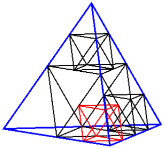
First Order System Least Squares for Elastic Contact Problems

Gerhard Starke

Fakultät für Mathematik, Universität Duisburg - Essen, Germany
joint work with Bernhard Kober (supported by DFG within SPP 1962)



4th Workshop on Minimum Residual & Least Squares Finite Element Methods
Berlin September 16, 2019

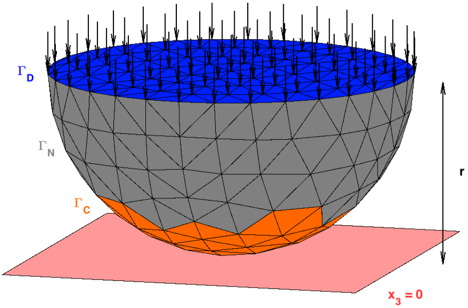
Overview

Overview

Friction-Less Contact: Variational Formulations

Physical Setting

Elastic deformation of a solid constrained by a rigid foundation



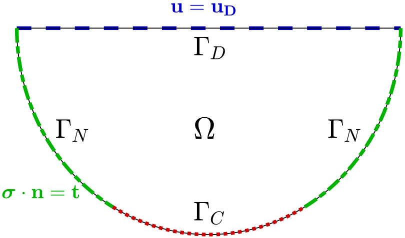
Small strain assumption $\rightarrow$ linear elasticity, lin. contact constraint

Problem is non-linear due to inequality constraints
(active contact zone not known a priori)

Antonio Signorini: Sopra alcune questioni di statica dei sistemi continui.
Annali della Scuola Normale Superiore di Pisa (1933)

Friction-Less Contact: Variational Formulations

Notations



Stress field $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{d \times d}$

Displacement field $\mathbf{u} : \Omega \to \mathbb{R}^{d}$

lin. Strain: $\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)$

Antisymmetric part: $\mathbf{as}\,\boldsymbol{\sigma} := \frac{1}{2}\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^T\right)$

Conservation equations:

$$\operatorname{div} \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \text{ in } \Omega$$

$$\mathbf{as}\,\boldsymbol{\sigma} = \mathbf{0} \text{ in } \Omega$$

Constitutive equation:

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u}) := 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda\,(\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}))\,\mathbf{I} \Longleftrightarrow$$

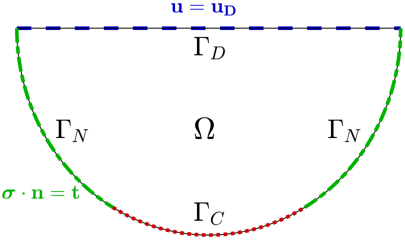
$$\boldsymbol{\varepsilon}(\mathbf{u}) = \mathcal{C}^{-1}\boldsymbol{\sigma} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{d\lambda + 2\mu}(\operatorname{tr}\boldsymbol{\sigma})\mathbf{I}\right)$$

μ, λ : Lamé Parameters

Friction-Less Contact: Variational Formulations

Displacement Variational Inequality

Minimize free energy $\frac{1}{2}(\mathcal{C}\varepsilon(\mathbf{u}),\varepsilon(\mathbf{u}))_\Omega - (\mathbf{f},\mathbf{u})_\Omega - \langle \mathbf{t},\mathbf{u}\rangle_{\Gamma_N}$



w.r.t. $\mathbf{u} \in H^1(\Omega)^d$

s.t. $\mathbf{u} = \mathbf{u}_D$ on Γ_D

$\mathbf{n}\cdot\mathbf{u} \le g$ on Γ_C

g : gap function

$$\mathcal{K}_u = \{\mathbf{u} \in H^1(\Omega)^d : \mathbf{u} = \mathbf{u}_D \text{ on } \Gamma_D\,,\ \mathbf{n}\cdot\mathbf{u} \le g \text{ on } \Gamma_C\}$$

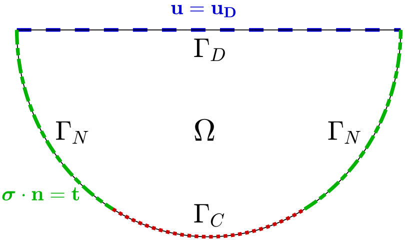
Optimality condition:

$$(\mathcal{C}\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}-\mathbf{u}))_\Omega \ge (\mathbf{f},\mathbf{v}-\mathbf{u})_\Omega + \langle \mathbf{t},\mathbf{v}-\mathbf{u}\rangle_{\Gamma_N} \text{ for all } \mathbf{v} \in \mathcal{K}_u$$

