A Posteriori Error Estimates by Weakly Symmetric Stress Reconstruction for the Biot Problem

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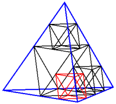
joint work with Fleurianne Bertrand (U Twente)



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Overview

Fleurianne Bertrand, Gerhard Starke: *A posteriori error estimates by weakly symmetric stress reconstruction for the Biot problem.*
Computers and Mathematics with Applications **91** (2021), 3–16.

A Three-Field Formulation of the Biot System

Find $\mathbf{u} \in H^1_0(\Omega)^d$, $p \in L^2(\Omega)$ and $\varphi \in H^1_0(\Omega)$ such that

$$\begin{aligned}
2\mu(\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v})) - (p, \operatorname{div} \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) \qquad && \forall \mathbf{v} \in H^1_0(\Omega)^d \,, \\
(\operatorname{div} \mathbf{u}, q) + \frac{1}{\lambda}(p - \varphi, q) &= 0 && \forall q \in L^2(\Omega) \,, \\
\frac{1}{\lambda}(\varphi - p, \psi) + \tau(\nabla \varphi, \nabla \psi) &= (g, \psi) && \forall \psi \in H^1_0(\Omega)
\end{aligned}$$

Lee/Mardal/Winther: SIAM J. Sci. Comput. (2017)

$\mathbf{V}_h \subset H^1_0(\Omega; \mathbb{R}^d)$: continuous piecewise quadratic
$Q_h \subset L^2(\Omega; \mathbb{R})$: continuous piecewise linear
$S_h \subset H^1_0(\Omega; \mathbb{R})$: continuous piecewise quadratic

Energy norm:

$$|||(\mathbf{u} - \mathbf{u}_h, p - p_h, \varphi - \varphi_h)|||$$
$$= \left(2\mu \|\boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}_h)\|^2 + \frac{1}{\lambda} \|p - p_h - (\varphi - \varphi_h)\|^2 + \tau \|\nabla(\varphi - \varphi_h)\|^2 \right)^{1/2}$$

A Three-Field Formulation of the Biot System

Stress and flux:

$$\begin{aligned} \boldsymbol{\theta}(\mathbf{u}, p, \varphi) &= 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) - (p - \varphi)\mathbf{I}\,, & \boldsymbol{\theta}(\mathbf{u}_h, p_h, \varphi_h) &= 2\mu\boldsymbol{\varepsilon}(\mathbf{u}_h) - (p_h - \varphi_h)\mathbf{I}\,, \\ \mathbf{w}(\varphi) &= -\nabla\varphi\,, & \mathbf{w}(\varphi_h) &= -\nabla\varphi_h\,, \end{aligned}$$

Stress reconstruction $\boldsymbol{\theta}_h^R \in H(\mathrm{div}, \Omega; \mathbb{R}^d)$ satisfying

$$\mathrm{div}\,\boldsymbol{\theta}_h^R = -\mathbf{f} + \nabla\varphi_h \text{ in } \Omega$$

Flux reconstruction $\mathbf{w}_h^R \in H(\mathrm{div}, \Omega; \mathbb{R})$ satisfying

$$\tau\,\mathrm{div}\,\mathbf{w}_h^R = g + \frac{p_h - P_h^1\varphi_h}{\lambda} \text{ in } \Omega$$

Earlier work: First-order system least squares formulation for the simultaneous computation of displacement/stress/pressure/flux:
Korsawe/St.: SIAM J. Numer. Anal. (2005)

A Posteriori Error Estimation

Stress-related error:

$$\begin{aligned}\eta_S &= \|\boldsymbol{\theta}_h^R - \boldsymbol{\theta}(\mathbf{u}_h, p_h, \varphi_h)\|_{\mathcal{A}} \\ &= (\boldsymbol{\theta}_h^R - \boldsymbol{\theta}(\mathbf{u}_h, p_h, \varphi_h), \mathcal{A}(\boldsymbol{\theta}_h^R - \boldsymbol{\theta}(\mathbf{u}_h, p_h, \varphi_h)))^{1/2}\end{aligned}$$

where $\mathcal{A} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ is defined by

$$\mathcal{A}\boldsymbol{\xi} = \frac{1}{2\mu}\left(\boldsymbol{\xi} - \frac{\lambda}{2\mu + d\lambda}(\mathrm{tr}\,\boldsymbol{\xi})\mathbf{I}\right)$$

