Stress-Based Adaptivity for Quasi-Variational Inequalities Associated With Frictional Contact

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Overview

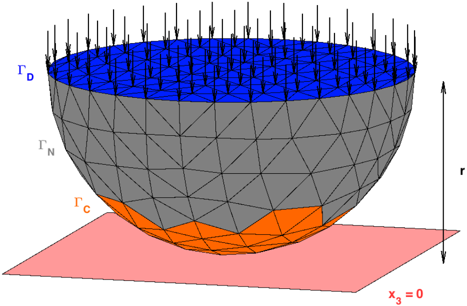
Dual stress-based formulation of linear elastic Contact with Friction

Displacement Reconstruction and A Posteriori Error Estimation

Numerical Experiments

Physical Setting of the Signorini Contact Problem

Elastic deformation of a solid constrained by a rigid foundation



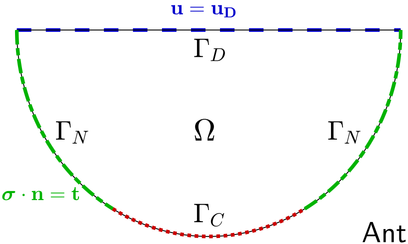
Small strain assumption → linear elasticity, lin. contact constraint

Problem is non-linear due to inequality constraints
(active contact zone not known a priori)

Antonio Signorini: Sopra alcune questioni di statica dei sistemi continui.
Annali della Scuola Normale Superiore di Pisa (1933)

Linear elasticity

Stress-based formulation



Stress field $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{d \times d}$

Displacement field $\mathbf{u} : \Omega \to \mathbb{R}^d$

lin. Strain: $\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}\left(\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^T\right)$

Antisymmetric part: $\mathbf{as}\,\boldsymbol{\sigma} := \frac{1}{2}\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^T\right)$

Conservation equations:

$$\operatorname{div}\boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \text{ in } \Omega$$

$$\mathbf{as}\,\boldsymbol{\sigma} = \mathbf{0} \text{ in } \Omega$$

Constitutive equation:

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \mathcal{A}\boldsymbol{\sigma} := \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{d\lambda + 2\mu}(\operatorname{tr}\boldsymbol{\sigma})\mathbf{I}\right)$$

μ, λ : Lamé Parameters

Contact conditions

Signorini contact conditions with gap-function $g \geq 0$:

$$\left\{ \begin{array}{ll} g - \mathbf{n} \cdot \mathbf{u} & \geq 0 \\ \mathbf{n} \cdot (\sigma \cdot \mathbf{n}) & \leq 0 \\ (\mathbf{n} \cdot \mathbf{u} - g)(\mathbf{n} \cdot (\sigma \cdot \mathbf{n})) & = 0 \end{array} \right\} \text{ on } \Gamma_C$$

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Coulomb friction law on Γ_C with friction coefficient $\mu_F \geq 0$:

$$\left\{ \begin{array}{l} |\mathbf{n} \times (\boldsymbol{\sigma} \cdot \mathbf{n})| \leq \mu_F |\mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n})| \\ |\mathbf{n} \times (\boldsymbol{\sigma} \cdot \mathbf{n})| < \mu_F |\mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n})| \Rightarrow \mathbf{n} \times \mathbf{u} = \mathbf{0} \quad \text{(sticking)} \\ |\mathbf{n} \times (\boldsymbol{\sigma} \cdot \mathbf{n})| = \mu_F |\mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n})| \Rightarrow \mathbf{n} \times \mathbf{u} = -c\, \mathbf{n} \times (\boldsymbol{\sigma} \cdot \mathbf{n}) \text{ with } c \geq 0 \quad \text{(sliding)} \end{array} \right\}$$

Contact conditions

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can be rewritten in this form:

$$\left.\begin{cases} |\mathbf{n} \times (\boldsymbol{\sigma} \cdot \mathbf{n})| & \leq \mu_F |\mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n})| \\ (\mathbf{n} \times (\boldsymbol{\sigma} \cdot \mathbf{n})) \cdot (\mathbf{n} \times \mathbf{u}) & = -\mu_F |\mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n})| |\mathbf{n} \times \mathbf{u}| \end{cases}\right\} \text{ on } \Gamma_C$$

