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Model reduction for geometrically nonlinear elasticity

Motivation:

Physical experiments show that the frequently used linear model for elastic deformation problems is only valid up to a certain strain. The usage of nonlinear models coincide also for larger strains better with these experimental results and should hence be used in numerical simulations. But also here a variety of nonlinear models are possible, e.g. a Neo-Hooke model, a more general Mooney-Rivlin model or even a more complex one.

Moreover, solving a nonlinear model needs in general more effort than solving a linear model, e.g. due to linearization techniques. The idea of model adaptivity is to use a nonlinear model only in regions of the domain where it is necessary and use the linear model on the other part.

Reduced mixed least squares method:

For the implementation of a model adaptivity technique we use a mixed least squares finite element method and choose a Neo-Hooke model with stored energy function

$$\psi_{NH}(\mathbf{C}) = \frac{\mu}{2}\mathrm{tr}(\mathbf{C}) + \frac{\lambda}{4}\det\mathbf{C} - \left(\frac{\mu}{2} + \frac{\lambda}{4}\right)\ln(\det\mathbf{C})$$

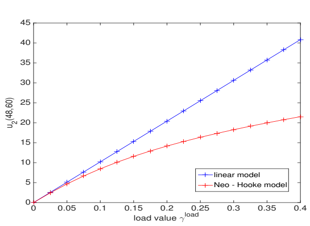
as nonlinear model.

We decompose a given body $\Omega \subset \mathbb{R}^n$ into a „nonlinear“ subdomain $\Omega_1 \subseteq \Omega$ and a „linear“ subdomain $\Omega_2 = \Omega \setminus \Omega_1$. For such an decomposition of Ω we seek a minimizer of a least squares functional of the form

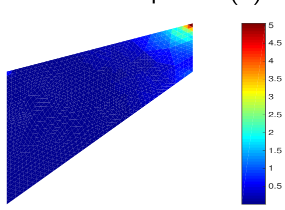
$$\mathcal{F}_{red}(\mathbf{P},\mathbf{u}) = \|\mathcal{R}_{NH}(\mathbf{P},\mathbf{u})\|^2_{L^2(\Omega_1)} + \|\mathcal{R}_{lin}(\mathbf{P},\mathbf{u})\|^2_{L^2(\Omega_2)}. \quad (1)$$

Essential steps in the algorithm:

- Start with a full linear model in step $i = 0$, i. e. $\Omega_1^{(0)} = \emptyset$, $\Omega_2^{(0)} = \Omega$ and solve (1) with solution $\left(\mathbf{P}^{(0)}_{red}, \mathbf{u}^{(0)}_{red}\right) = (\mathbf{P}_{lin}, \mathbf{u}_{lin})$.
- Adapt the nonlinear subdomain by choosing $\Omega_1^{(i+1)}$ with the help of the nonlinear least squares functional $\mathcal{F}_{NH}\left(\mathbf{P}^{(i)}_{red}, \mathbf{u}^{(i)}_{red}\right)$ and a suitable marking strategy. Set $\Omega_2^{(i+1)} = \Omega \setminus \Omega_1^{(i+1)}$ and determine the solution $\left(\mathbf{P}^{(i+1)}_{red}, \mathbf{u}^{(i+1)}_{red}\right)$ of (1) on the new decomposition, set $i = i+1$ and continue in the same manner.
- Nonlinear elements in previous step remain nonlinear.
- Use Gauss-Newton method for nonlinear part in (1).



Displacements $u_2(48,60)$

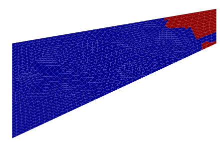


Distribution of $\mathcal{F}_{NH}(\mathbf{P}_{lin}, \mathbf{u}_{lin})$

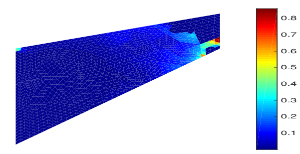
Numerical results for model adaptivity (Cook's membrane in 2d)

Given:
$\mathbf{u} = \mathbf{0}$ (left boundary), $\mathbf{P}\cdot\mathbf{n} = \mathbf{0}$ (top/bottom boundary) and $\mathbf{P}\cdot\mathbf{n} = (0, \gamma^{\text{load}})^T$ (right boundary) with load parameter $\gamma^{\text{load}} = 0.25$, Lamé constants: $\lambda = \infty$, $\mu = 1$

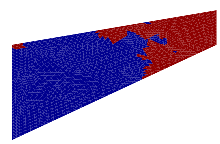
Below the decomposition of the domain Ω into the linear part $\Omega_2^{(i)}$ and the nonlinear part $\Omega_1^{(i)}$ (left) and the distribution $\mathcal{F}_{NH}\left(\mathbf{P}^{(i)}_{red}, \mathbf{u}^{(i)}_{red}\right)$ (right) is depicted:



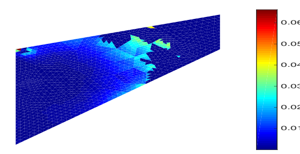
$|\Omega_1^{(1)}| = 107$, $|\Omega_2^{(1)}| = 1989$



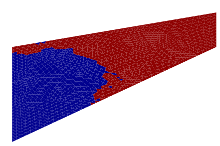
Distribution of $\mathcal{F}_{NH}\left(\mathbf{P}^{(1)}_{red}, \mathbf{u}^{(1)}_{red}\right)$



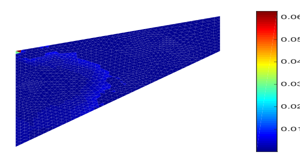
$|\Omega_1^{(3)}| = 545$, $|\Omega_2^{(3)}| = 1551$



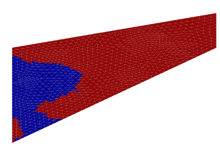
Distribution of $\mathcal{F}_{NH}\left(\mathbf{P}^{(3)}_{red}, \mathbf{u}^{(3)}_{red}\right)$



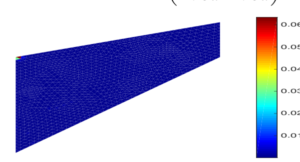
$|\Omega_1^{(6)}| = 1106$, $|\Omega_2^{(6)}| = 990$



Distribution of $\mathcal{F}_{NH}\left(\mathbf{P}^{(6)}_{red}, \mathbf{u}^{(6)}_{red}\right)$



$|\Omega_1^{(9)}| = 1551$, $|\Omega_2^{(9)}| = 545$



Distribution of $\mathcal{F}_{NH}\left(\mathbf{P}^{(9)}_{red}, \mathbf{u}^{(9)}_{red}\right)$

Benefits:

- Model adaptivity techniques, also in the context of LSFEM, may help for speeding up an algorithm.
- Quadrature error for the reduced model in comparison to the nonlinear model becomes smaller.
- Combining model adaptivity and usual mesh refinement is possible.

References:

[1] *B. Müller, G. Starke, A. Schwarz, J. Schröder*: A First-Order System Least Squares Method for Hyperelasticity. SIAM J. Sci. Comput. 36(5): B795-B816, 2014

[2] *B. Müller*: Mixed Least Squares Finite Element Methods Based on Inverse Stress-Strain Relations in Hyperelasticity. Ph.D. Thesis, Universität Duisburg-Essen, 2014.

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