Weakly symmetric stress equilibration in computational solid mechanics

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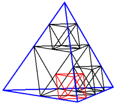

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Overview

Overview

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Stress Equilibration: Basic Idea

The Linear Elasticity Model

$$\varepsilon(\mathbf{v}) := \frac{\nabla \mathbf{v} + (\nabla \mathbf{v})^T}{2} = \nabla \mathbf{v} - \frac{\nabla \mathbf{v} - (\nabla \mathbf{v})^T}{2} =: \nabla \mathbf{v} - \mathbf{as}\,\nabla \mathbf{v}$$

Determine $\mathbf{u} \in H^1_{\Gamma_D}(\Omega)^d$, $d = 2, 3$, such that

$$2\mu\,(\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{L^2(\Omega)} + \lambda\,(\operatorname{tr}\varepsilon(\mathbf{u}), \operatorname{tr}\varepsilon(\mathbf{v}))_{L^2(\Omega)} = (\mathbf{f}, \mathbf{v})_{L^2(\Omega)} + \langle \mathbf{t}, \mathbf{v} \rangle_{\Gamma_N}$$

holds for all $\mathbf{v} \in H^1_{\Gamma_D}(\Omega)^d$

Scaling: $\mu \approx 1$ (by appropriate choice of units)
Incompressibility: $\lambda \to \infty$, new variable (pressure): $p = \lambda \operatorname{tr}\varepsilon(\mathbf{u}) = \lambda \operatorname{div}\mathbf{u}$
Treatment of (near-)incompressibility: Fleurianne Bertrand on Monday!

Finite element space $\mathbf{V}_h \subset H^1_{\Gamma_D}(\Omega)^d$

Determine $\mathbf{u}_h \in \mathbf{V}_h$ such that

$$2\mu\,(\varepsilon(\mathbf{u}_h), \varepsilon(\mathbf{v}_h))_{L^2(\Omega)} + \lambda(\operatorname{tr}\varepsilon(\mathbf{u}_h), \operatorname{tr}\varepsilon(\mathbf{v}_h))_{L^2(\Omega)} = (\mathbf{f}, \mathbf{v}_h)_{L^2(\Omega)} + \langle \mathbf{t}, \mathbf{v}_h \rangle_{L^2(\Gamma_N)}$$

holds for all $\mathbf{v}_h \in \mathbf{V}_h$

Stress Equilibration: Basic Idea

Raviart-Thomas Finite Element Spaces

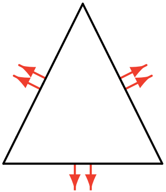
The stress tensor $\boldsymbol{\sigma} = 2\mu\varepsilon(\mathbf{u}) + \lambda(\mathrm{div}\,\mathbf{u})\,\mathbf{I}$ satisfies

$$\mathrm{div}\,\boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \text{ in } \Omega$$
$$\boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{t} \text{ on } \Gamma_N$$

Finite element space $\boldsymbol{\Sigma}_h$ for stress approximation should be in $H(\mathrm{div},\Omega)^d$
Raviart-Thomas space $(\mathcal{RT}_k)^d$:

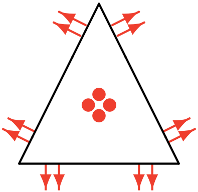
$$\boldsymbol{\Sigma}_h \subset H(\mathrm{div},\Omega)^d \ : \ \boldsymbol{\sigma}_h|_T \in \mathcal{P}_k^{d\times d}(T) + \mathcal{P}_k^d(T)\,\mathbf{x}^T \text{ for all } T \in \mathcal{T}_h\}$$

$k = 0$:



$(\mathcal{RT}_0)^2$

$k = 1$:



$(\mathcal{RT}_1)^2$

Stress Equilibration: Basic Idea

In general: $\boldsymbol{\sigma}(\mathbf{u}_h) = 2\mu\boldsymbol{\varepsilon}(\mathbf{u}_h) + \lambda(\operatorname{div}\mathbf{u}_h)\,\mathbf{I} \notin H(\operatorname{div},\Omega)^d$

Reconstruct $\boldsymbol{\sigma}_h^R \in H(\operatorname{div},\Omega)^d$ with $\operatorname{div}\boldsymbol{\sigma}_h^R + \mathbf{f} = \mathbf{0}$ in Ω, $\boldsymbol{\sigma}_h^R \cdot \mathbf{n} = \mathbf{t}$ on Γ_N

Flux/stress reconstruction algorithms, equilibrated fluxes/stresses:

Ern/Vohralík (2015, ...) ...

... Hannukainen/Stenberg/Vohralík (2012) ...

... Cai/Zhang (2012, ...) ...

... Braess/Schöberl (2008) ...

... Nicaise/Witowski/Wohlmuth (2008) ...

... Parés/Diez/Huerta (2006) ...

... Ainsworth/Oden (1993) ...

... Ladevéze/Leguillon (1983) ...

... Prager/Synge (1947)

Stress Equilibration: Basic Idea

Partition of unity with vertex basis functions: $$1 \equiv \sum_{z \in \mathcal{V}_h} \phi_z$$

Vertex patch: $$\omega_z = \bigcup \{ T \in \mathcal{T}_h : z \text{ is a vertex of } T \}$$

$$\boldsymbol{\sigma}_h^R = \boldsymbol{\sigma}(\mathbf{u}_h) + \boldsymbol{\sigma}_h^\Delta = \boldsymbol{\sigma}(\mathbf{u}_h) + \sum_{z \in \mathcal{V}_h} \boldsymbol{\sigma}_{h,z}^\Delta \qquad \text{with}$$

$$\boldsymbol{\sigma}_{h,z}^\Delta \in \boldsymbol{\Sigma}_{h,z} := \{ \boldsymbol{\tau} \in L^2(\omega_z) : \boldsymbol{\tau}| \in RT_k(T)^d, \boldsymbol{\tau} \cdot \mathbf{n} = \mathbf{0} \text{ on } \partial\omega_z \}$$

