehat{\mathbf{N}} \in \mathbb{R}^{2\times 2}$ with $|\widehat{\mathbf{N}}| = 1$

$$\widehat{\mathbf{N}} = \alpha_d \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \alpha_e \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_a \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

with $2(\alpha_d^2 + \alpha_e^2 + \alpha_a^2) = 1\,,\ \alpha_d \neq 0$



Sufficient: Inf-sup stability on vertex patch

$z \in \mathcal{V}$: $\omega_z = \operatorname{supp}\phi_z$

$\boldsymbol{\Sigma}_z = \{\boldsymbol{\sigma} \in \boldsymbol{\Sigma} : \operatorname{supp}\boldsymbol{\sigma} \subseteq \omega_z\,,\ \boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{0} \text{ on } \partial\omega_z\}$

$$\inf_{\mathbf{v},\boldsymbol{\vartheta},\chi} \sup_{\boldsymbol{\tau}\in\boldsymbol{\Sigma}_z} \frac{(\operatorname{div}\boldsymbol{\tau},\mathbf{v})_{\omega_z} + (\mathbf{as}\,\boldsymbol{\tau},\boldsymbol{\vartheta})_{\omega_z} + (\widehat{\mathbf{N}} : \mathbf{dev}\,\boldsymbol{\tau},\chi)_{\omega_z}}{\|\boldsymbol{\tau}\|_{H(\operatorname{div},\omega_z)}\left(\|\mathbf{v}\|_{\omega_z} + \|\boldsymbol{\vartheta}\|_{\omega_z} + \|\chi\|_{\omega_z}\right)} \geq \beta > 0$$

Verification of the Inf-Sup Condition on Vertex Patches

Inf-Sup Condition on Vertex Patch

Dimension of subspace in $\mathbf{V} \times \mathbf{R} \times X$ with

$$\sup_{\boldsymbol{\tau} \in \boldsymbol{\Sigma}_z} \frac{(\operatorname{div} \boldsymbol{\tau}, \mathbf{v})_{\omega_z} + (\mathbf{as}\, \boldsymbol{\tau}, \boldsymbol{\vartheta})_{\omega_z} + (\widehat{\mathbf{N}} : \mathbf{dev}\, \boldsymbol{\tau}, \chi)_{\omega_z}}{\|\boldsymbol{\tau}\|_{H(\operatorname{div},\omega_z)} \left(\|\mathbf{v}\|_{\omega_z} + \|\boldsymbol{\vartheta}\|_{\omega_z} + \|\chi\|_{\omega_z}\right)} = 0 \ :$$

$\boldsymbol{\Sigma}_z \setminus \mathbf{V}/\mathbf{R}/X$:	$DP_1/DP_1/DP_1$	$DP_1/P_1/DP_1$	$DP_1/P_1/P_1$	$P_1/P_1/P_1$
RT_1^2	24	13	8	4

Verification of the Inf-Sup Condition on Vertex Patches

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RT_1^2	24	13	8	4
$RT_1^2 + \nabla^\perp B_3^{\mathrm{cf}}$	12	8	6	4

$$B_3^{\mathrm{cf}} = \{\mathbf{b} \in P_3(\mathcal{T})^2 : \mathbf{b}|_E = 0 \text{ for all edges } E\}$$

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$RT_1^2 + \nabla^\perp B_3^{\mathrm{nc}}$	4	4	4	4

$$B_3^{\mathrm{cf}} = \{\mathbf{b} \in P_3(\mathcal{T})^2 : \mathbf{b}|_E = 0 \text{ for all edges } E\}$$
$$B_3^{\mathrm{nc}} = \{\mathbf{b} \in P_3(\mathcal{T})^2 : \langle \mathbf{b}, \mathbf{q} \rangle_E = 0 \text{ for all } \mathbf{q} \in P_1(E)^2 \text{ and all edges } E\}$$

Related to symmetric stress method by
Gopalakrishnan/Guzmán (SIAM J. Numer. Anal., 2011)
See also: Kober/GS (2018, ENUMATH-Proceedings)

