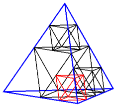
Constrained First-Order System Least Squares for Elastoplasticity

Gerhard Starke

Fakultät für Mathematik, Universität Duisburg-Essen
joint work with Henrik Schneider



93rd Annual GAMM Meeting, Dresden Section 18 June 1, 2023

Overview

Linear Elasticity: Constrained First-Order System Least Squares

Elastoplasticity: Constrained First-Order System Least Squares

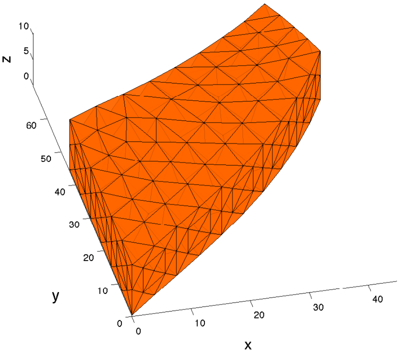
Semi-Smooth Gauß-Newton Iteration

Computational Experiments

Extension to Elastoplasticity with Hardening

Linear Elasticity: Constrained First-Order System Least Sq

Displacement field $\underline{u} : \Omega \to \mathbb{R}^3$
Stress tensor $\underline{\underline{\sigma}} : \Omega \to \mathbb{R}^{3\times 3}$



$$\underline{\underline{\varepsilon}}(\underline{u}) = \frac{\nabla \underline{u} + \nabla \underline{u}^T}{2} = \begin{bmatrix} \partial_1 u_1 & \frac{\partial_2 u_1 + \partial_1 u_2}{2} & \frac{\partial_3 u_1 + \partial_1 u_3}{2} \\ \frac{\partial_1 u_2 + \partial_2 u_1}{2} & \partial_2 u_2 & \frac{\partial_3 u_2 + \partial_2 u_3}{2} \\ \frac{\partial_1 u_3 + \partial_3 u_1}{2} & \frac{\partial_2 u_3 + \partial_3 u_2}{2} & \partial_3 u_3 \end{bmatrix}$$

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \mathcal{C}\, \underline{\underline{\varepsilon}}(\underline{u}) := 2\mu \underline{\underline{\varepsilon}}(\underline{u}) + \lambda \left(\operatorname{tr} \underline{\underline{\varepsilon}}(\underline{u})\right) \underline{\underline{I}}$$

$$\underline{\underline{\varepsilon}}(\underline{u}) = \mathcal{A}\underline{\underline{\sigma}} := \frac{1}{2\mu}\left(\underline{\underline{\sigma}} - \frac{\lambda}{3\lambda + 2\mu} (\operatorname{tr} \underline{\underline{\sigma}})\, \underline{\underline{I}}\right) = \frac{1+\nu}{E}\left(\underline{\underline{\sigma}} - \frac{\nu}{1+\nu} (\operatorname{tr} \underline{\underline{\sigma}})\, \underline{\underline{I}}\right)$$

Linear Elasticity: Constrained First-Order System Least Sq

$$\|\underline{\underline{\varepsilon}}(\underline{v}) - \mathcal{A}\underline{\underline{\tau}}\|^2_{L^2(\Omega)} \to \min!$$

$$\text{among all } (\underline{\underline{\tau}}, \underline{v}) \in H(\text{div}, \Omega; \mathbb{R}^{3\times 3}) \times H^1_{\Gamma_D}(\Omega; \mathbb{R}^3)$$

$$\text{with } \underline{\underline{\tau}} \cdot \underline{n} = \underline{g} \text{ on } \Gamma_N$$

$$\text{subject to } (\text{div}\, \underline{\underline{\tau}}, \underline{z})_{L^2(\Omega)} = 0 \text{ for all } \underline{z} \in L^2(\Omega; \mathbb{R}^3)$$