(appropriately modified for patches with $\partial\omega_z \cap \partial\Omega \neq \emptyset$)

$\boldsymbol{\sigma}_{h,z}^\Delta \in \boldsymbol{\Sigma}_{h,z}^\Delta$ s. t.

$$(\operatorname{div} \boldsymbol{\sigma}_{h,z}^\Delta, \mathbf{z})_T = -((\mathbf{f} + \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}_h))\phi_z, \mathbf{z})_T \qquad \forall \mathbf{z} \in P_k(T)^d \ , \ T \subset \omega_z$$

$$\langle [\![\boldsymbol{\sigma}_{h,z}^\Delta \cdot \mathbf{n}]\!]_S, \boldsymbol{\zeta} \rangle_S = -\langle [\![\boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}]\!]_S \phi_z, \boldsymbol{\zeta} \rangle_S \qquad \forall \boldsymbol{\zeta} \in P_k(S)^d \ , \ S \subset \omega_z$$

ensures that $\boldsymbol{\sigma}_h^R \in \boldsymbol{\Sigma}_h$ and $\operatorname{div} \boldsymbol{\sigma}_h^R + \mathbf{f} = \mathbf{0}$

Stress Equilibration: Basic Idea

$\boldsymbol{\sigma}_{h,z}^{\Delta} \in \boldsymbol{\Sigma}_{h,z}^{\Delta}$ s. t.

$$(\operatorname{div} \boldsymbol{\sigma}_{h,z}^{\Delta}, \mathbf{z})_T = -((\mathbf{f} + \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}_h))\phi_z, \mathbf{z})_T \qquad \forall \mathbf{z} \in P_k(T)^d \ , \ T \subset \omega_z$$

$$\langle [\![\boldsymbol{\sigma}_{h,z}^{\Delta} \cdot \mathbf{n}]\!]_S, \boldsymbol{\zeta} \rangle_S = -\langle [\![\boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}]\!]_S \phi_z, \boldsymbol{\zeta} \rangle_S \qquad \forall \boldsymbol{\zeta} \in P_k(S)^d \ , \ S \subset \omega_z$$

Constraints are not linearly independent since, for all constant $\mathbf{e} \in \mathbb{R}^d$:

$$\sum_{T \subset \omega_z} (\operatorname{div} \boldsymbol{\sigma}_{h,z}^{\Delta}, \mathbf{e})_T + \sum_{S \subset \omega_z} \langle [\![\boldsymbol{\sigma}_{h,z}^{\Delta} \cdot \mathbf{n}]\!]_S, \mathbf{e} \rangle_S = 0$$

Kernel of the adjoint operator: $\operatorname{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$

Compatibility condition: Right-hand side $\perp$ to $\operatorname{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$, i.e.,

$$\sum_{T \subset \omega_z} ((\mathbf{f} + \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}_h))\phi_z, \mathbf{e})_T + \sum_{S \subset \omega_z} \langle [\![\boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}]\!]_S \phi_z, \mathbf{e} \rangle_S = 0$$

follows from $(\mathbf{f}, \phi_z \mathbf{e})_{\omega_z} - (\boldsymbol{\sigma}(\mathbf{u}_h), \varepsilon(\phi_z \mathbf{e}))_{\omega_z} = 0$ (since $\phi_z \mathbf{e} \in \mathbf{V}_h$)

Stress Equilibration: Basic Idea

Concentrate on stress symmetry and ignore the incompressibility issue!

$\boldsymbol{\sigma}(\mathbf{u}_h) = 2\mu\boldsymbol{\varepsilon}(\mathbf{u}_h) + \lambda(\operatorname{div}\mathbf{u}_h)\,\mathbf{I} =: \mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u}_h)$

$$\Longrightarrow \quad \boldsymbol{\varepsilon}(\mathbf{u}_h) = \mathcal{C}^{-1}\boldsymbol{\sigma}(\mathbf{u}_h) = \frac{1}{2\mu}\left(\boldsymbol{\sigma}(\mathbf{u}_h) - \frac{\lambda}{d\lambda + 2\mu}(\operatorname{tr}\boldsymbol{\sigma}(\mathbf{u}_h))\,\mathbf{I}\right)$$

$\boldsymbol{\sigma}_h^R \in \boldsymbol{\Sigma}_h$ with $\operatorname{div}\boldsymbol{\sigma}_h^R + \mathbf{f} = \mathbf{0}$

$$\begin{aligned}
\|\boldsymbol{\sigma}_h^R - \boldsymbol{\sigma}(\mathbf{u}_h)\|_{\mathcal{C}^{-1}}^2 &= \|\boldsymbol{\sigma}_h^R - \boldsymbol{\sigma} + \boldsymbol{\sigma} - \boldsymbol{\sigma}(\mathbf{u}_h)\|_{\mathcal{C}^{-1}}^2 \\
&= \|\boldsymbol{\sigma}_h^R - \boldsymbol{\sigma}\|_{\mathcal{C}^{-1}}^2 + \|\boldsymbol{\sigma}(\mathbf{u}) - \boldsymbol{\sigma}(\mathbf{u}_h)\|_{\mathcal{C}^{-1}}^2 \\
&\qquad + 2(\boldsymbol{\sigma}_h^R - \boldsymbol{\sigma}, \mathcal{C}^{-1}(\boldsymbol{\sigma}(\mathbf{u}) - \boldsymbol{\sigma}(\mathbf{u}_h))) \\
&= \|\boldsymbol{\sigma}_h^R - \boldsymbol{\sigma}\|_{\mathcal{C}^{-1}}^2 + \|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\mathbf{u}_h)\|_{\mathcal{C}}^2 \\
&\qquad + 2\underbrace{(\boldsymbol{\sigma}_h^R - \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\mathbf{u}_h))}
\end{aligned}$$

$$\begin{aligned}
&(\boldsymbol{\sigma}_h^R - \boldsymbol{\sigma}, \nabla\mathbf{u} - \nabla\mathbf{u}_h) - (\boldsymbol{\sigma}_h^R - \boldsymbol{\sigma}, \mathbf{as}\,\nabla\mathbf{u} - \mathbf{as}\,\nabla\mathbf{u}_h) \\
&= -(\operatorname{div}(\boldsymbol{\sigma}_h^R - \boldsymbol{\sigma}), \mathbf{u} - \mathbf{u}_h) - (\mathbf{as}\,\boldsymbol{\sigma}_h^R, \nabla\mathbf{u} - \nabla\mathbf{u}_h) \\
&= \qquad\qquad - (\mathbf{as}\,\boldsymbol{\sigma}_h^R, \nabla\mathbf{u} - \nabla\mathbf{u}_h)
\end{aligned}$$