Verification of the Inf-Sup Condition on Vertex Patches

The Singular Constraint Subspace

Linearly dependent Lagrange multiplier space:

$$\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \ \boldsymbol{\vartheta} = \mathbf{0} , \ \chi = 0$$

$$\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \ \boldsymbol{\vartheta} = \mathbf{0} , \ \chi = 0$$

$$\mathbf{v} = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} , \ \boldsymbol{\vartheta} = 2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \ \chi = 0 \qquad \text{(rigid body modes)}$$

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and an additional plastic mode

$$\mathbf{v} = \alpha_d \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} + \alpha_e \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \, , \; \boldsymbol{\vartheta} = \mathbf{0} \, , \; \chi = 2$$

Extension to Elasto-Plasticity with Hardening

Treatment of Hardening

Determine $\boldsymbol{\sigma} \in \boldsymbol{\sigma}^N + \boldsymbol{\Sigma}$, $\mathbf{u} \in \mathbf{V}$, $\boldsymbol{\rho} \in \mathbf{R}$, $\chi \in X^{\mathrm{a}}$ and $\alpha \in X^{\mathrm{a}}$ s.t.

$$
\begin{aligned}
(\mathcal{A}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{old}}), \boldsymbol{\tau}) + (\mathbf{u}, \operatorname{div} \boldsymbol{\tau}) + (\boldsymbol{\rho}, \mathbf{as}\, \boldsymbol{\tau}) & \\
+(\chi, \frac{\mathbf{dev}\widehat{\boldsymbol{\sigma}}}{|\mathbf{dev}\widehat{\boldsymbol{\sigma}}|} : \mathbf{dev}\, \boldsymbol{\tau}) &= 0 \\
(\operatorname{div} \boldsymbol{\sigma} + \mathbf{f}, \mathbf{v}) &= 0 \\
(\mathbf{as}\, \boldsymbol{\sigma}, \boldsymbol{\vartheta}) &= 0 \\
(\frac{\mathbf{dev}\, \widehat{\boldsymbol{\sigma}}}{|\mathbf{dev}\, \widehat{\boldsymbol{\sigma}}|} : \mathbf{dev}\, \boldsymbol{\sigma} - \kappa(\alpha), \omega) &= 0 \\
(\alpha - \alpha^{\mathrm{old}} - \chi, \beta) &= 0
\end{aligned}
$$

for all $\boldsymbol{\tau} \in \boldsymbol{\Sigma}$, $\mathbf{v} \in \mathbf{V}$, $\boldsymbol{\vartheta} \in \mathbf{R}$, $\omega \in X^{\mathrm{a}}$ and $\beta \in X^{\mathrm{a}}$

$\kappa(\alpha) \approx \kappa(\hat{\alpha}) + \kappa'(\hat{\alpha})(\alpha - \hat{\alpha})$ with $\kappa'(\hat{\alpha}) > 0$ has a stabilizing effect:
Any stable combination from the linear elasticity case can be used!

Reconstruction-Based A Posteriori Error Estimation

Error Estimation for Hellinger-Reissner from Displacement Reconstruction

$$
\begin{aligned}
-(\mathbf{u}_h, \operatorname{div} \boldsymbol{\tau}_h) &= (\mathcal{A}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}^{\mathrm{old}}, \boldsymbol{\tau}_h) + (\boldsymbol{\rho}_h, \mathbf{as}\, \boldsymbol{\tau}_h) + (\chi_h, \frac{\mathbf{dev}\, \boldsymbol{\sigma}_h}{|\mathbf{dev}\, \boldsymbol{\sigma}_h|} : \mathbf{dev}\, \boldsymbol{\tau}_h) \\
&= (\mathcal{A}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}^{\mathrm{old}}) + \boldsymbol{\rho}_h + \chi_h \frac{\mathbf{dev}\, \boldsymbol{\sigma}_h}{|\mathbf{dev}\, \boldsymbol{\sigma}_h|}, \boldsymbol{\tau}_h)
\end{aligned}
$$

$$
\mathbf{G}_h = \mathcal{A}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}^{\mathrm{old}}) + \boldsymbol{\rho}_h + \chi_h \frac{\mathbf{dev}\, \boldsymbol{\sigma}_h}{|\mathbf{dev}\, \boldsymbol{\sigma}_h|}
$$

discontinuous piecewise defined approximation to $\boldsymbol{\nabla} \mathbf{u}_h$