Friction-Less Contact: Variational Formulations

Stress Variational Inequality

Minimize free energy $\frac{1}{2}(\mathcal{C}^{-1}\boldsymbol{\sigma},\boldsymbol{\sigma})_\Omega - \langle \mathbf{u}_D, \boldsymbol{\sigma}\cdot\mathbf{n}\rangle_{\Gamma_D} - \langle g, \mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n})\rangle_{\Gamma_C}$



w.r.t. $\boldsymbol{\sigma} \in H(\mathrm{div},\Omega)^d$

s.t.

$$\begin{aligned} \mathrm{div}\,\boldsymbol{\sigma} + \mathbf{f} &= \mathbf{0} \text{ in } \Omega \\ \mathbf{as}\,\boldsymbol{\sigma} &= \mathbf{0} \text{ in } \Omega \\ \boldsymbol{\sigma}\cdot\mathbf{n} &= \mathbf{t} \text{ on } \Gamma_N \\ \mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n}) &\le 0 \text{ on } \Gamma_C \\ \mathbf{n}\times(\boldsymbol{\sigma}\cdot\mathbf{n}) &= \mathbf{0} \text{ on } \Gamma_C \end{aligned}$$

$$\begin{aligned} \mathcal{K}_\sigma = \{\boldsymbol{\sigma} \in H(\mathrm{div},\Omega)^d : \mathrm{div}\,\boldsymbol{\sigma} + \mathbf{f} = \mathbf{0}\,,\ \mathbf{as}\,\boldsymbol{\sigma} = \mathbf{0} \text{ in } \Omega\,, \\ \boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{t} \text{ on } \Gamma_N\,,\ \mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n}) \le 0 \text{ and } \mathbf{n}\times(\boldsymbol{\sigma}\cdot\mathbf{n}) = \mathbf{0} \text{ on } \Gamma_C\} \end{aligned}$$

Optimality condition:

$$(\mathcal{C}^{-1}\boldsymbol{\sigma},\boldsymbol{\tau})_\Omega \ge (\mathbf{u}_D, \boldsymbol{\tau}-\boldsymbol{\sigma})_{\Gamma_D} + \langle g, \mathbf{n}\cdot((\boldsymbol{\tau}-\boldsymbol{\sigma})\cdot\mathbf{n}\rangle_{\Gamma_C} \text{ for all } \boldsymbol{\tau} \in \mathcal{K}_\sigma$$

Friction-Less Contact: Variational Formulations

Strong duality

For all $\mathbf{v} \in \mathcal{K}_u$, $\boldsymbol{\tau} \in \mathcal{K}_\sigma$:

$$
\begin{aligned}
J(\mathbf{v}) :=& \frac{1}{2}(\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{v}), \boldsymbol{\varepsilon}(\mathbf{v}))_\Omega - (\mathbf{f}, \mathbf{v})_\Omega - \langle \mathbf{t}, \mathbf{v} \rangle_{\Gamma_N} \\
&\geq \frac{1}{2}(\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{u}))_\Omega - (\mathbf{f}, \mathbf{u})_\Omega - \langle \mathbf{t}, \mathbf{u} \rangle_{\Gamma_N} \\
&\quad = -\left(\frac{1}{2}(\mathcal{C}^{-1}\boldsymbol{\sigma}, \boldsymbol{\sigma})_\Omega - \langle \mathbf{u}_D, \boldsymbol{\sigma} \cdot \mathbf{n} \rangle_{\Gamma_D} - \langle g, \mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}) \rangle_{\Gamma_C} \right) \\
&\quad \geq -\left(\frac{1}{2}(\mathcal{C}^{-1}\boldsymbol{\tau}, \boldsymbol{\tau})_\Omega - \langle \mathbf{u}_D, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\Gamma_D} - \langle g, \mathbf{n} \cdot (\boldsymbol{\tau} \cdot \mathbf{n}) \rangle_{\Gamma_C} \right) =: J^*(\boldsymbol{\tau})
\end{aligned}
$$

$$
\left. \begin{array}{c} J(\mathbf{v}) - J(\mathbf{u}) \\ J^*(\boldsymbol{\sigma}) - J^*(\boldsymbol{\tau}) \end{array} \right\} \leq J(\mathbf{v}) - J^*(\boldsymbol{\tau})
$$

$$
\begin{aligned}
&= \frac{1}{2}(\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{v}), \boldsymbol{\varepsilon}(\mathbf{v}))_\Omega - (\mathbf{f}, \mathbf{v})_\Omega - \langle \mathbf{t}, \mathbf{v} \rangle_{\Gamma_N} + \frac{1}{2}(\mathcal{C}^{-1}\boldsymbol{\tau}, \boldsymbol{\tau})_\Omega \\
&\qquad - \langle \mathbf{u}_D, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\Gamma_D} - \langle g, \mathbf{n} \cdot (\boldsymbol{\tau} \cdot \mathbf{n}) \rangle_{\Gamma_C}
\end{aligned}
$$