Stress Equilibration: Basic Idea

$\Longrightarrow$ Upper bound for the energy norm of the error:

$$\begin{aligned}\|\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)\|_{\mathcal{C}}^2 &\leq \|\boldsymbol{\sigma}_h^R - \boldsymbol{\sigma}(\mathbf{u}_h)\|_{\mathcal{C}^{-1}}^2 + 2\|\mathbf{as}\,\boldsymbol{\sigma}_h^R\|\,\|\boldsymbol{\nabla}(\mathbf{u}-\mathbf{u}_h)\| \\ &\leq \|\boldsymbol{\sigma}_h^R - \boldsymbol{\sigma}(\mathbf{u}_h)\|_{\mathcal{C}^{-1}}^2 + 2\|\mathbf{as}\,\boldsymbol{\sigma}_h^R\|\,C_K\|\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)\| \\ &= \|\boldsymbol{\sigma}_h^\Delta\|_{\mathcal{C}^{-1}}^2 + 2\|\mathbf{as}\,\boldsymbol{\sigma}_h^\Delta\|\,C_K\|\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)\|\end{aligned}$$

using Korn's inequality $\|\boldsymbol{\nabla}(\mathbf{u}-\mathbf{u}_h)\| \leq C_K\|\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)\|$

With $\mathbf{dev}\,\boldsymbol{\tau} := \boldsymbol{\tau} - \frac{1}{d}(\mathrm{tr}\,\boldsymbol{\tau})\mathbf{I}$:

$$\begin{aligned}\|\boldsymbol{\sigma}_h^\Delta\|_{\mathcal{C}^{-1}}^2 = (\mathcal{C}^{-1}\boldsymbol{\sigma}_h^\Delta, \boldsymbol{\sigma}_h^\Delta) &= \frac{1}{2\mu}(\mathbf{dev}\,\boldsymbol{\sigma}_h^\Delta + \frac{2\mu}{d(d\lambda+2\mu)}(\mathrm{tr}\,\boldsymbol{\sigma}_h^\Delta)\mathbf{I}, \boldsymbol{\sigma}_h^\Delta) \\ &= \frac{1}{2\mu}\|\mathbf{dev}\,\boldsymbol{\sigma}_h^\Delta\|^2 + \frac{2\mu}{d(d\lambda+2\mu)}\|\mathrm{tr}\,\boldsymbol{\sigma}_h^\Delta\|^2 \\ &\geq \frac{1}{2\mu}\|\mathbf{dev}\,\boldsymbol{\sigma}_h^\Delta\|^2 \geq \frac{1}{2\mu}\|\mathbf{as}\,\boldsymbol{\sigma}_h^\Delta\|^2\end{aligned}$$

$$\Longrightarrow \quad \|\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)\|_{\mathcal{C}}^2 \leq \|\boldsymbol{\sigma}_h^\Delta\|_{\mathcal{C}^{-1}}^2 + 2C_K\|\boldsymbol{\sigma}_h^\Delta\|_{\mathcal{C}^{-1}}\|\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)\|_{\mathcal{C}}$$

$$\Longrightarrow \quad \|\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)\|_{\mathcal{C}}^2 \leq 2\left(1+4C_K^2\right)\|\boldsymbol{\sigma}_h^\Delta\|_{\mathcal{C}^{-1}}^2$$

Overview

Stress Equilibration: The Role of Weak Symmetry

$$\|\varepsilon(\mathbf{u}-\mathbf{u}_h)\|_{\mathcal{C}}^2 = \|\boldsymbol{\sigma}_h^\Delta\|_{\mathcal{C}^{-1}}^2 - \|\boldsymbol{\sigma}_h^R - \boldsymbol{\sigma}\|_{\mathcal{C}^{-1}}^2 + 2(\mathbf{as}\,\boldsymbol{\sigma}_h^\Delta, \nabla(\mathbf{u}-\mathbf{u}_h))$$

What can we do with the term $(\mathbf{as}\,\boldsymbol{\sigma}_h^\Delta, \nabla\mathbf{u} - \nabla\mathbf{u}_h)$?

$$\begin{aligned}(\mathbf{as}\,\boldsymbol{\sigma}_h^\Delta, \nabla\mathbf{u} - \nabla\mathbf{u}_h) &\le \|\mathbf{as}\,\boldsymbol{\sigma}_h^\Delta\|\,\|\nabla(\mathbf{u}-\mathbf{u}_h)\| \le \|\mathbf{as}\,\boldsymbol{\sigma}_h^\Delta\|\,C_K^{\mathrm{glob}}\|\varepsilon(\mathbf{u}-\mathbf{u}_h)\| \\ &\le \frac{(C_K^{\mathrm{glob}})^2}{2\delta}\|\mathbf{as}\,\boldsymbol{\sigma}_h^\Delta\|^2 + \frac{\delta}{2}\|\varepsilon(\mathbf{u}-\mathbf{u}_h)\|^2\end{aligned}$$

with a global Korn constant C_K^{glob} depending on the problem (domain, boundary conditions)

$\mathbf{RM} = \mathrm{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_d, \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_{d(d-1)/2}\}$ (rigid body modes)