Optimality (KKT-) System:
$\underline{\underline{\sigma}} \in H(\text{div}, \Omega; \mathbb{R}^{3\times 3})$ with $\underline{\underline{\sigma}} \cdot \underline{n} = \underline{g}$, $\underline{u} \in H^1_{\Gamma_D}(\Omega; \mathbb{R}^3)$, $\underline{w} \in L^2(\Omega; \mathbb{R}^3)$
such that

$$\begin{aligned}
(\mathcal{A}\underline{\underline{\sigma}}, \mathcal{A}\underline{\underline{\tau}})_{L^2(\Omega)} - (\underline{\underline{\varepsilon}}(\underline{u}), \mathcal{A}\underline{\underline{\tau}})_{L^2(\Omega)} - (\underline{w}, \text{div}\, \underline{\underline{\tau}})_{L^2(\Omega)} &= 0 \;\forall\, \underline{\underline{\tau}} \in H(\text{div}, \Omega; \mathbb{R}^{3\times 3}) \\
-(\mathcal{A}\underline{\underline{\sigma}}, \underline{\underline{\varepsilon}}(\underline{v}))_{L^2(\Omega)} + (\underline{\underline{\varepsilon}}(\underline{u}), \underline{\underline{\varepsilon}}(\underline{v}))_{L^2(\Omega)} &= 0 \;\forall\, \underline{v} \in H^1_{\Gamma_D}(\Omega; \mathbb{R}^3) \\
-(\text{div}\, \underline{\underline{\sigma}}, \underline{z})_{L^2(\Omega)} &= 0 \;\forall\, \underline{z} \in L^2(\Omega; \mathbb{R}^3)
\end{aligned}$$

Well-posedness from Cai/St.: SIAM J. Numer. Anal. **42** (2004), 826–842

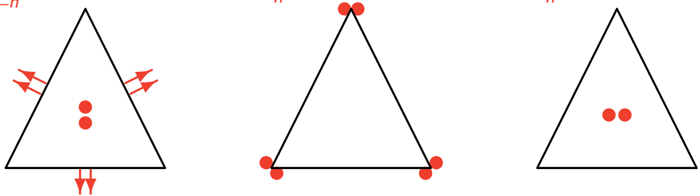
Linear Elasticity: Constrained First-Order System Least Sq

$\mathcal{T}_h = \{T : T \subset \mathbb{R}^d\}$: Triangulation (member of a non-degenerate family)

Suitable finite element combinations (for $k \geq 0$ and $d = 2$ or 3):

$$\begin{aligned} \underline{\underline{\Sigma}}_h \subset H(\mathrm{div}, \Omega; \mathbb{R}^{d\times d}) &: \underline{\underline{\sigma}}_h\Big|_T \in RT_k(T) = P_k(T;\mathbb{R}^{d\times d}) + P_k(T;\mathbb{R}^d)\,\underline{x} \\ \underline{V}_h \subset H^1(\Omega;\mathbb{R}^d) &: \underline{u}_h|_T \in P_{k+1}(T;\mathbb{R}^d) \\ \underline{Z}_h \subset L^2(\Omega;\mathbb{R}^d) &: \underline{w}_h|_T \in P_k(T;\mathbb{R}^d) \end{aligned}$$

$k = 0$: $\underline{\underline{\sigma}}_h$ $\quad$ $\underline{u}_h$ $\quad$ $\underline{w}_h$



$RT_0(\mathcal{T}_h;\mathbb{R}^{2\times 2})$ $\quad$ $P_1(\mathcal{T}_h;\mathbb{R}^2)$ $\quad$ $DP_0(\mathcal{T}_h;\mathbb{R}^2)$