Friction-Less Contact: Variational Formulations

Strong duality

For all $\mathbf{v} \in \mathcal{K}_u$, $\boldsymbol{\tau} \in \mathcal{K}_\sigma$: $\mathbf{t} = \mathbf{0}$ for simplicity

$$\begin{aligned}
J(\mathbf{v}) - J^*(\boldsymbol{\tau}) &= \frac{1}{2}(\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{v}), \boldsymbol{\varepsilon}(\mathbf{v}))_\Omega - (\mathbf{f}, \mathbf{v})_\Omega + \frac{1}{2}(\mathcal{C}^{-1}\boldsymbol{\tau}, \boldsymbol{\tau})_\Omega \\
&\qquad - \langle \mathbf{u}_D, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\Gamma_D} - \langle g, \mathbf{n} \cdot (\boldsymbol{\tau} \cdot \mathbf{n}) \rangle_{\Gamma_C} \\
&= \frac{1}{2}\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{v}) - \mathcal{C}^{-1/2}\boldsymbol{\tau}\|_\Omega^2 + (\boldsymbol{\varepsilon}(\mathbf{v}), \boldsymbol{\tau})_\Omega - (\mathbf{f}, \mathbf{v})_\Omega \\
&\qquad - \langle \mathbf{u}_D, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\Gamma_D} - \langle g, \mathbf{n} \cdot (\boldsymbol{\tau} \cdot \mathbf{n}) \rangle_{\Gamma_C} \\
&= \frac{1}{2}\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{v}) - \mathcal{C}^{-1/2}\boldsymbol{\tau}\|_\Omega^2 + \langle \mathbf{v}, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\partial\Omega} - (\mathbf{v}, \operatorname{div} \boldsymbol{\tau})_\Omega - (\mathbf{f}, \mathbf{v})_\Omega \\
&\qquad - \langle \mathbf{u}_D, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\Gamma_D} - \langle g, \mathbf{n} \cdot (\boldsymbol{\tau} \cdot \mathbf{n}) \rangle_{\Gamma_C} \\
&= \frac{1}{2}\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{v}) - \mathcal{C}^{-1/2}\boldsymbol{\tau}\|_\Omega^2 - (\operatorname{div} \boldsymbol{\tau} + \mathbf{f}, \mathbf{v})_\Omega \\
&\qquad + \langle \mathbf{v} - \mathbf{u}_D, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\Gamma_D} + \langle \mathbf{n} \cdot \mathbf{v} - g, \mathbf{n} \cdot (\boldsymbol{\tau} \cdot \mathbf{n}) \rangle_{\Gamma_C} \\
&= \frac{1}{2}\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{v}) - \mathcal{C}^{-1/2}\boldsymbol{\tau}\|_\Omega^2 + \langle \mathbf{n} \cdot \mathbf{v} - g, \mathbf{n} \cdot (\boldsymbol{\tau} \cdot \mathbf{n}) \rangle_{\Gamma_C}
\end{aligned}$$

Overview

Friction-Less Contact: Variational Formulations

Friction-Less Contact: First Order System Least Squares

Contact with Coulomb Friction: Variational Formulations

Contact with Coulomb Friction: First Order System Least Squares

Error Estimation and Adaptive Computations

Friction-Less Contact: First Order System Least Squares

Minimize

$$\mathcal{F}_C(\mathbf{v},\boldsymbol{\tau}) := \frac{1}{2}\left(\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{v}) - \mathcal{C}^{-1/2}\boldsymbol{\tau}\|_\Omega^2 + \|\operatorname{div}\boldsymbol{\tau} + \mathbf{f}\|_\Omega^2\right) + \langle \mathbf{n}\cdot\mathbf{v} - g, \mathbf{n}\cdot(\boldsymbol{\tau}\cdot\mathbf{n})\rangle_{\Gamma_C}$$

among all $(\mathbf{v},\boldsymbol{\tau}) \in \mathcal{K}_{u,h} \times \widetilde{\mathcal{K}}_{\sigma,h} \subset \mathcal{K}_u \times \widetilde{\mathcal{K}}_\sigma \subset H^1(\Omega)^d \times H(\operatorname{div},\Omega)^d$ with

$$\widetilde{\mathcal{K}}_\sigma = \{\boldsymbol{\tau} \in H(\operatorname{div},\Omega)^d : \boldsymbol{\tau}\cdot\mathbf{n} = \mathbf{0} \text{ on } \Gamma_N\,,\ \mathbf{n}\cdot(\boldsymbol{\tau}\cdot\mathbf{n}) \le 0 \text{ on } \Gamma_C\} \supset \mathcal{K}_\sigma$$