Use a local Korn inequality instead:

$$\inf_{\boldsymbol{\rho}\in\mathbf{RM}} \|\nabla(\mathbf{u}-\mathbf{u}_h-\boldsymbol{\rho})\|_{\omega_z} \le C_K^{\mathrm{loc}}\|\varepsilon(\mathbf{u}-\mathbf{u}_h)\|_{\omega_z}$$

with C_K^{loc} only dependent on the shape regularity of the triangulation [Horgan: SIAM Review (1995)]

Stress Equilibration: The Role of Weak Symmetry

$\nabla \boldsymbol{\rho} = \mathbf{J}^d(\boldsymbol{\alpha})$ with

$$\mathbf{J}^2(\alpha) = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} , \ \mathbf{J}^3(\boldsymbol{\alpha}) = \begin{pmatrix} 0 & \alpha_3 & -\alpha_2 \\ -\alpha_3 & 0 & \alpha_1 \\ \alpha_2 & -\alpha_1 & 0 \end{pmatrix}$$

$$(\mathbf{as}\, \boldsymbol{\sigma}_h^\Delta, \nabla \mathbf{u} - \nabla \mathbf{u}_h)_{\omega_z} = (\mathbf{as}\, \boldsymbol{\sigma}_h^\Delta, \nabla(\mathbf{u} - \mathbf{u}_h - \boldsymbol{\rho})_{\omega_z} \text{ for all } \boldsymbol{\rho} \in \mathbf{RM}$$
$$\iff \ (\mathbf{as}\, \boldsymbol{\sigma}_h^\Delta, \mathbf{J}^d(\boldsymbol{\alpha}))_{\omega_z} = 0 \text{ for all } \boldsymbol{\alpha} \in \mathbb{R}^{d(d-1)/2}$$

Add this constraint to the local stress equilibration problem:

$$\begin{aligned}
&\boldsymbol{\sigma}_{h,z}^\Delta \in \boldsymbol{\Sigma}_{h,z}^\Delta \text{ s. t.} \\
&\quad (\operatorname{div} \boldsymbol{\sigma}_{h,z}^\Delta, \mathbf{z})_T = -((\mathbf{f} + \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}_h))\phi_z, \mathbf{z})_T && \forall \mathbf{z} \in P_k(T)^d \ , \ T \subset \omega_z \\
&\quad \langle [\![\boldsymbol{\sigma}_{h,z}^\Delta \cdot \mathbf{n}]\!]_S, \boldsymbol{\zeta} \rangle_S = -\langle [\![\boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}]\!]_S \phi_z, \boldsymbol{\zeta} \rangle_S && \forall \boldsymbol{\zeta} \in P_k(S)^d \ , \ S \subset \omega_z \\
&(\mathbf{as}\, \boldsymbol{\sigma}_{h,z}^\Delta, \mathbf{J}^d(\boldsymbol{\alpha}))_{\omega_z} = 0 && \forall \boldsymbol{\alpha} \in \mathbb{R}^{d(d-1)/2}
\end{aligned}$$

Stress Equilibration: The Role of Weak Symmetry

Still well-posed and we may constrain symmetry even more:

$$\boldsymbol{\sigma}_{h,z}^{\Delta} \in \boldsymbol{\Sigma}_{h,z}^{\Delta} \text{ s. t.}$$

$$(\operatorname{div} \boldsymbol{\sigma}_{h,z}^{\Delta}, \mathbf{z})_T = -((\mathbf{f} + \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}_h))\phi_z, \mathbf{z})_T \quad \forall \mathbf{z} \in P_k(T)^d \, , \; T \subset \omega_z$$

$$\langle [\![\boldsymbol{\sigma}_{h,z}^{\Delta} \cdot \mathbf{n}]\!]_S, \boldsymbol{\zeta} \rangle_S = -\langle [\![\boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}]\!]_S \phi_z, \boldsymbol{\zeta} \rangle_S \qquad \forall \boldsymbol{\zeta} \in P_k(S)^d \, , \; S \subset \omega_z$$

$$(\mathbf{as}\, \boldsymbol{\sigma}_{h,z}^{\Delta}, \mathbf{J}^d(\boldsymbol{\gamma}))_{\omega_z} = 0 \qquad \forall \boldsymbol{\gamma} \in P_1(\mathcal{T}_h)|_{\omega_z} \cap H^1(\omega_z)$$

Kernel of the adjoint operator associated with **RM**

Well-posedness follows from inf-sup stability of the finite element triple $RT_k(\mathcal{T}_h)/P_k(\mathcal{T}_h)$ (discont.)$/P_k(\mathcal{T}_h) \cap H^1$ for $k \geq 1$

[Boffi/Brezzi/Fortin: Commun. Pure Appl. Anal. **8** (2009)]

$$\Longrightarrow \;\; \forall \boldsymbol{\alpha}_h = \sum_{z \in \mathcal{V}_h} \boldsymbol{\alpha}_z \phi_z \text{ with } \boldsymbol{\alpha}_z \in \mathbb{R}^{d(d-1)/2} \;\; :$$

$$(\mathbf{as}\, \boldsymbol{\sigma}_h^{\Delta}, \mathbf{J}^d(\boldsymbol{\alpha}_h)) = \sum_{z \in \mathcal{V}_h} (\mathbf{as}\, \boldsymbol{\sigma}_h^{\Delta}, \mathbf{J}^d(\boldsymbol{\alpha}_z)\phi_z) = \sum_{z \in \mathcal{V}_h} (\mathbf{as}\, \boldsymbol{\sigma}_h^{\Delta}, \mathbf{J}^d(\boldsymbol{\alpha}_z)\phi_z)_{\omega_z} = 0$$

Stress Equilibration: The Role of Weak Symmetry

Lemma [Bertrand/Kober/Moldenhauer/GS (2018)]:
If $(\mathbf{as}\,\boldsymbol{\sigma}_h^\Delta, \mathbf{J}^d(\boldsymbol{\alpha}_h)) = 0$ for all $\boldsymbol{\alpha}_h \in P_1(\mathcal{T}_h) \cap H^1(\Omega)$, then

$$\left|(\mathbf{as}\,\boldsymbol{\sigma}_h^\Delta, \boldsymbol{\nabla}(\mathbf{u} - \mathbf{u}_h))\right| \leq C_K^{\mathrm{loc}} \|\mathbf{as}\,\boldsymbol{\sigma}_h^\Delta\| \, \|\boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}_h)\|$$

holds with C_K^{loc} depending only on the shape regularity of $\mathcal{T}_h$.