Dual stress-based formulation of Contact with Friction

Compute $\boldsymbol{\sigma}$ such that it minimizes the energy functional

$$\mathcal{J}(\boldsymbol{\tau}) := \frac{1}{2}\int_\Omega (\mathcal{A}\boldsymbol{\tau}) : \boldsymbol{\tau}\, dx - \int_{\Gamma_D} \mathbf{u}_D \cdot (\boldsymbol{\tau}\cdot\mathbf{n})\, ds - \int_{\Gamma_C} g\,(\mathbf{n}\cdot(\boldsymbol{\tau}\cdot\mathbf{n}))\, ds$$

among all $\boldsymbol{\tau} \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}) := \{\boldsymbol{\tau}\in\boldsymbol{\Sigma} :\ \operatorname{div}\boldsymbol{\tau} + \mathbf{f} = \mathbf{0},\ \mathbf{as}\,\boldsymbol{\tau} = \mathbf{0},$

$$\mathbf{n}\cdot(\boldsymbol{\tau}\cdot\mathbf{n}) \le 0,\ |\mathbf{n}\times(\boldsymbol{\tau}\cdot\mathbf{n})| \le \mu_F|\mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n})| \text{ on } \Gamma_C\}$$

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$$\Longleftrightarrow$$

Determine $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}(\boldsymbol{\sigma})$ such that:

$$(\mathcal{A}\boldsymbol{\sigma}, \boldsymbol{\tau} - \boldsymbol{\sigma})_\Omega \ge \langle \mathbf{u}_D, (\boldsymbol{\tau} - \boldsymbol{\sigma}) \cdot \mathbf{n} \rangle_{\Gamma_D} + \langle g, \mathbf{n} \cdot ((\boldsymbol{\tau} - \boldsymbol{\sigma}) \cdot \mathbf{n}) \rangle_{\Gamma_C}$$

holds for all $\boldsymbol{\tau} \in \boldsymbol{\Sigma}(\boldsymbol{\sigma})$.

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holds for all $\boldsymbol{\tau} \in \boldsymbol{\Sigma}(\boldsymbol{\sigma})$.

- **Continuous problem**: Existence for small μ_F
- **Discrete problem**: Existence for all μ_F, uniqueness for small μ_F

N. Kikuchi, J. T. Oden: SIAM, 1988

I. Hlaváček, J. Haslinger, J. Nečas, J. Lovíšek: Springer, 1988

Displacement Reconstruction in H^1

$$-(\mathbf{u}_h, \operatorname{div} \boldsymbol{\tau}_h) = (\mathcal{A}\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (\boldsymbol{\theta}_h, \mathbf{as}\, \boldsymbol{\tau}_h) = (\mathcal{A}\boldsymbol{\sigma}_h + \boldsymbol{\theta}_h, \boldsymbol{\tau}_h) =: (\mathbf{z}_h, \boldsymbol{\tau}_h)$$

$\mathbf{z}_h$: discontinuous piecewise polynomial approximation of $\nabla \mathbf{u}_h$

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Gradient Reconstruction Algorithm:

Step 1. For each $T \in \mathcal{T}_h$, determine $\mathbf{u}_h^\circ|_T \in \mathcal{P}_k(T)^d$

$$(\nabla \mathbf{u}_h^\circ, \nabla \mathbf{v}_h)_{L^2(T)} = (\mathbf{z}_h, \nabla \mathbf{v}_h)_{L^2(T)} \text{ for all } \mathbf{v}_h \in \mathcal{P}_k(T)^d$$

$$(\mathbf{u}_h^\circ, \mathbf{e}_i)_{L^2(T)} = (\mathbf{u}_h, \mathbf{e}_i)_{L^2(T)} \text{ for } 1 \leq i \leq d$$

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Step 2. Construct $\mathbf{u}_h^R \in \mathcal{P}_k(\mathcal{T}_h)^d \subset H^1(\Omega)^d$ by averaging:

$$\mathbf{u}_h^R(\mathbf{x}) = \frac{1}{\#\{T : \mathbf{x} \in T\}} \left(\sum_{T : \mathbf{x} \in T} \mathbf{u}_h^\circ|_T (\mathbf{x}) \right)$$

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Step 3. Enforce boundary conditions on $\mathbf{u}_h^R$.