Reconstruction-Based A Posteriori Error Estimation

Error Estimation for Hellinger-Reissner from Displacement Reconstruction

$$-(\mathbf{u}_h, \operatorname{div} \boldsymbol{\tau}_h) = (\mathcal{A}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}^{\mathrm{old}}, \boldsymbol{\tau}_h) + (\boldsymbol{\rho}_h, \mathbf{as}\, \boldsymbol{\tau}_h) + (\chi_h, \frac{\mathbf{dev}\, \boldsymbol{\sigma}_h}{|\mathbf{dev}\, \boldsymbol{\sigma}_h|} : \mathbf{dev}\, \boldsymbol{\tau}_h)$$

$$= (\mathcal{A}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}^{\mathrm{old}}) + \boldsymbol{\rho}_h + \chi_h \frac{\mathbf{dev}\, \boldsymbol{\sigma}_h}{|\mathbf{dev}\, \boldsymbol{\sigma}_h|}, \boldsymbol{\tau}_h)$$

$$\mathbf{G}_h = \mathcal{A}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}^{\mathrm{old}}) + \boldsymbol{\rho}_h + \chi_h \frac{\mathbf{dev}\, \boldsymbol{\sigma}_h}{|\mathbf{dev}\, \boldsymbol{\sigma}_h|}$$

discontinuous piecewise defined approximation to $\boldsymbol{\nabla} \mathbf{u}_h$

Gradient Reconstruction Algorithm:
(cf. Kim (2012), Vohralík (2010), Stenberg (1991))
Step 1. For each $T \in \mathcal{T}_h$, determine $\mathbf{u}_h^\circ|_T \in \mathcal{P}_\ell(T)^d$

$$(\boldsymbol{\nabla} \mathbf{u}_h^\circ, \boldsymbol{\nabla} \mathbf{v}_h)_{L^2(T)} = (\mathbf{G}_h, \boldsymbol{\nabla} \mathbf{v}_h)_{L^2(T)} \text{ for all } \mathbf{v}_h \in \mathcal{P}_\ell(T)^d$$
$$(\mathbf{u}_h^\circ, \mathbf{e}_i)_{L^2(T)} = (\mathbf{u}_h, \mathbf{e}_i)_{L^2(T)} \text{ for } 1 \le i \le d$$

Step 2. Construct $\mathbf{u}_h^R \in \mathcal{P}_\ell(\mathcal{T}_h)^d \subset H^1(\Omega)^d$ by averaging:

$$\mathbf{u}_h^R(\mathbf{x}) = \frac{1}{\#\{T : \mathbf{x} \in T\}} \left(\sum_{T : \mathbf{x} \in T} \mathbf{u}_h^\circ|_T (\mathbf{x}) \right)$$

Reconstruction-Based A Posteriori Error Estimation

Error Estimation for Hellinger-Reissner from Displacement Reconstruction

In the case of positive hardening, it can be shown that

$$\|\mathcal{A}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}^{\mathrm{old}}) + \chi_h \frac{\mathbf{dev}\,\boldsymbol{\sigma}_h}{|\mathbf{dev}\,\boldsymbol{\sigma}_h|} - \boldsymbol{\varepsilon}(\mathbf{u}_h^R)\|_{\mathcal{C}}^2 \gtrsim \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathcal{A}}^2$$

holds.

Proof follows along the lines of the coercivity proof for the first-order system least squares formulation in

GS: SIAM J. Numer. Anal. **45**: 371–388 (2007)

Other adaptive refinement approaches to elastoplasticity:

Han/Reddy/Schroeder (1997)
Barthold/Schmidt/Stein (1998)
Rannacher/Suttmeier (1999)
Alberty/Carstensen/Zarrabi (1999)
...
Carstensen/Schröder/Wiedemann (2016)

Conclusions

Weakly symmetric stress-based method for elasto-plasticity:

- provides exact balance of momentum
- simplifies the treatment of the plasticity constraint
- perfect plasticity:
  enlarged stress space in relation to linear elasticity required
- plasticity with hardening:
  any stress space stable for linear elasticiity may be used
- plasticity with hardening:
  a posteriori error estimation by displacement reconstruction