Proof:

$$\begin{aligned}
|(\mathbf{as}\,\boldsymbol{\sigma}_h^\Delta, \boldsymbol{\nabla}(\mathbf{u} - \mathbf{u}_h))| &= |(\mathbf{as}\,\boldsymbol{\sigma}_h^\Delta, \boldsymbol{\nabla}(\mathbf{u} - \mathbf{u}_h) - \mathbf{J}^d(\boldsymbol{\alpha}_h))| \\
&= \left| \sum_{z \in \mathcal{V}_h} (\mathbf{as}\,\boldsymbol{\sigma}_h^\Delta, \left(\boldsymbol{\nabla}(\mathbf{u} - \mathbf{u}_h) - \mathbf{J}^d(\boldsymbol{\alpha}_z)\right) \phi_z)_{\omega_z} \right| \\
&= \left| \sum_{z \in \mathcal{V}_h} ((\mathbf{as}\,\boldsymbol{\sigma}_h^\Delta)\phi_z, \boldsymbol{\nabla}(\mathbf{u} - \mathbf{u}_h) - \mathbf{J}^d(\boldsymbol{\alpha}_z))_{\omega_z} \right| \\
&\leq \sum_{z \in \mathcal{V}_h} \|(\mathbf{as}\,\boldsymbol{\sigma}_h^\Delta)\phi_z\|_{\omega_z} \|\boldsymbol{\nabla}(\mathbf{u} - \mathbf{u}_h) - \mathbf{J}^d(\boldsymbol{\alpha}_z)\|_{\omega_z} \\
&\leq \sum_{z \in \mathcal{V}_h} \|\mathbf{as}\,\boldsymbol{\sigma}_h^\Delta\|_{\omega_z} \|\boldsymbol{\nabla}(\mathbf{u} - \mathbf{u}_h) - \mathbf{J}^d(\boldsymbol{\alpha}_z)\|_{\omega_z}
\end{aligned}$$

Stress Equilibration: The Role of Weak Symmetry

Choose $\boldsymbol{\alpha}_z$ such that

$$\|\nabla(\mathbf{u}-\mathbf{u}_h)-\mathbf{J}^d(\boldsymbol{\alpha}_z)\|_{\omega_z} = \inf_{\boldsymbol{\rho}\in\mathbf{RM}} \|\nabla(\mathbf{u}-\mathbf{u}_h-\boldsymbol{\rho})\|_{\omega_z} \leq C^{\mathrm{loc}}_{K,z}\|\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)\|_{\omega_z}$$

With $C^{\mathrm{loc}}_K := (d+1)\max\{C^{\mathrm{loc}}_{K,z} : z \in \mathcal{V}_h\}$:

$$\begin{aligned}
|(\mathbf{as}\,\boldsymbol{\sigma}^\Delta_h, \nabla(\mathbf{u}-\mathbf{u}_h))| &\leq \frac{C^{\mathrm{loc}}_K}{d+1}\sum_{z\in\mathcal{V}_h}\|\mathbf{as}\,\boldsymbol{\sigma}^\Delta_h\|_{\omega_z}\|\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)\|_{\omega_z} \\
&\leq C^{\mathrm{loc}}_K\left(\frac{1}{d+1}\sum_{z\in\mathcal{V}_h}\|\mathbf{as}\,\boldsymbol{\sigma}^\Delta_h\|^2_{\omega_z}\right)^{1/2}\left(\frac{1}{d+1}\sum_{z\in\mathcal{V}_h}\|\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)\|^2_{\omega_z}\right)^{1/2} \\
&= C^{\mathrm{loc}}_K\|\mathbf{as}\,\boldsymbol{\sigma}^\Delta_h\|\,\|\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)\|
\end{aligned}$$

□

Weakly Symmetric Stress Equilibration: Upper Bound

Weakly Symmetric Stress Equilibration: Upper Bound

$$
\begin{aligned}
(\textbf{as}\,\boldsymbol{\sigma}_h^\Delta, \nabla \mathbf{u} - \nabla \mathbf{u}_h) &\leq C_K^{\mathrm{loc}} \|\textbf{as}\,\boldsymbol{\sigma}_h^\Delta\| \, \|\boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}_h)\| \\
&\leq \frac{(C_K^{\mathrm{loc}})^2}{2\delta} \|\textbf{as}\,\boldsymbol{\sigma}_h^\Delta\|^2 + \frac{\delta}{2} \|\boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}_h)\|^2
\end{aligned}
$$

$\Longrightarrow$

$$
\begin{aligned}
\|\boldsymbol{\sigma}_h^\Delta\|_{\mathcal{C}^{-1}}^2 &\geq \|\boldsymbol{\sigma}_h^R - \boldsymbol{\sigma}\|_{\mathcal{C}^{-1}}^2 + \|\boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}_h)\|_{\mathcal{C}}^2 \\
&\quad - \frac{(C_K^{\mathrm{loc}})^2}{\delta} \|\textbf{as}\,\boldsymbol{\sigma}_h^\Delta\|^2 - \delta \|\boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}_h)\|^2 \\
&\geq \left(1 - \frac{\delta}{2\mu}\right) \|\boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}_h)\|_{\mathcal{C}}^2 - \frac{(C_K^{\mathrm{loc}})^2}{\delta} \|\textbf{as}\,\boldsymbol{\sigma}_h^\Delta\|^2
\end{aligned}
$$