Linear Elasticity: Constrained First-Order System Least Sq

Discretized Optimality (KKT-) System:
$\underline{\underline{\sigma}}_h \in \underline{\underline{\Sigma}}_h$ with $\underline{\underline{\sigma}}_h \cdot \underline{n} = \underline{g}$, $\underline{u}_h \in \underline{V}_h$, $\underline{w}_h \in \underline{Z}_h$ such that

$$\begin{aligned}
(\mathcal{A}\underline{\underline{\sigma}}_h, \mathcal{A}\underline{\underline{\tau}}_h)_{L^2(\Omega)} - (\underline{\underline{\varepsilon}}(\underline{u}_h), \mathcal{A}\underline{\underline{\tau}}_h)_{L^2(\Omega)} - (\underline{w}_h, \operatorname{div} \underline{\underline{\tau}}_h)_{L^2(\Omega)} &= 0 \; \forall\, \underline{\underline{\tau}}_h \in \underline{\underline{\Sigma}}_h \\
-(\mathcal{A}\underline{\underline{\sigma}}_h, \underline{\underline{\varepsilon}}(\underline{v}_h))_{L^2(\Omega)} + (\underline{\underline{\varepsilon}}(\underline{u}_h), \underline{\underline{\varepsilon}}(\underline{v}_h))_{L^2(\Omega)} &= 0 \; \forall\, \underline{v}_h \in \underline{V}_h \\
-(\operatorname{div} \underline{\underline{\sigma}}_h, \underline{z}_h)_{L^2(\Omega)} &= 0 \; \forall\, \underline{z}_h \in \underline{Z}_h
\end{aligned}$$

Well-posedness from Cai/St.: SIAM J. Numer. Anal. **42** (2004), 826–842
and inf-sup stability of RT_k/DP_k ($\underline{\underline{\Sigma}}_h/\underline{V}_h$) combination

Error equivalence:

$$\begin{aligned}
&\|\underline{\underline{\varepsilon}}(\underline{u}_h) - \mathcal{A}\underline{\underline{\sigma}}_h\|^2_{L^2(\Omega)} \\
&= \|\underline{\underline{\varepsilon}}(\underline{u}_h - \underline{u}) - \mathcal{A}(\underline{\underline{\sigma}}_h - \underline{\underline{\sigma}})\|^2_{L^2(\Omega)} \;\substack{\lesssim \\ \gtrsim}\; \|\underline{\underline{\varepsilon}}(\underline{u}_h - \underline{u})\|^2_{L^2(\Omega)} + \|\underline{\underline{\sigma}}_h - \underline{\underline{\sigma}}\|^2_{L^2(\Omega)}
\end{aligned}$$

and momentum balance satisfied exactly: $\operatorname{div}(\underline{\underline{\sigma}}_h - \underline{\underline{\sigma}}) = 0$

Elastoplasticity: Constrained First-Order System Least Sq

Time stepping (quasi-static problem):

$$\underline{\underline{\varepsilon}}(\underline{u}) = \mathcal{A}\underline{\underline{\sigma}} + \underline{\underline{p}}$$

$$\underline{\underline{p}} = \underline{\underline{p}}^{\rm old} + \gamma(\underline{u})\,\mathsf{dev}\,\underline{\underline{\sigma}} \text{ s.t. } |\mathsf{dev}\,\underline{\underline{\sigma}}| \le \sqrt{\frac{2}{3}}K_0 \text{ in } \Omega$$

(perfect plasticity)

Explicit formula: $$\gamma(\underline{u}) = \max\left\{0, \frac{|\mathsf{dev}\,\underline{\underline{\varepsilon}}(\underline{u}) - \underline{\underline{p}}^{\rm old}|}{\sqrt{\frac{2}{3}}K_0} - \frac{1}{2\mu}\right\}$$

$$\|\underline{\underline{\varepsilon}}(\underline{v}) - (\mathcal{A}\underline{\underline{\tau}} + \underline{\underline{p}}^{\rm old} + \gamma(\underline{v})\,\mathsf{dev}\,\underline{\underline{\tau}})\|^2_{L^2(\Omega)} \to \min!$$

among all $(\underline{\underline{\tau}}, \underline{v}) \in H(\mathsf{div}, \Omega; \mathbb{R}^{3\times 3}) \times H^1_{\Gamma_D}(\Omega; \mathbb{R}^3)$

with $\underline{\underline{\tau}} \cdot \underline{n} = \underline{g}$ on Γ_N

subject to $(\mathsf{div}\,\underline{\underline{\tau}}, \underline{z})_{L^2(\Omega)} = 0$ for all $\underline{z} \in L^2(\Omega; \mathbb{R}^3)$