Kim, Appl. Math. Comp. (2012)

Vohralík, Math. Comp. (2010)

Stenberg, RAIRO Math. Model. Numer. Anal. (1991)

Reconstruction-Based A Posteriori Error Estimator

- **Linear elasticity:**

$$\eta(\boldsymbol{\sigma}_h, \mathbf{u}_h^R) := \|\mathcal{A}\boldsymbol{\sigma}_h - \varepsilon(\mathbf{u}_h^R)\|_{L^2(\Omega)} \begin{array}{l} \gtrsim \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\mathrm{div},\Omega)} \text{ (reliability)} \\ \lesssim \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\mathrm{div},\Omega)} \text{ (efficiency)} \end{array}$$

with constants which are independent of $\lambda \to \infty$

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with constants which are independent of $\lambda \to \infty$

- **Frictional contact:**

Add complementarity terms to the error estimator:

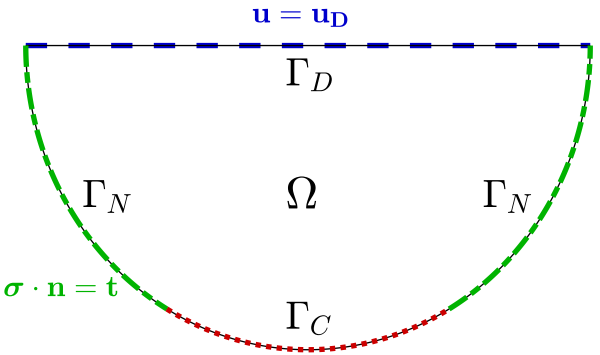
$$\eta^{CF}(\boldsymbol{\sigma}_h, \mathbf{u}_h^R)^2 := \|\mathcal{A}\boldsymbol{\sigma}_h - \varepsilon(\mathbf{u}_h^R)\|_{L^2(\Omega)}^2 + \langle \mathbf{n} \cdot \mathbf{u}_h^R - g, \mathbf{n} \cdot (\boldsymbol{\sigma}_h \cdot \mathbf{n})\rangle_{\Gamma_C}$$
$$+ \langle \mathbf{n} \times \mathbf{u}_h^R, \mathbf{n} \times (\boldsymbol{\sigma}_h \cdot \mathbf{n})\rangle_{\Gamma_C} + \langle |\mathbf{n} \times \mathbf{u}_h^R|, \mu_F |\mathbf{n} \cdot (\boldsymbol{\sigma}_h \cdot \mathbf{n})|\rangle_{\Gamma_C}$$

The idea of using such a functional for error estimation of variational inequalities goes (at least) back to

I. Hlaváček, J. Haslinger, J. Nečas, J. Lovíšek: Springer, 1988

Numerical Results

Hertzian Contact - Setting

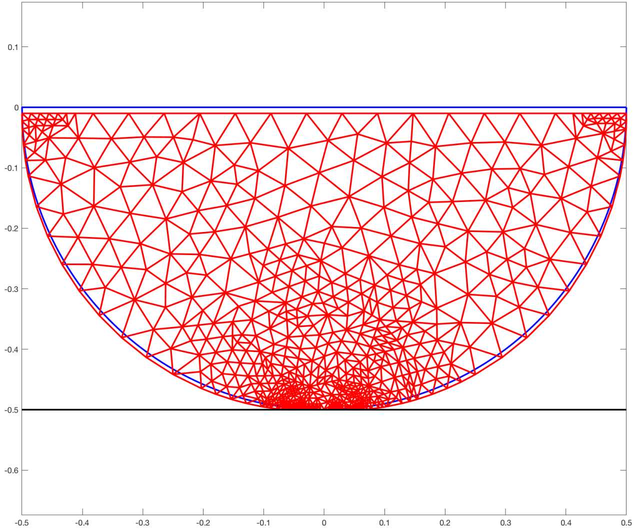


$\mathbf{u}_D = (0, -0.01)^T$, $\mathbf{t} = \mathbf{0}$, $\mathbf{f} = \mathbf{0}$, $E = 20600$, $\mu_F = 0.4$,

$\boldsymbol{\Sigma}_h = \mathcal{RT}_1(\mathcal{T}_h)^2$, $\mathbf{U}_h = \mathcal{DP}_1(\mathcal{T}_h)^2$, $\boldsymbol{\Theta}_h = \mathcal{P}_1(\mathcal{T}_h)^{2\times 2, as}$