$\Longrightarrow \quad (\delta = \mu)$

$$
\|\boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}_h)\|_{\mathcal{C}}^2 \leq 2 \|\boldsymbol{\sigma}_h^\Delta\|_{\mathcal{C}^{-1}}^2 + \frac{2(C_K^{\mathrm{loc}})^2}{\mu} \|\textbf{as}\,\boldsymbol{\sigma}_h^\Delta\|^2
$$

Weakly Symmetric Stress Equilibration: Lower Bound

Weakly Symmetric Stress Equilibration: Lower Bound

h_z: local mesh size on ω_z

$\|\boldsymbol{\sigma}^{\Delta}_{h,z}\|_{\omega_z} \longrightarrow$ min! s. t.

$$\begin{aligned} (\operatorname{div} \boldsymbol{\sigma}^{\Delta}_{h,z}, \mathbf{z})_T &= -((\mathbf{f} + \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}_h))\phi_z, \mathbf{z})_T \quad \forall \mathbf{z} \in P_k(T)^d \,,\ T \subset \omega_z \\ \langle [\![\boldsymbol{\sigma}^{\Delta}_{h,z} \cdot \mathbf{n}]\!]_S, \boldsymbol{\zeta} \rangle_S &= -\langle [\![\boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}]\!]_S \phi_z, \boldsymbol{\zeta} \rangle_S \qquad \forall \boldsymbol{\zeta} \in P_k(S)^d \,,\ S \subset \omega_z \\ (\boldsymbol{\sigma}^{\Delta}_{h,z}, \mathbf{J}^d(\gamma))_{\omega_z} &= 0 \qquad\qquad \forall \boldsymbol{\gamma} \in P_1(\mathcal{T}_h)|_{\omega_z} \cap H^1(\omega_z) \end{aligned}$$

Then, $\|\boldsymbol{\sigma}^{\Delta}_{h,z}\|_{\omega_z}$ is bounded by the residual error estimator:

$$\|\boldsymbol{\sigma}^{\Delta}_{h,z}\|_{\omega_z} \lesssim h_z \|\mathbf{f} + \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}_h)\|_{\omega_z} + \sum_{S \subset \omega_z} h_z^{1/2} \| [\![\boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}]\!]_S \|_S$$

Idea of Proof:
$\|\boldsymbol{\sigma}^{\Delta}_{h,z}\|_{\omega_z}$ bounded by suitably scaled linear combination of
$|(\operatorname{div} \boldsymbol{\sigma}^{\Delta}_{h,z}, \mathbf{z})_T|$, $T \in \mathcal{T}_h$ and $\| [\![\boldsymbol{\sigma}^{\Delta}_{h} \cdot \mathbf{n}]\!]_S \|_S$, $S \in \mathcal{S}_h$
(finite dimensional problem)

Efficiency of residual estimator $\Longrightarrow$ $\|\boldsymbol{\sigma}^{\Delta}_{h,z}\|_{\omega_z} \lesssim \|\boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}_h)\|_{\mathcal{C},\omega_z}$

Extensions to Nonlinear Deformation Models

Extensions to Nonlinear Deformation Models

Hyperelasticity

Deformation gradient $\mathbf{F}(\mathbf{u}) = \mathbf{I} + \nabla\mathbf{u}$

1st Piola-Kirchhoff stress tensor $\mathbf{P} = \partial_{\mathbf{F}}\psi(\mathbf{F}(\mathbf{u}))$

For example: Neo-Hooke material (with $J(\mathbf{u}) = \det \mathbf{F}(\mathbf{u})$):

$$\mathbf{P} = \mu\,(\mathbf{F}(\mathbf{u}) - \mathbf{F}(\mathbf{u})^{-T}) + \frac{\lambda}{2}\,(J(\mathbf{u})^2 - 1)\mathbf{F}(\mathbf{u})^{-T}$$

Symmetry of Cauchy stress $\mathbf{P}\mathbf{F}(\mathbf{u})^T/J(\mathbf{u})$ (w.r.t. deformed configuration):

$$\mathbf{P}\mathbf{F}(\mathbf{u})^T = \mu\,(\mathbf{F}(\mathbf{u})\mathbf{F}(\mathbf{u})^T - \mathbf{I}) + \frac{\lambda}{2}\,(J(\mathbf{u})^2 - 1)\mathbf{I}$$

$\mathbf{P}_h^R = \mathbf{P}(\mathbf{u}_h) + \mathbf{P}_h^\Delta$ with $\mathbf{P}_{h,z}^\Delta \in \mathbf{\Sigma}_{h,z}^\Delta$ s.t.

$$\begin{aligned}
(\operatorname{div}\mathbf{P}_{h,z}^\Delta, \mathbf{z})_T &= -((\mathbf{f} + \operatorname{div}\mathbf{P}(\mathbf{u}_h))\phi_z, \mathbf{z})_T \quad && \forall \mathbf{z} \in P_k(T)^d\,,\ T \subset \omega_z \\
\langle [\![\mathbf{P}_{h,z}^\Delta \cdot \mathbf{n}]\!]_S, \boldsymbol{\zeta}\rangle_S &= -\langle [\![\mathbf{P}(\mathbf{u}_h)\cdot\mathbf{n}]\!]_S \phi_z, \boldsymbol{\zeta}\rangle_S && \forall \boldsymbol{\zeta} \in P_k(S)^d\,,\ S \subset \omega_z \\
(\mathbf{P}_{h,z}^\Delta \mathbf{F}(\mathbf{u}_h)^T, \mathbf{J}^d(\gamma))_{\omega_z} &= 0 && \forall \gamma \in P_1(\mathcal{T}_h)|_{\omega_z} \cap H^1(\omega_z)
\end{aligned}$$

Extensions to Nonlinear Deformation Models

Hyperelasticity

Since $\mathbf{P}(\mathbf{u}_h) \notin \mathbf{\Sigma}_h$, in general, compute a projection $\widehat{\mathbf{P}}(\mathbf{u}_h) \in \mathbf{\Sigma}_h$ first

$\mathbf{P}^\Delta_{h,z} \in \mathbf{\Sigma}^\Delta_{h,z}$ s.t.