Missing conditions from $\mathcal{K}_\sigma$ controlled by ls functional, note that

$$\frac{1}{2\mu}\|\mathbf{as}\,\boldsymbol{\tau}\|_\Omega^2 = \|\mathbf{as}(\mathcal{C}^{-1/2}\boldsymbol{\tau})\|_\Omega^2 \le \|\mathcal{C}^{-1/2}\boldsymbol{\tau}\|_\Omega^2 = \|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{v}) - \mathcal{C}^{-1/2}\boldsymbol{\tau}\|_\Omega^2$$

Coercivity:
$\mathcal{F}_C(\mathbf{v},\boldsymbol{\tau}) \gtrsim \|\mathbf{u}-\mathbf{v}\|_{H^1(\Omega)}^2 + \|\boldsymbol{\sigma}-\boldsymbol{\tau}\|_{H(\operatorname{div},\Omega)}^2$ for all $(\mathbf{v},\boldsymbol{\tau}) \in \mathcal{K}_u \times \widetilde{\mathcal{K}}_\sigma$

Uses complementarity condition: $\langle \mathbf{n}\cdot\mathbf{u} - g, \mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n})\rangle_{\Gamma_C} = 0$

Attia/Cai/GS: SIAM J. Numer. Anal. **47**, 3027–3043 (2009)

Friction-Less Contact: First Order System Least Squares

Robust formulation with respect to the incompressible limit $\lambda \to \infty$: Minimize

$$\widetilde{\mathcal{F}}_C(\mathbf{v}, \boldsymbol{\tau}) := \frac{1}{2}\left(\|\boldsymbol{\varepsilon}(\mathbf{v}) - \mathcal{C}^{-1}\boldsymbol{\tau}\|_{\Omega}^2 + \|\operatorname{div}\boldsymbol{\tau} + \mathbf{f}\|_{\Omega}^2\right) + \langle \mathbf{n}\cdot\mathbf{v} - g, \mathbf{n}\cdot(\boldsymbol{\tau}\cdot\mathbf{n})\rangle_{\Gamma_C}$$

among all $(\mathbf{v}, \boldsymbol{\tau}) \in \mathcal{K}_{u,h} \times \widetilde{\mathcal{K}}_{\sigma,h} \subset \mathcal{K}_u \times \widetilde{\mathcal{K}}_\sigma$

Krause/Müller/GS: Numer. Methods Partial Differential Eq. **33**: 276–289 (2017)

based on elasticity formulation studied in

Cai/GS: SIAM J. Numer. Anal. **42**: 826–842 (2004)

Related work:

Führer: First-Order Least-Squares Method for the Obstacle Problem (arXiv, 2018)

Friction-Less Contact: First Order System Least Squares

A Posteriori Error Estimation

Coercivity of the functional

$$\widetilde{\mathcal{F}}_C(\mathbf{v}, \boldsymbol{\tau}) \gtrsim \|\mathbf{u} - \mathbf{v}\|^2_{H^1(\Omega)} + \|\boldsymbol{\sigma} - \boldsymbol{\tau}\|^2_{H(\mathrm{div},\Omega)} \text{ for all } (\mathbf{v}, \boldsymbol{\tau}) \in \mathcal{K}_u \times \widetilde{\mathcal{K}}_\sigma$$

implies reliability of its use as an a posteriori error estimator

Applicable for any finite element approximation
(not limited to least squares)!

The use of such a functional (involving complementarity terms) for error estimation can already be found in

I. Hlaváček, J. Haslinger, J. Nečas, J. Lovíšek: Springer, 1988

and

L. Demkowicz, M. Swierczek: Italian-Polish Meeting on Selected Problems of Modern Continuum Theory, Bologna, June 1987

Friction-Less Contact: First Order System Least Squares

A Posteriori Error Estimation

Adaptive refinement specifically for first-order system least-squares:

Berndt/Manteuffel/McCormick, Electr. Trans. Numer. Anal. (1997)

⋮

Adler/Manteuffel/McCormick/Nolting/Ruge/Tang, SIAM J. Sci. Comput. (2011)

Other a posteriori error estimators for Signorini contact problems:

Carstensen/Scherf/Wriggers, SIAM J. Sci. Comput. (1999)

⋮

Nochetto/Siebert/Veeser, SIAM J. Numer. Anal. (2005)

⋮

A. Schröder, Comput. Methods Appl. Mech. Engrg. (2012)

⋮

Krause/Veeser/Walloth, Numer. Math. (2015)