Asymmetry: $\eta_A = \|\mathbf{as}\boldsymbol{\theta}_h^R\| = \|\frac{1}{2}\left(\boldsymbol{\theta}_h^R - (\boldsymbol{\theta}_h^R)^T\right)\|$

Fluid-related error:

$$\eta_F = \tau^{1/2}\|\mathbf{w}_h^R - \mathbf{w}(\varphi_h)\| \ \ , \ \ \eta_P = \frac{1}{\lambda}\tau^{1/2}\|\varphi_h - P_h^1\varphi_h\|$$

Coupling error: $\eta_C = \|\frac{p_h - \varphi_h}{\lambda} + \mathrm{div}\mathbf{u}_h\|$

A Posteriori Error Estimation

Energy norm:

$$
\begin{aligned}
&|||(\mathbf{u}-\mathbf{u}_h, p-p_h, \varphi-\varphi_h)||| \\
&\quad = \left(2\mu\|\varepsilon(\mathbf{u}-\mathbf{u}_h)\|^2 + \frac{1}{\lambda}\|p-p_h-(\varphi-\varphi_h)\|^2 + \tau\|\nabla(\varphi-\varphi_h)\|^2\right)^{1/2}
\end{aligned}
$$

Theorem 1. The total error associated with the Biot system satisfies

$$
\begin{aligned}
&|||(\mathbf{u}-\mathbf{u}_h, p-p_h, \varphi-\varphi_h)|||^2 \\
&\leq 2\left(\eta_S^2 + \frac{C_K^2}{4\mu}\eta_A^2 + \left(\mu\left(\frac{C_D\lambda}{2\mu+d\lambda} + \frac{1}{C_D}\right)^2 + \frac{4C_F^2}{\tau}\right)\eta_C^2 + \eta_F^2 + 4C_F^2\eta_P^2\right).
\end{aligned}
$$

Remark. With our construction (to be presented later) the Korn constant C_K and the "dev-div" constant C_D are local, i.e. only dependent on the mesh quality (not on the global problem)

C_F is the Poincaré-Friedrichs constant for the (flow pressure) space S

Reconstruction of Stress and Flux

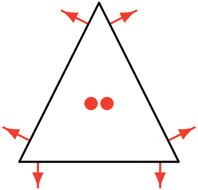
Raviart-Thomas Finite Element Spaces

$$\boldsymbol{\theta}_h^R \in \boldsymbol{\Theta}_h^R \subset H(\mathrm{div}, \Omega; \mathbb{R}^d) : \qquad \boldsymbol{\theta}_h^R = \boldsymbol{\theta}(\mathbf{u}_h, p_h, \varphi_h) + \boldsymbol{\theta}_h^\Delta \text{ with}$$
$$\boldsymbol{\theta}_h^\Delta \in \boldsymbol{\Theta}_h^\Delta = \{\boldsymbol{\xi}_h \in L^2(\Omega; \mathbb{R}^{d\times d}) : \boldsymbol{\xi}_h|_T \in RT_1(T; \mathbb{R}^{d\times d})\}$$

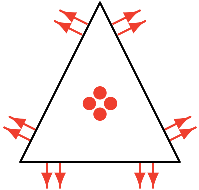
$$\mathbf{w}_h^R \in \mathbf{W}_h^R \subset H(\mathrm{div}, \Omega) : \qquad \mathbf{w}_h^R = \mathbf{w}(\varphi) + \mathbf{w}_h^\Delta \text{ with}$$
$$\mathbf{w}_h^\Delta \in \mathbf{W}_h^\Delta = \{\boldsymbol{\xi}_h \in L^2(\Omega; \mathbb{R}^d) : \boldsymbol{\xi}_h|_T \in RT_1(T; \mathbb{R}^d)\}$$