$$
\begin{aligned}
(\operatorname{div} \mathbf{P}^\Delta_{h,z}, \mathbf{z})_T &= -((\mathbf{f} + \operatorname{div} \widehat{\mathbf{P}}(\mathbf{u}_h))\phi_z, \mathbf{z})_T && \forall \mathbf{z} \\
\langle [\![\mathbf{P}^\Delta_{h,z} \cdot \mathbf{n}]\!]_S, \boldsymbol{\zeta} \rangle_S &= -\langle [\![\widehat{\mathbf{P}}(\mathbf{u}_h) \cdot \mathbf{n}]\!]_S \phi_z, \boldsymbol{\zeta} \rangle_S && \forall \boldsymbol{\zeta} \\
(\mathbf{P}^\Delta_{h,z} \mathbf{F}(\mathbf{u}_h)^T, \mathbf{J}^d(\boldsymbol{\gamma}))_{\omega_z} &= -(\widehat{\mathbf{P}}(\mathbf{u}_h) \mathbf{F}(\mathbf{u}_h)^T \phi_z, \mathbf{J}^d(\boldsymbol{\gamma}))_{\omega_z} && \forall \boldsymbol{\gamma}
\end{aligned}
$$

Symmetric stress elements would not be appropriate here!

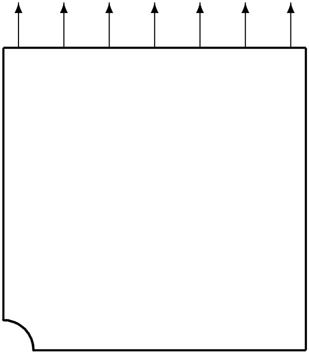
Kernel of the adjoint operator (linearly dependent constraints):

$$
\mathbf{RM}(\mathbf{u}_h) = \operatorname{span}\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ x_3 + (u_h)_3 \\ -(x_2 + (u_h)_2) \end{pmatrix}, \begin{pmatrix} -(x_3 + (u_h)_3) \\ 0 \\ x_1 + (u_h)_1 \end{pmatrix}, \begin{pmatrix} x_2 + (u_h)_2 \\ -(x_1 + (u_h)_1) \\ 0 \end{pmatrix} \}
$$

[Bertrand/Moldenhauer/GS (2019), arXiv: 1903.05888]

Extensions to Nonlinear Deformation Models

(Perfect) Plasticity



Poisson ratio: $\nu = 0.29$

Boundary conditions:
$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0}$ at right bdy and circle
$\boldsymbol{\sigma} \cdot \mathbf{n} = (0, \gamma)$ at upper bdy

Symmetry conditions:
$(\sigma_{11}, \sigma_{12}) \cdot \mathbf{n} = 0, u_2 = 0$ at bottom
$u_1 = 0, (\sigma_{21}, \sigma_{22}) \cdot \mathbf{n} = 0$ at left bdy

Load cycle: γ from 0 to 4.5 and then back
Load step size: $\delta\gamma = 0.025$

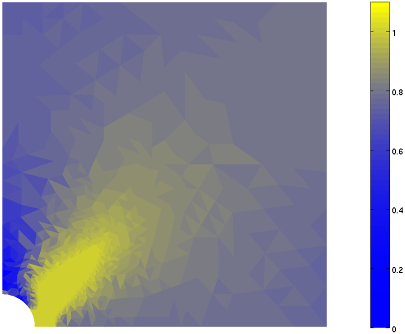
Plane strain

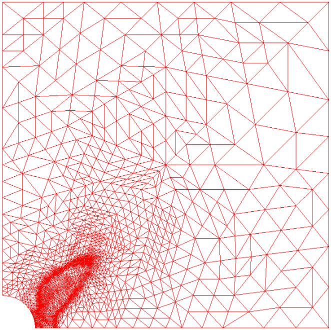
Collection of benchmark problems for elastoplasticity from Stein/Wriggers/Rieger/Schmidt and Lang/Wieners/Wittum in E. Stein (editor): Error-controlled Adaptive Finite Elements in Solid Mechanics. Wiley, 2002

Extensions to Nonlinear Deformation Models

(Perfect) Plasticity

$\gamma = 4.0$: $|\mathbf{dev}\, \boldsymbol{\sigma}|$ triangulation





Extensions to Nonlinear Deformation Models

(Perfect) Plasticity

Admissible set for the stresses:

$$\mathcal{K}(t) = \{\boldsymbol{\sigma} \in H(\mathrm{div},\Omega)^d : \mathrm{div}\,\boldsymbol{\sigma} + \mathbf{f} = \mathbf{0}\,,\ \mathbf{as}\,\boldsymbol{\sigma} = \mathbf{0} \text{ in } \Omega\,,\\ \boldsymbol{\sigma}\cdot\mathbf{n} = \boldsymbol{\ell}(t) \text{ on } \Gamma_N\,,\ |\mathbf{dev}\,\boldsymbol{\sigma}| \le \kappa \text{ in } \Omega\}$$

$\boldsymbol{\sigma}(\mathbf{u}_h) = \mathcal{P}_{\mathcal{K}(t)}(\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u}_h))$ not $H(\mathrm{div})$-conforming, in general

$\boldsymbol{\sigma}_h^R = \boldsymbol{\sigma}(\mathbf{u}_h) + \boldsymbol{\sigma}_h^\Delta$ with $\boldsymbol{\sigma}_{h,z}^\Delta \in \boldsymbol{\Sigma}_{h,z}^\Delta$ s.t.