⋮

Overview

Contact with Coulomb Friction: Variational Formulations

Displacement Quasi-Variational Inequality

$$\mathcal{K}_u = \{\mathbf{v} \in H^1(\Omega)^d : \mathbf{v} = \mathbf{u}_D \text{ on } \Gamma_D \,,\ \mathbf{n} \cdot \mathbf{v} \leq g \text{ on } \Gamma_C\}$$

Compute $\mathbf{u} \in \mathcal{K}_u$ such that it minimizes

the free energy $\dfrac{1}{2}(\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{v}), \boldsymbol{\varepsilon}(\mathbf{v}))_\Omega + \langle \nu_F |\mathbf{n} \cdot (\boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n})|, |\mathbf{n} \times \mathbf{v}| \rangle_{\Gamma_C} - (\mathbf{f}, \mathbf{v})_\Omega$

among all $\mathbf{v} \in \mathcal{K}_u$

Optimality condition:

$$\begin{aligned}
&(\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}))_\Omega \\
&\quad + \langle \nu_F |\mathbf{n} \cdot (\boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n})|, |\mathbf{n} \times \mathbf{v}| \rangle_{\Gamma_C} - \langle \nu_F |\mathbf{n} \cdot (\boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n})|, |\mathbf{n} \times \mathbf{u}| \rangle_{\Gamma_C} \\
&\quad \geq (\mathbf{f}, \mathbf{v} - \mathbf{u})_\Omega \text{ for all } \mathbf{v} \in \mathcal{K}_u
\end{aligned}$$

Contact with Coulomb Friction: Variational Formulations

Stress Quasi-Variational Inequality

$$\mathcal{K}_\sigma(\boldsymbol{\sigma}) = \{\boldsymbol{\tau} \in H(\mathrm{div},\Omega)^d : \mathrm{div}\,\boldsymbol{\tau} + \mathbf{f} = \mathbf{0}\,,\ \mathbf{as}\,\boldsymbol{\tau} = \mathbf{0} \text{ in } \Omega\,,\ \boldsymbol{\tau}\cdot\mathbf{n} = \mathbf{0} \text{ on } \Gamma_N\,,$$
$$\mathbf{n}\cdot(\boldsymbol{\tau}\cdot\mathbf{n}) \le 0 \text{ and } |\mathbf{n}\times(\boldsymbol{\tau}\cdot\mathbf{n})| \le -\mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n}) \text{ on } \Gamma_C\}$$

Compute $\boldsymbol{\sigma} \in \mathcal{K}_\sigma(\boldsymbol{\sigma})$ such that it minimizes

the free energy $\quad \dfrac{1}{2}(\mathcal{C}^{-1}\boldsymbol{\tau},\boldsymbol{\tau})_\Omega - \langle \mathbf{u}_D, \boldsymbol{\tau}\cdot\mathbf{n}\rangle_{\Gamma_D} - \langle g, \mathbf{n}\cdot(\boldsymbol{\tau}\cdot\mathbf{n})\rangle_{\Gamma_C}$

among all $\boldsymbol{\tau} \in K_\sigma(\boldsymbol{\sigma})$

Optimality condition: $\boldsymbol{\sigma} \in \mathcal{K}_\sigma(\boldsymbol{\sigma})$ such that

$(\mathcal{C}^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau}-\boldsymbol{\sigma})_\Omega \ge (\mathbf{u}_D, \boldsymbol{\tau}-\boldsymbol{\sigma})_{\Gamma_D} + \langle g, \mathbf{n}\cdot((\boldsymbol{\tau}-\boldsymbol{\sigma})\cdot\mathbf{n}\rangle_{\Gamma_C}$ for all $\boldsymbol{\tau} \in \mathcal{K}_\sigma(\boldsymbol{\sigma})$

Contact with Coulomb Friction: Variational Formulations

Strong duality

For all $\mathbf{v} \in \mathcal{K}_u$, $\boldsymbol{\tau} \in \mathcal{K}_\sigma$:

$$\left.\begin{array}{c} J(\mathbf{v}) - J(\mathbf{u}) \\ J^*(\boldsymbol{\sigma}) - J^*(\boldsymbol{\tau}) \end{array}\right\} \leq J(\mathbf{v}) - J^*(\boldsymbol{\tau})$$