Elastoplasticity: Constrained First-Order System Least Sq

Plane strain assumption: Reduction to two-dimensional domain Ω

$$\underline{\underline{\varepsilon}}(\underline{u}) = \begin{bmatrix} \partial_1 u_1 & \frac{\partial_2 u_1 + \partial_1 u_2}{2} & 0 \\ \frac{\partial_1 u_2 + \partial_2 u_1}{2} & \partial_2 u_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ , but: } \underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

with $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}(x_1, x_2)$, $\underline{u} = \underline{u}(x_1, x_2)$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \in H(\text{div}, \Omega; \mathbb{R}^{2\times 2}) \text{ , } \sigma_{33} \in L^2(\Omega; \mathbb{R}) \text{ , } \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in H^1(\Omega; \mathbb{R}^2)$$

Semi-Smooth Gauß-Newton Iteration

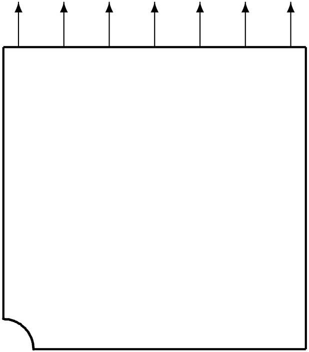
Newton derivative:

$$D_{\underline{u}}\gamma(\underline{u})[\underline{v}] = \begin{cases} \sqrt{\frac{3}{2}}\frac{1}{K_0}\frac{\operatorname{dev}\underline{\underline{\varepsilon}}(\underline{u})-\underline{\underline{p}}^{\rm old}}{|\operatorname{dev}\underline{\underline{\varepsilon}}(\underline{u})-\underline{\underline{p}}^{\rm old}|} : \underline{\underline{\varepsilon}}(\underline{v}) & , \ |\operatorname{dev}\underline{\underline{\varepsilon}}(\underline{u}) - \underline{\underline{p}}^{\rm old}| \geq \frac{1}{2\mu}\sqrt{\frac{2}{3}}K_0 \,, \\ 0 & , \text{ otherwise.} \end{cases}$$

$(\hat{\underline{\underline{\sigma}}}, \hat{\underline{u}})$ current iterate

$$\begin{aligned} &\|\underline{\underline{\varepsilon}}(\hat{\underline{u}}) - (\mathcal{A}\hat{\underline{\underline{\sigma}}} + \underline{\underline{p}}^{\rm old} + \gamma(\hat{\underline{u}}) \operatorname{dev}\hat{\underline{\underline{\sigma}}}) \\ &\qquad + \underline{\underline{\varepsilon}}(\underline{v}) - (\mathcal{A}\underline{\underline{\tau}} + \gamma(\hat{\underline{u}}) \operatorname{dev}\underline{\underline{\tau}} + D_{\underline{u}}\gamma(\hat{\underline{u}})[\underline{v}] \operatorname{dev}\hat{\underline{\underline{\sigma}}})\|^2_{L^2(\Omega)} \to \min! \\ &\qquad \text{among all } (\underline{\underline{\tau}}, \underline{v}) \in H(\operatorname{div}, \Omega; \mathbb{R}^{3\times 3}) \times H^1_{\Gamma_D}(\Omega; \mathbb{R}^3) \\ &\qquad \text{with } \underline{\underline{\tau}} \cdot \underline{n} = \underline{g} \text{ on } \Gamma_N \\ &\qquad \text{subject to } (\operatorname{div}\underline{\underline{\tau}}, \underline{z})_{L^2(\Omega)} = 0 \text{ for all } \underline{z} \in L^2(\Omega; \mathbb{R}^3) \end{aligned}$$