$$\begin{aligned}
(\mathrm{div}\,\boldsymbol{\sigma}_{z,h}^\Delta, \mathbf{z})_{\omega_z} &= -((\mathbf{f} + \mathrm{div}\,\boldsymbol{\sigma}(\mathbf{u}_h))\phi_z, \mathbf{z})_{\omega_z} && \forall \mathbf{z} \\
\langle [\![\boldsymbol{\sigma}_{z,h}^\Delta \cdot \mathbf{n}]\!]_S, \boldsymbol{\zeta}\rangle_S &= -\langle [\![\boldsymbol{\sigma}(\mathbf{u}_h)\cdot\mathbf{n}]\!]_S \phi_z, \boldsymbol{\zeta}\rangle_S && \forall \boldsymbol{\zeta} \\
(\boldsymbol{\sigma}_{z,h}^\Delta, \mathbf{J}^d(\boldsymbol{\gamma}))_{\omega_z} &= 0 && \forall \boldsymbol{\gamma} \\
(\widehat{\mathbf{N}} : (\mathbf{dev}\,\boldsymbol{\sigma}_{h,z}^\Delta), \vartheta)_{\omega_z} &= -((\kappa + \mathbf{dev}\,\boldsymbol{\sigma}(\mathbf{u}_h))\phi_z, \vartheta)_{\omega_z} && \forall \vartheta \in X^a
\end{aligned}$$

with $\widehat{\mathbf{N}} = \dfrac{\mathbf{dev}\,\widehat{\boldsymbol{\sigma}}}{|\mathbf{dev}\,\widehat{\boldsymbol{\sigma}}|}$, X^a : set of active constraints

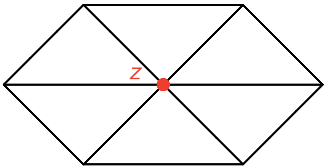
Extensions to Nonlinear Deformation Models

(Perfect) Plasticity

$\widehat{\mathbf{N}} \in \mathbb{R}^{2\times 2}$ with $|\widehat{\mathbf{N}}| = 1$

$$\widehat{\mathbf{N}} = \alpha_d \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \alpha_e \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_a \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\text{with } 2(\alpha_d^2 + \alpha_e^2 + \alpha_a^2) = 1\,,\ \alpha_d \neq 0$$



$$\inf_{\mathbf{z},\boldsymbol{\gamma},\vartheta} \sup_{\boldsymbol{\tau}\in\boldsymbol{\Sigma}^{\Delta}_{z,h}} \frac{(\operatorname{div}\boldsymbol{\tau},\mathbf{z})_{\omega_z} + (\boldsymbol{\tau},\mathbf{J}^2(\boldsymbol{\gamma}))_{\omega_z} + (\widehat{\mathbf{N}} : \mathbf{dev}\,\boldsymbol{\tau},\vartheta)_{\omega_z}}{\|\boldsymbol{\tau}\|_{H(\operatorname{div},\omega_z)}\left(\|\mathbf{z}\|_{\omega_z} + \|\boldsymbol{\gamma}\|_{\omega_z} + \|\vartheta\|_{\omega_z}\right)} \geq \beta > 0$$

Extensions to Nonlinear Deformation Models

(Perfect) Plasticity

Dimension of subspace with

$$\sup_{\boldsymbol{\tau} \in \boldsymbol{\Sigma}^{\Delta}_{z,h}} \frac{(\operatorname{div} \boldsymbol{\tau}, \mathbf{z})_{\omega_z} + (\boldsymbol{\tau}, \mathbf{J}^2(\gamma))_{\omega_z} + (\widehat{\mathbf{N}} : \mathbf{dev}\, \boldsymbol{\tau}, \vartheta)_{\omega_z}}{\|\boldsymbol{\tau}\|_{H(\operatorname{div},\omega_z)} \left(\|\mathbf{z}\|_{\omega_z} + \|\gamma\|_{\omega_z} + \|\vartheta\|_{\omega_z}\right)} = 0 \ :$$

$\boldsymbol{\Sigma}_z \setminus \mathbf{Z}/\mathbf{R}/X$:	$DP_1/DP_1/DP_1$	$DP_1/P_1/DP_1$	$DP_1/P_1/P_1$	$P_1/P_1/P_1$
RT_1^2	24	13	8	4
$RT_1^2 + \nabla^{\perp} B_3^{\mathrm{cf}}$	12	8	6	4
$RT_1^2 + \nabla^{\perp} B_3^{\mathrm{nc}}$	4	4	4	4

$$B_3^{\mathrm{cf}} = \{\mathbf{b} \in P_3(\mathcal{T})^2 : \mathbf{b}|_E = 0 \text{ for all edges } E\}$$

$$B_3^{\mathrm{nc}} = \{\mathbf{b} \in P_3(\mathcal{T})^2 : \langle \mathbf{b}, \mathbf{q} \rangle_E = 0 \text{ for all } \mathbf{q} \in P_1(E)^2 \text{ and all edges } E\}$$

Related to symmetric stress method by

Gopalakrishnan/Guzmán (SIAM J. Numer. Anal., 2011)

See also: Kober/GS (2018, ENUMATH-Proceedings)

Extensions to Nonlinear Deformation Models

(Perfect) Plasticity

Null space of the adjoint operator (linearly dependent constraints):

$$\mathbf{z} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \; \boldsymbol{\gamma} = \mathbf{0} \, , \; \vartheta = 0$$

$$\mathbf{z} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \; \boldsymbol{\gamma} = \mathbf{0} \, , \; \vartheta = 0$$

$$\mathbf{z} = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} , \; \boldsymbol{\gamma} = 2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \; \vartheta = 0 \qquad \text{(rigid body modes)}$$

and an additional plastic mode

$$\mathbf{z} = \alpha_d \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} + \alpha_e \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} , \; \boldsymbol{\gamma} = \mathbf{0} \, , \; \vartheta = 2$$

Conclusions and References

Conclusions and References

Weakly symmetric stress equilibration

Computable (and reasonably close) upper bounds for linear elasticity

Extension to nonlinear material models (hyperelasticity, plasticity) requires modifications!

Bertrand/Kober/Moldenhauer/GS:
Weakly Symmetric Stress Equilibration and A Posteriori Error Estimation for Linear Elasticity.
arXiv: 1808.02655

Bertrand/Moldenhauer/GS:
Weakly Symmetric Stress Equilibration for Hyperelastic Material Models.
arXiv: 1903.05888