$$= \frac{1}{2}(\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{v}), \boldsymbol{\varepsilon}(\mathbf{v}))_\Omega + \langle \nu_F |\mathbf{n} \cdot (\boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n})|, |\mathbf{n} \times \mathbf{v}| \rangle_{\Gamma_C} - (\mathbf{f}, \mathbf{v})_\Omega + \frac{1}{2}(\mathcal{C}^{-1}\boldsymbol{\tau}, \boldsymbol{\tau})_\Omega$$
$$- \langle \mathbf{u}_D, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\Gamma_D} - \langle g, \mathbf{n} \cdot (\boldsymbol{\tau} \cdot \mathbf{n}) \rangle_{\Gamma_C}$$

$$= \frac{1}{2}\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{v}) - \mathcal{C}^{-1/2}\boldsymbol{\tau}\|_\Omega^2 + (\boldsymbol{\varepsilon}(\mathbf{v}), \boldsymbol{\tau})_\Omega + \langle \nu_F |\mathbf{n} \cdot (\boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n})|, |\mathbf{n} \times \mathbf{v}| \rangle_{\Gamma_C}$$
$$- (\mathbf{f}, \mathbf{v})_\Omega - \langle \mathbf{u}_D, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\Gamma_D} - \langle g, \mathbf{n} \cdot (\boldsymbol{\tau} \cdot \mathbf{n}) \rangle_{\Gamma_C}$$

$$= \frac{1}{2}\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{v}) - \mathcal{C}^{-1/2}\boldsymbol{\tau}\|_\Omega^2 + \langle \mathbf{n} \cdot \mathbf{v} - g, \mathbf{n} \cdot (\boldsymbol{\tau} \cdot \mathbf{n}) \rangle_{\Gamma_C}$$
$$+ \langle \nu_F |\mathbf{n} \cdot (\boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n})|, |\mathbf{n} \times \mathbf{v}| \rangle_{\Gamma_C} + \langle \mathbf{n} \times (\boldsymbol{\tau} \cdot \mathbf{n}), \mathbf{n} \times \mathbf{v} \rangle_{\Gamma_C}$$

N. Kikuchi, J. T. Oden: SIAM, 1988

A. Capatina: Springer, 2014

Overview

Contact with Coulomb Friction: First Order System LS

Minimize

$$\widetilde{\mathcal{F}}_{CF}(\mathbf{v},\boldsymbol{\tau}) := \frac{1}{2}\|\boldsymbol{\varepsilon}(\mathbf{v}) - \mathcal{C}^{-1}\boldsymbol{\tau}\|_{\Omega}^2 + \frac{1}{2}\|\operatorname{div}\boldsymbol{\tau} + \mathbf{f}\|_{\Omega}^2 + \langle \mathbf{n}\cdot\mathbf{v} - g, \mathbf{n}\cdot(\boldsymbol{\tau}\cdot\mathbf{n})\rangle_{\Gamma_C}$$
$$+ \langle \nu_F|\mathbf{n}\cdot(\boldsymbol{\tau}\cdot\mathbf{n})|, |\mathbf{n}\times\mathbf{v}|\rangle_{\Gamma_C} + \langle \mathbf{n}\times(\boldsymbol{\tau}\cdot\mathbf{n}), \mathbf{n}\times\mathbf{v}\rangle_{\Gamma_C}$$

among all $(\mathbf{v},\boldsymbol{\tau}) \in \mathcal{K}_{u,h} \times \widetilde{\mathcal{K}}_{\sigma,h}(\boldsymbol{\sigma}_h) \subset \mathcal{K}_u \times \widetilde{\mathcal{K}}_\sigma(\boldsymbol{\sigma})$ with

$$\widetilde{\mathcal{K}}_\sigma(\boldsymbol{\sigma}) = \{\boldsymbol{\tau} \in H(\operatorname{div},\Omega)^d : \boldsymbol{\tau}\cdot\mathbf{n} = \mathbf{0} \text{ on } \Gamma_N ,$$
$$\mathbf{n}\cdot(\boldsymbol{\tau}\cdot\mathbf{n}) \le 0 \text{ and } |\mathbf{n}\times(\boldsymbol{\tau}\cdot\mathbf{n})| \le -\nu_F\mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n}) \text{ on } \Gamma_C\} \supset \mathcal{K}_\sigma(\boldsymbol{\sigma})$$

Coercivity? No uniqueness result for Coulomb friction, in general!

Counterexample, for sufficiently large ν_F:
P. Hild: C. R. Acad. Sci. Paris (2003)
Uniqueness under additional assumptions for small ν_F:
Y. Renard: SIAM J. Numer. Anal. (2006)

Contact with Coulomb Friction: First Order System LS

Where does the problem arise?

