Faculty of Mathematics
Work group: Numerische Mathematik
Prof. Dr. Gerhard Starke, Dipl.-Math. Benjamin Müller

UNIVERSITÄT DUISBURG ESSEN

Open-Minded

Mixed Least Squares Finite Element Methods for Hyperelastic Materials

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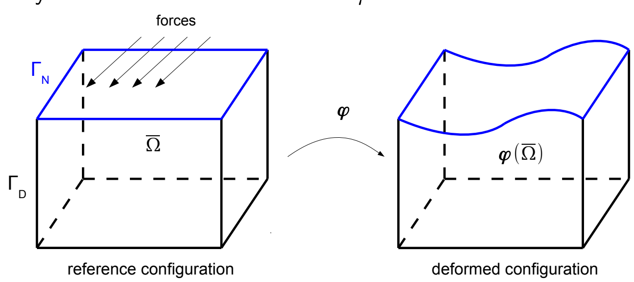
Motivation:

The finite element method is an important tool for the simulation of elasticity problems in solid mechanics. It is well known that the linear elastic theory does not cover arising real life problems. Physically more realistic models lead to nonlinear partial differential equations. The least squares finite element method is until now only rarely investigated for nonlinear deformation processes despite its advantages, e.g. providing a candidate for a-posteriori error estimation and not being restricted to certain finite element spaces (inf-sup condition).

Problem description:

Possible forces are volume forces, acting on the whole body Ω, and surface forces, acting on the Neumann boundary Γ_N. Applied forces on an unloaded body lead to a deformation φ and related stresses.



The balance of momentum and the (nonlinear) constitutive equation are basic ingredients for modelling deformation processes mathematically.

We use a homogeneous isotropic hyperelastic material of Neo-Hookean type with stored energy function

$$\psi(\mathbf{C}) = \frac{\mu}{2}\,\mathrm{tr}(\mathbf{C}) + \frac{\lambda}{4}\det\mathbf{C} - \left(\frac{\lambda}{2}+\mu\right)\ln\sqrt{\det\mathbf{C}}$$

and Lamé constants λ, μ for minimizing the nonlinear least squares functional

$$\mathcal{F}(\mathbf{P},\mathbf{u}) = \omega_1^2\|\mathrm{div}\,\mathbf{P}+\mathbf{f}\|^2 + \omega_2^2\|\mathcal{A}(\mathbf{P}\mathbf{F}(\mathbf{u})^T) - \mathbf{B}(\mathbf{u})\|^2$$

in appropriate function spaces. Here $\mathbf{P} = \partial_{\mathbf{F}}\psi(\mathbf{C})$ denotes the first Piola-Kirchhoff stress tensor, $\mathbf{u}$ the pointwise displacement, ω_1 and ω_2 suitable weighting parameters and $\mathbf{f}$ the acting volume force.

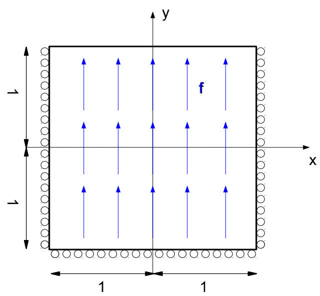
Algorithmic implementation:

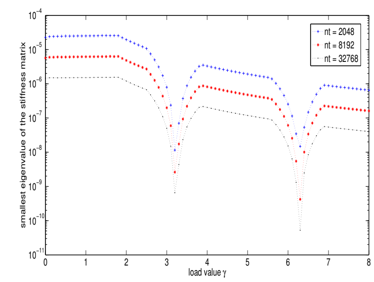
- Gauss-Newton method for minimization of $\mathcal{F}(\mathbf{P},\mathbf{u})$
- Raviart-Thomas elements $\mathcal{RT}_1$ for each row of $\mathbf{P}$ and continuous piecewise quadratic elements for each component of $\mathbf{u}$
- backtracking line search as damping strategy

Numerical results for the incompressible case:

1. Detection of critical load values:

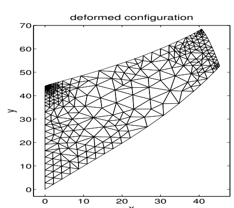
Given: $\Omega = (-1,1)\times(-1,1)$, $\mathbf{f} = (0,\gamma)^T$ with $\gamma \in \mathbb{R}$, $\mathbf{u}\cdot\mathbf{n} = 0$ and $(\mathbf{P}\cdot\mathbf{n})\cdot\mathbf{t} = 0$ (left, bottom, right boundary), $\mathbf{P}\cdot\mathbf{n} = \mathbf{0}$ (top boundary)





2. Adaptivity for a plane strain model (Cook's membrane in 2d):

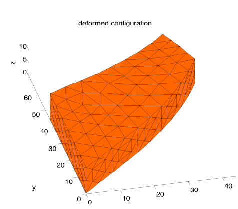
Given: $\mathbf{f} = \mathbf{0}$, $\mathbf{u} = \mathbf{0}$ (left boundary), $\mathbf{P}\cdot\mathbf{n} = \mathbf{0}$ (top/bottom boundary) and $\mathbf{P}\cdot\mathbf{n} = (0,\gamma)^T$ (right boundary) with load parameter $\gamma = 0.1$

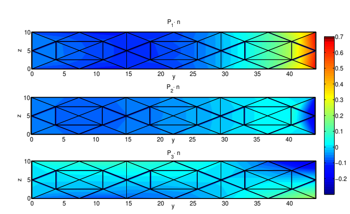


nt	$\mathcal{F}(\mathbf{P}_h,\mathbf{u}_h)$	(order)	$\mathbf{u}_2(48,60)$
131	$1.27\cdot10^{-1}$		8.2461
192	$5.89\cdot10^{-2}$	(2.02)	8.4005
271	$2.85\cdot10^{-2}$	(2.11)	8.4638
400	$1.38\cdot10^{-2}$	(1.87)	8.4932
593	$6.43\cdot10^{-3}$	(1.94)	8.5077
868	$2.90\cdot10^{-3}$	(2.09)	8.5139
1267	$1.24\cdot10^{-3}$	(2.25)	8.5168
1893	$5.09\cdot10^{-4}$	(2.22)	8.5178

3. Cook's membrane in 3d:

Given: $\mathbf{f} = \mathbf{0}$, $\mathbf{u} = \mathbf{0}$ (left boundary), $\mathbf{P}\cdot\mathbf{n} = (0,\gamma,0)^T$ (right boundary) with load parameter $\gamma = 0.1$, $\mathbf{P}\cdot\mathbf{n} = \mathbf{0}$ (other four boundaries)





Conclusion:

Our least squares finite element approach has an essential advantage compared to other methods, namely the full stress tensor is approximated next to the displacement. Our approach is suitable for the incompressible limit and provides good results for 2d and 3d simulations. The least squares functional as error estimator tends to identify the regions of interest in our considered numerical experiments.

Prof. Dr. Gerhard Starke
Dipl.-Math. Benjamin Müller
Faculty of Mathematics
University of Duisburg-Essen

Prof. Dr.-Ing. Jörg Schröder
Dr.-Ing. Alexander Schwarz, M. Sc. Karl Steeger
Institute of Mechanics
University of Duisburg-Essen