Computational Experiments



Poisson ratio: $\nu = 0.29$

Boundary conditions:
$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{0}$ at right bdy and circle
$\underline{\underline{\sigma}} \cdot \underline{n} = (0, \gamma)$ at upper bdy

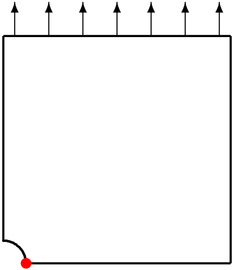
Symmetry conditions:
$(\sigma_{11}, \sigma_{12}) \cdot \underline{n} = 0, u_2 = 0$ at bottom
$u_1 = 0, (\sigma_{21}, \sigma_{22}) \cdot \underline{n} = 0$ at left bdy

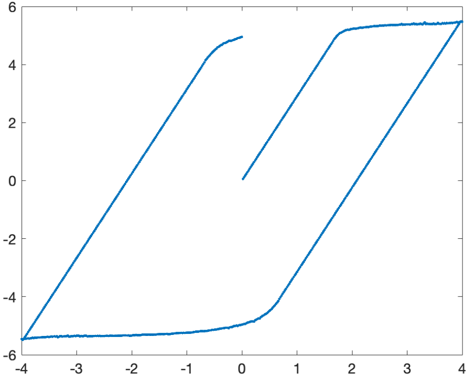
Load cycle: γ from 0 to 4 to -4 to 0
Load step size: $\delta\gamma = 0.02$

Plane strain assumption

Collection of benchmark problems for elastoplasticity from
Stein/Wriggers/Rieger/Schmidt and Lang/Wieners/Wittum in
E. Stein: Error-controlled Adaptive Finite Elements in Solid Mechanics. Wiley, 2002

Computational Experiments



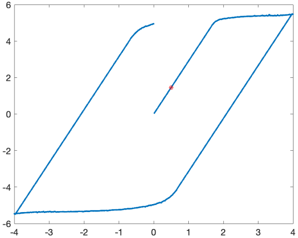


σ_{22} at lower left corner for one complete load cycle

Computational Experiments

$\gamma = 0.5$:

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
d.o.f.	364	1456	5824	23296	93184
$\mathcal{F}(\underline{\underline{\sigma}}_h, \underline{u}_h)^{1/2}$	1.3480 e-1	8.6422 e-2	4.8478 e-2	2.5198 e-2	1.2745 e-2

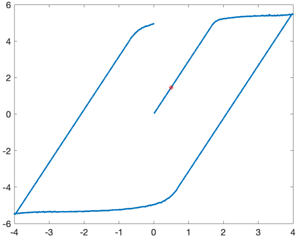


Plastic strain

Computational Experiments

$\gamma = 1.0$:

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
d.o.f.	364	1456	5824	23296	93184
$\mathcal{F}(\underline{\underline{\sigma}}_h, \underline{u}_h)^{1/2}$	2.6960 e-1	1.7284 e-1	9.6956 e-2	5.0396 e-2	2.5491 e-2

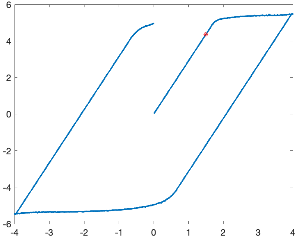


Plastic strain

Computational Experiments

$\gamma = 1.5$:

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
d.o.f.	364	1456	5824	23296	93184
$\mathcal{F}(\underline{\underline{\sigma}}_h, \underline{u}_h)^{1/2}$	4.0441 e-1	2.5926 e-1	1.4543 e-1	7.5593 e-2	3.8236 e-2