Following the same line of proof as in the friction-less case, we arrive at

$$\nu_F \langle \mathbf{n} \cdot ((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \mathbf{n}), |\mathbf{n} \times \mathbf{u}| - |\mathbf{n} \times \mathbf{u}_h| \rangle_{\Gamma_C}$$

which needs to be bounded by error norms

But

$$\| |\mathbf{n} \times \mathbf{u}| - |\mathbf{n} \times \mathbf{u}_h| \|_{1/2,\Gamma_C} \lesssim \| \mathbf{n} \times \mathbf{u} - \mathbf{n} \times \mathbf{u}_h \|_{1/2,\Gamma_C}$$

does not hold in general!

Remedy: Restrict $\mathbf{n} \times \mathbf{u}$ to vanish in an interval of some length whenever a sign change occurs, motivated by the uniqueness investigation in
Y. Renard: SIAM J. Numer. Anal. (2006)

Coercivity for sufficiently small ν_F
Kober/GS/Krause/Rovi (2019)

Overview

Friction-Less Contact: Variational Formulations

Friction-Less Contact: First Order System Least Squares

Contact with Coulomb Friction: Variational Formulations

Contact with Coulomb Friction: First Order System Least Squares

Error Estimation and Adaptive Computations

Error Estimation and Adaptive Computations

Use

$$\widetilde{\mathcal{F}}_{CF}(\mathbf{u}_h, \boldsymbol{\sigma}_h) := \frac{1}{2}\|\varepsilon(\mathbf{u}_h) - \mathcal{C}^{-1}\boldsymbol{\sigma}_h\|_\Omega^2 + \frac{1}{2}\|\mathrm{div}\,\boldsymbol{\sigma}_h + \mathbf{f}\|_\Omega^2$$
$$+ \langle \mathbf{n}\cdot\mathbf{u}_h - g, \mathbf{n}\cdot(\boldsymbol{\sigma}_h\cdot\mathbf{n})\rangle_{\Gamma_C}$$
$$+ \langle \nu_F|\mathbf{n}\cdot(\boldsymbol{\sigma}_h\cdot\mathbf{n})|, |\mathbf{n}\times\mathbf{u}_h|\rangle_{\Gamma_C} + \langle \mathbf{n}\times(\boldsymbol{\sigma}_h\cdot\mathbf{n}), \mathbf{n}\times\mathbf{u}_h\rangle_{\Gamma_C}$$

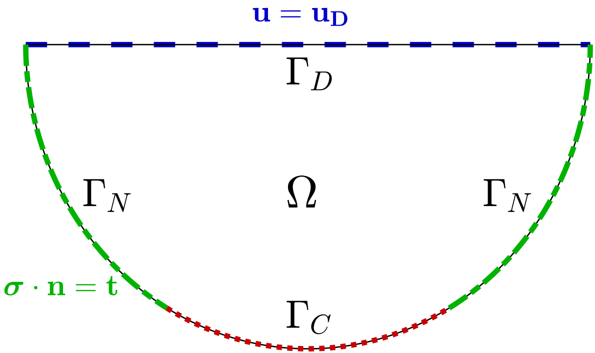
as a posteriori error estimator

$\boldsymbol{\sigma}_h \in \mathbf{\Sigma}_h \subset \mathcal{RT}_1(\mathcal{T}_h)^2$ from a discretized stress variational inequality

$\mathbf{u}_h \in \mathbf{V}_h \subset \mathcal{P}_2(\mathcal{T}_h)^2$ displacement reconstruction

Error Estimation and Adaptive Computations

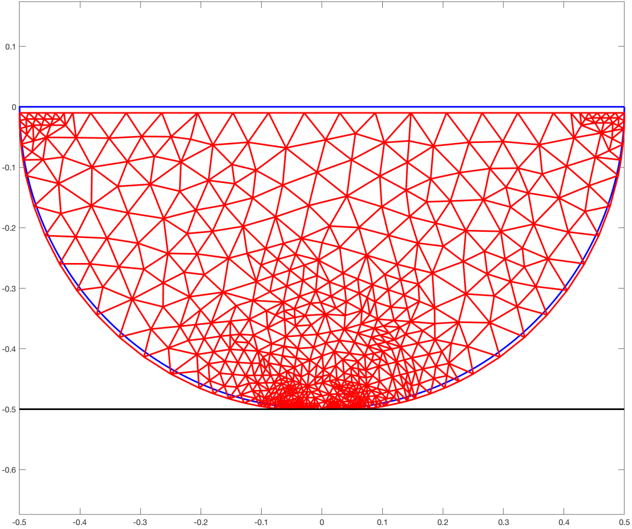
Hertzian Contact - Setting



$\mathbf{u}_D = (0, -0.01)^T$, $\mathbf{t} = \mathbf{0}$, $\mathbf{f} = \mathbf{0}$, $E = 20600$, $\nu_F = 0.4$,