Raviart-Thomas space $RT_k(T; \mathbb{R}^{d\times d})$:

$$\boldsymbol{\theta}_h|_T \in P_k(T)^{d\times d} + P_k(T)^d\, \mathbf{x}^T \text{ for all } T \in \mathcal{T}_h$$



$RT_1(T; \mathbb{R}^2)$



$RT_1(T; \mathbb{R}^{2\times 2})$

Reconstruction of Stress and Flux

Incomplete History

Flux/stress reconstruction algorithms, equilibrated fluxes/stresses:

Ern/Vohralík (2015, . . .) . . .

. . . Dörsek/Melenk (2013) . . .

. . . Hannukainen/Stenberg/Vohralík (2012) . . .

. . . Cai/Zhang (2012, . . .) . . .

. . . Braess/Schöberl (2008) . . .

. . . Nicaise/Witowski/Wohlmuth (2008) . . .

. . . Ainsworth/Oden (1993) . . .

. . . Ladevéze/Leguillon (1983) . . .

. . . Prager/Synge (1947)

Reconstruction of Stress and Flux

Weakly Symmetric Stress Reconstruction

Add symmetry of $\boldsymbol{\theta}_h^R$ in a weak sense:

$$\mathbf{J}^2(\alpha) = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}, \ \mathbf{J}^3(\boldsymbol{\alpha}) = \begin{pmatrix} 0 & \alpha_3 & -\alpha_2 \\ -\alpha_3 & 0 & \alpha_1 \\ \alpha_2 & -\alpha_1 & 0 \end{pmatrix}$$

$$(\boldsymbol{\theta}_h^\Delta, \mathbf{J}^d(\boldsymbol{\alpha})) = (\boldsymbol{\theta}_h^R, \mathbf{J}^d(\boldsymbol{\alpha})) = 0 \text{ for suitable functions } \boldsymbol{\alpha}$$

E.g., $\boldsymbol{\alpha} \in P_1(\mathcal{T}_h)^{d(d-1)/2}$ (continuous piecewise linear elements) in the case of the Boffi/Brezzi/Fortin (2009) weakly symmetric stress element

F. Bertrand/B. Kober/M. Moldenhauer/G. S.: Weakly symmetric stress equilibration and a posteriori error estimation for linear elasticity. Numerical Methods for PDEs (2021)

F. Bertrand/M. Moldenhauer/G. S.: Weakly symmetric stress equilibration for hyperelastic material models. GAMM-Mitteilungen (2020)

Reconstruction of Stress and Flux

Partition of unity with vertex basis functions: $$1 \equiv \sum_{z \in \mathcal{V}_h} \phi_z$$

Vertex patch: $$\omega_z = \bigcup \{ T \in \mathcal{T}_h : z \text{ is a vertex of } T \}$$

$$\boldsymbol{\theta}_h^R = \boldsymbol{\theta}(\mathbf{u}_h, p_h, \varphi_h) + \boldsymbol{\theta}_h^\Delta = \boldsymbol{\theta}(\mathbf{u}_h, p_h, \varphi_h) + \sum_{z \in \mathcal{V}_h} \boldsymbol{\theta}_{h,z}^\Delta \qquad \text{with}$$

$$\boldsymbol{\theta}_{h,z}^\Delta \in \boldsymbol{\Theta}_{h,z}^\Delta := \{ \boldsymbol{\xi} \in L^2(\omega_z; \mathbb{R}^{d \times d}) : \boldsymbol{\xi}|_T \in RT_1(T; \mathbb{R}^{d \times d}), \boldsymbol{\xi} \cdot \mathbf{n} = \mathbf{0} \text{ on } \partial\omega_z \}$$

(appropriately modified for patches with $\partial\omega_z \cap \partial\Omega \neq \emptyset$)