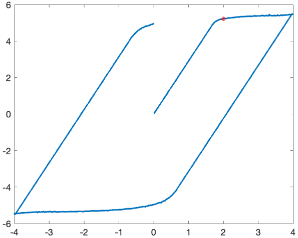


Plastic strain

Computational Experiments

$\gamma = 2.0$:

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
d.o.f.	364	1456	5824	23296	93184
$\mathcal{F}(\underline{\underline{\sigma}}_h, \underline{u}_h)^{1/2}$	5.3921 e-1	3.4569 e-1	1.9184 e-1	9.9821 e-2	5.0562 e-2

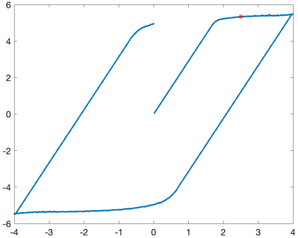


Plastic strain

Computational Experiments

$\gamma = 2.5$:

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
d.o.f.	364	1456	5824	23296	93184
$\mathcal{F}(\underline{\underline{\sigma}}_h, \underline{u}_h)^{1/2}$	6.7401 e-1	4.1946 e-1	2.3559 e-1	1.2350 e-1	6.2887 e-2

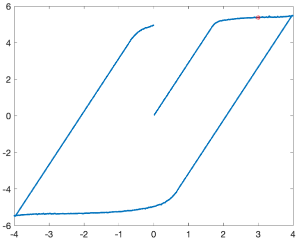


Plastic strain

Computational Experiments

$\gamma = 3.0$:

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
d.o.f.	364	1456	5824	23296	93184
$\mathcal{F}(\underline{\underline{\sigma}}_h, \underline{u}_h)^{1/2}$	8.0881 e-1	5.0295 e-1	2.8280 e-1	1.4815 e-1	7.5942 e-2

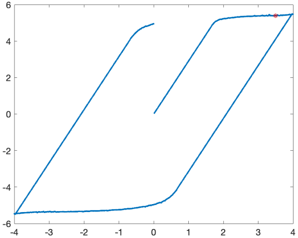


Plastic strain

Computational Experiments

$\gamma = 3.5$:

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
d.o.f.	364	1456	5824	23296	93184
$\mathcal{F}(\underline{\underline{\sigma}}_h, \underline{u}_h)^{1/2}$	9.3216 e-1	5.8870 e-1	3.2643 e-1	1.7262 e-1	8.9075 e-2

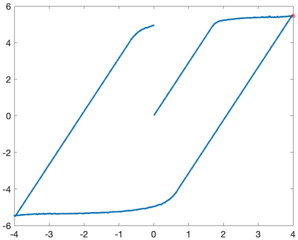


Plastic strain

Computational Experiments

$\gamma = 4.0$:

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
d.o.f.	364	1456	5824	23296	93184
$\mathcal{F}(\underline{\underline{\sigma}}_h, \underline{u}_h)^{1/2}$	1.0796 e-0	6.6973 e-1	3.7910 e-1	2.0473 e-1	1.0769 e-1

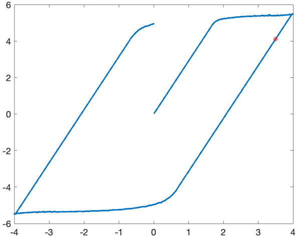


Plastic strain

Computational Experiments

$\gamma = 3.5$:

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
d.o.f.	364	1456	5824	23296	93184
$\mathcal{F}(\underline{\underline{\sigma}}_h, \underline{u}_h)^{1/2}$	9.3998 e-1	5.8021 e-1	3.2646 e-1	1.7950 e-1	9.6805 e-2

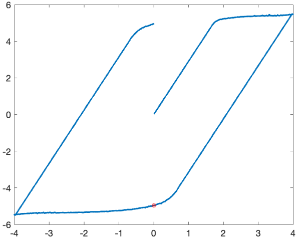


Plastic strain

Computational Experiments

$\gamma = 0$:

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
d.o.f.	364	1456	5824	23296	93184
$\mathcal{F}(\underline{\underline{\sigma}}_h, \underline{u}_h)^{1/2}$	2.3686 e-1	2.7633 e-1	1.9434 e-1	1.2336 e-1	7.1906 e-2

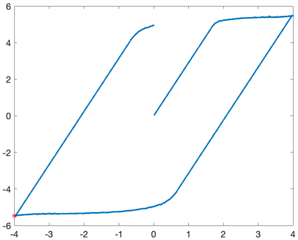


Plastic strain

Computational Experiments

$\gamma = -4.0$:

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
d.o.f.	364	1456	5824	23296	93184
$\mathcal{F}(\underline{\underline{\sigma}}_h, \underline{u}_h)^{1/2}$	1.0801 e-0	6.7038 e-1	3.7957 e-1	2.0477 e-1	1.0680 e-1

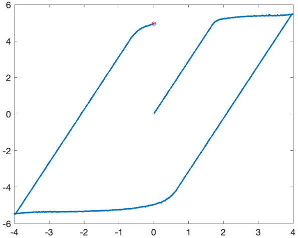


Plastic strain

Computational Experiments

$\gamma = 0$:

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
d.o.f.	364	1456	5824	23296	93184
$\mathcal{F}(\underline{\underline{\sigma}}_h, \underline{u}_h)^{1/2}$	2.3597 e-1	2.7057 e-1	1.8862 e-1	1.1777 e-1	6.6823 e-2



Plastic strain

Extension to Elastoplasticity with Hardening

W. Han, B. D. Reddy: Plasticity: Mathematical Theory and Numerical Analysis. 2013.

$$\underline{\underline{\varepsilon}}(\underline{u}) = \mathcal{A}\underline{\underline{\sigma}} + \underline{\underline{p}}$$

$$\underline{\underline{p}} = \underline{\underline{p}}^{\mathrm{old}} + \gamma(\underline{u},\alpha)\,\mathrm{dev}\,\underline{\underline{\sigma}} \text{ s.t. } |\mathrm{dev}\,\underline{\underline{\sigma}}| \le \sqrt{\frac{2}{3}}K(\alpha) \text{ in } \Omega$$

Implicit formula:
$$\gamma(\underline{u},\alpha) = \max\left\{0, \frac{|\mathrm{dev}\,\underline{\underline{\varepsilon}}(\underline{u}) - \underline{\underline{p}}^{\mathrm{old}}|}{\sqrt{\frac{2}{3}}K(\alpha)} - \frac{1}{2\mu}\right\}$$

$$\alpha = \alpha^{\mathrm{old}} + \gamma(\underline{u},\alpha)$$

$$\|\underline{\underline{\varepsilon}}(\underline{v}) - (\mathcal{A}\underline{\underline{\tau}} + \underline{\underline{p}}^{\mathrm{old}} + \gamma(\underline{v},\alpha)\,\mathrm{dev}\,\underline{\underline{\tau}})\|^2_{L^2(\Omega)} \to \text{min!}$$

among all $(\underline{\underline{\tau}}, \underline{v}) \in H(\mathrm{div},\Omega;\mathbb{R}^{3\times 3}) \times H^1_{\Gamma_D}(\Omega;\mathbb{R}^3)$

with $\underline{\underline{\tau}} \cdot \underline{n} = \underline{g}$ on Γ_N

subject to $(\mathrm{div}\,\underline{\underline{\tau}}, \underline{z})_{L^2(\Omega)} = 0$ for all $\underline{z} \in L^2(\Omega;\mathbb{R}^3)$

Error equivalence guaranteed if (uniformly) $K'(\alpha) > 0$:

St.: SIAM J. Numer. Anal. **45** (2007), 371–388