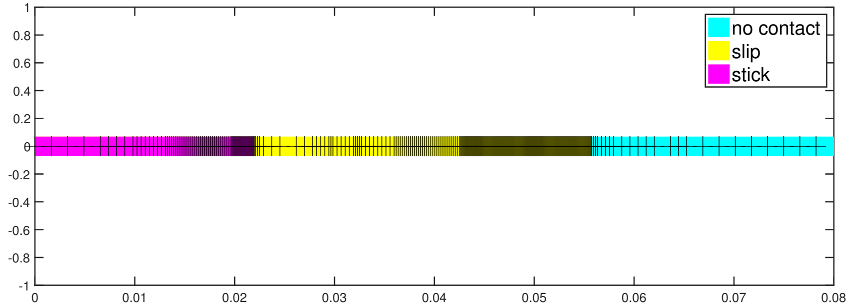
Error Estimation and Adaptive Computations

Deformed Mesh (Compressible case - 5 refinements)



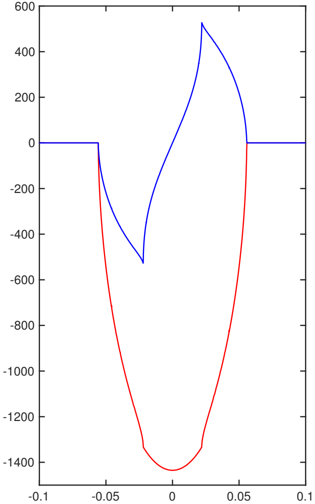
Error Estimation and Adaptive Computations

Gridpoints on half of Γ_C after 10 refinements

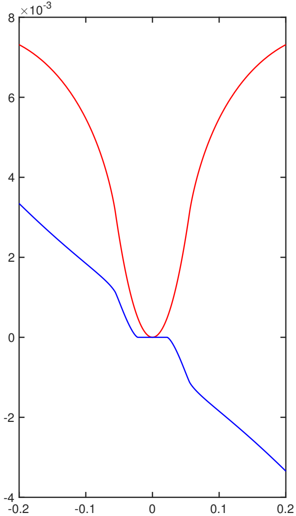


Error Estimation and Adaptive Computations

Contact stress (Compressible case: $\nu = 0.28$)



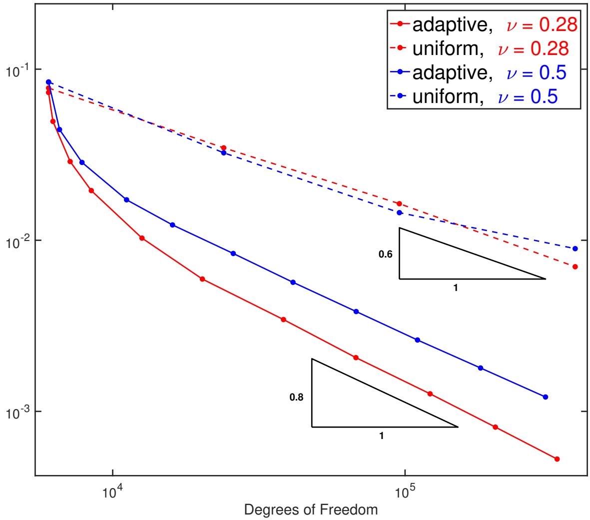
normal and tangential stress



reconstructed displacement

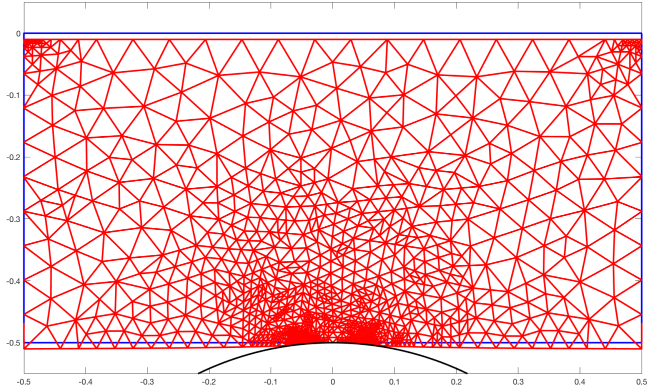
Error Estimation and Adaptive Computations

Convergence comparison: Values of $\widetilde{\mathcal{F}}_{CF}(\mathbf{u}_h, \boldsymbol{\sigma}_h)^{1/2}$



Error Estimation and Adaptive Computations

Straight boundaries



Error Estimation and Adaptive Computations

Convergence comparison: Values of $\widetilde{\mathcal{F}}_{CF}(\mathbf{u}_h, \boldsymbol{\sigma}_h)^{1/2}$