$\boldsymbol{\theta}_{h,z}^\Delta \in \boldsymbol{\Theta}_{h,z}^\Delta$ such that

$$(\operatorname{div} \boldsymbol{\theta}_{h,z}^\Delta, \mathbf{z})_T = -((\mathbf{f} - \nabla \varphi_h + \operatorname{div} \boldsymbol{\theta}(\mathbf{u}_h, p_h, \varphi_h))\phi_z, \mathbf{z})_T \quad \forall \mathbf{z} \in P_1 \, , \; T \subset \omega_z$$

$$\langle [\![\boldsymbol{\theta}_{h,z}^\Delta \cdot \mathbf{n}]\!]_S, \boldsymbol{\zeta} \rangle_S = -\langle [\![\boldsymbol{\theta}(\mathbf{u}_h, p_h, \varphi_h) \cdot \mathbf{n}]\!]_S \phi_z, \boldsymbol{\zeta} \rangle_S \qquad \forall \boldsymbol{\zeta} \in P_1(S)^d \, , \; S \subset \omega_z$$

$$(\boldsymbol{\theta}_{h,z}^\Delta, \mathbf{J}^d(\gamma))_{\omega_z} = 0 \qquad \forall \boldsymbol{\gamma} \in P_1(\mathcal{T}_h)|_{\omega_z} \cap H^1(\omega_z)$$

Reconstruction of Stress and Flux

Why enforce weak symmetry for the stress?

Theorem 2.
Let $\boldsymbol{\theta}_h^R \in \boldsymbol{\Theta}_h^R$ be a stress reconstruction satisfying the weak symmetry condition $(\boldsymbol{\theta}_h^R, \mathbf{J}^d(\gamma_h)) = 0$ for all $\gamma_h \in \mathbf{Q}_h$. Then, for all $\mathbf{v} \in H_0^1(\Omega)^d$,

$$\left|(\mathbf{as}\,\boldsymbol{\theta}_h^R, \nabla \mathbf{v})\right| \leq C_K \|\mathbf{as}\,\boldsymbol{\theta}_h^R\| \, \|\varepsilon(\mathbf{v})\|$$

holds with a constant C_K which depends only on (the largest interior angle in) the triangulation $\mathcal{T}_h$.
Moreover, for any $s \in L^2(\Omega)$ with $(s, q_h) = 0$ for all $q_h \in Q_h$ (the space of conforming piecewise linear functions),

$$\left|(\mathrm{tr}(\boldsymbol{\theta}(\mathbf{u}, p, \varphi)) - \boldsymbol{\theta}_h^R), s)\right| \leq C_D \mu^{1/2} \|\boldsymbol{\theta}(\mathbf{u}, p, \varphi)) - \boldsymbol{\theta}_h^R\|_{\mathcal{A}} \, \|s\|$$

holds, where C_D again depends only on (the largest interior angle in) the triangulation $\mathcal{T}_h$.

Reconstruction of Stress and Flux

Flux reconstruction in the standard way:

$$\mathbf{w}_h^R = \mathbf{w}(\varphi_h) + \mathbf{w}_h^\Delta = \mathbf{w}(\varphi_h) + \sum_{z \in \mathcal{V}_h} \mathbf{w}_{h,z}^\Delta \qquad \text{with}$$

$$\mathbf{w}_{h,z}^\Delta \in \mathbf{W}_{h,z}^\Delta := \{\boldsymbol{\xi} \in L^2(\omega_z; \mathbb{R}^d) : \boldsymbol{\xi}|_T \in RT_1(T; \mathbb{R}^d), \boldsymbol{\xi} \cdot \mathbf{n} = \mathbf{0} \text{ on } \partial\omega_z\}$$

(appropriately modified for patches with $\partial\omega_z \cap \partial\Omega \neq \emptyset$)

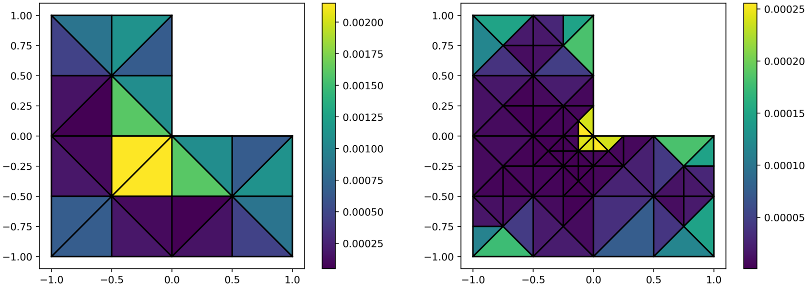
$\mathbf{w}_{h,z}^\Delta \in \mathbf{W}_{h,z}^\Delta$ such that

$$(\tau \operatorname{div} \mathbf{w}_{h,z}^\Delta, z)_T = ((g - \tau \operatorname{div} \mathbf{w}(\varphi_h) + \frac{p_h - P_h^1 \varphi_h}{\lambda})\phi_z, z)_T \;\; \forall z \in P_1 \,, \; T \subset \omega_z$$

$$\langle [\![\mathbf{w}_{h,z}^\Delta \cdot \mathbf{n}]\!]_S, \boldsymbol{\zeta} \rangle_S = -\langle [\![\mathbf{w}(\varphi_h) \cdot \mathbf{n}]\!]_S \phi_z, \boldsymbol{\zeta} \rangle_S \qquad \forall \boldsymbol{\zeta} \in P_1(S)^d \,, \; S \subset \omega_z$$

Numerical Results

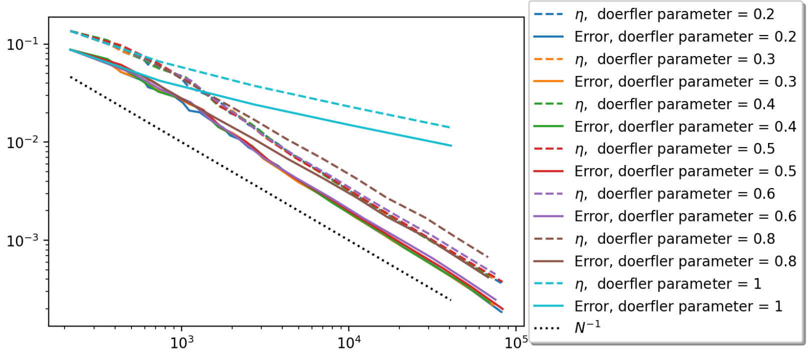
L-shape domain $\mathbf{f} = (1, 1)$ $g = 1$



Error distribution on initial and fifth mesh for $\lambda = \tau = \mu = 1$

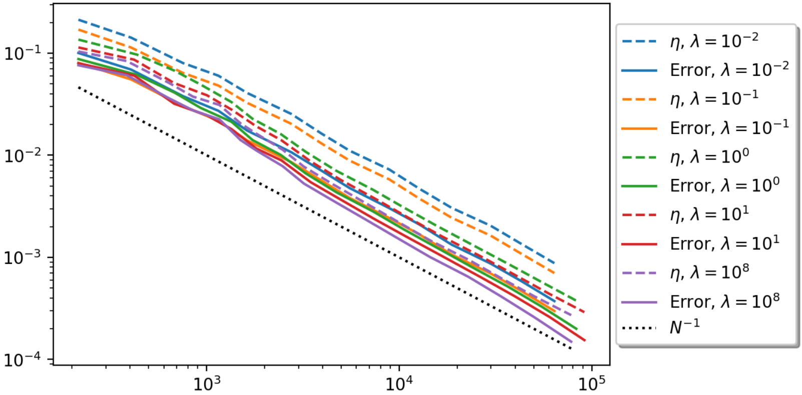
Numerical Results

$$\eta^2 = \eta_S^2 + \eta_A^2 + \eta_F^2 + \eta_P^2 + \eta_C^2$$



Comparison of estimator and error for different Willy Dörfler parameters

Numerical Results



Error estimator and error for adaptive refinement, Dörfler parameter 0.5