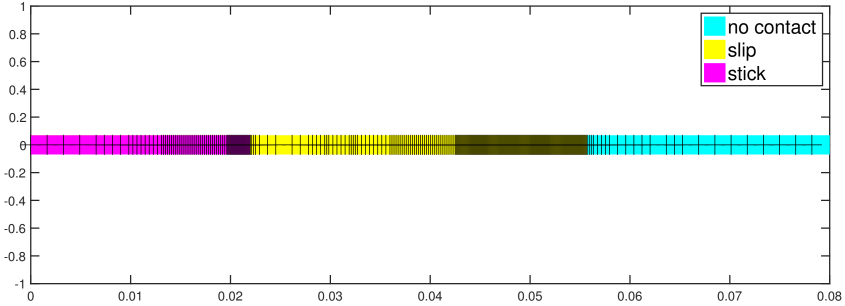
Deformed Mesh

Compressible case - 5 refinements



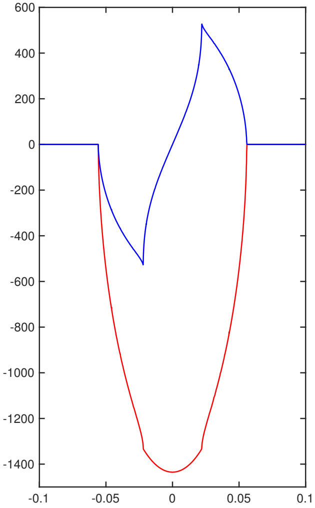
Refinement of Contact zone

Gridpoints on half of Γ_C after 10 refinements

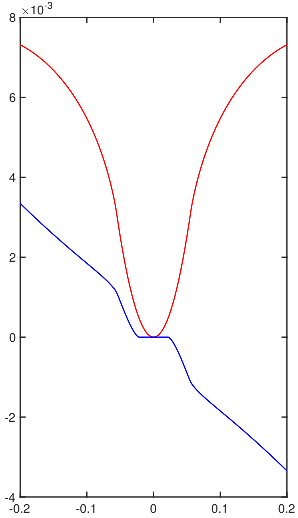


Contact Stress and Displacement

Compressible case: $\nu = 0.28$



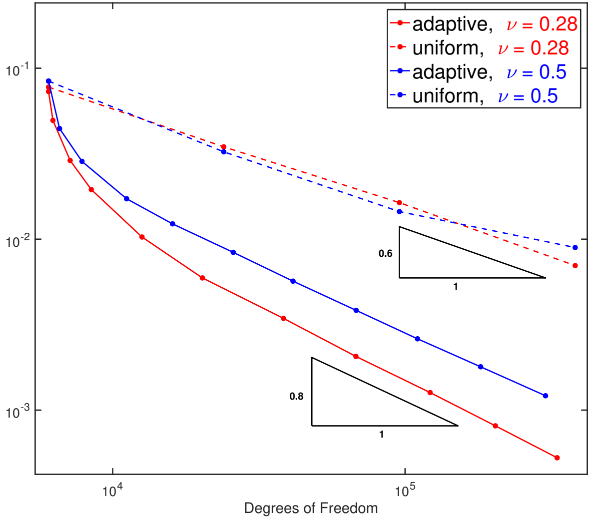
normal and tangential stress



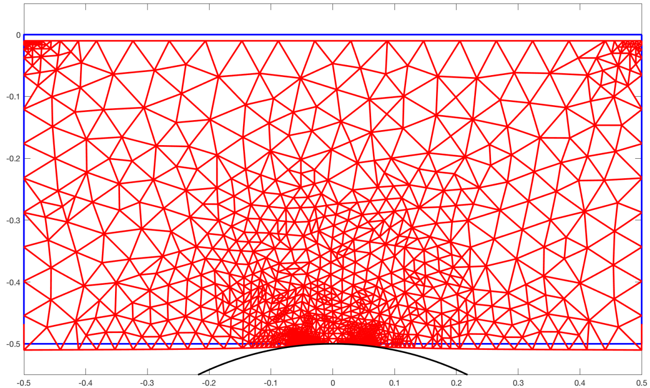
reconstructed displacement

Convergence comparison

Values of η^{CF}



Straight boundaries



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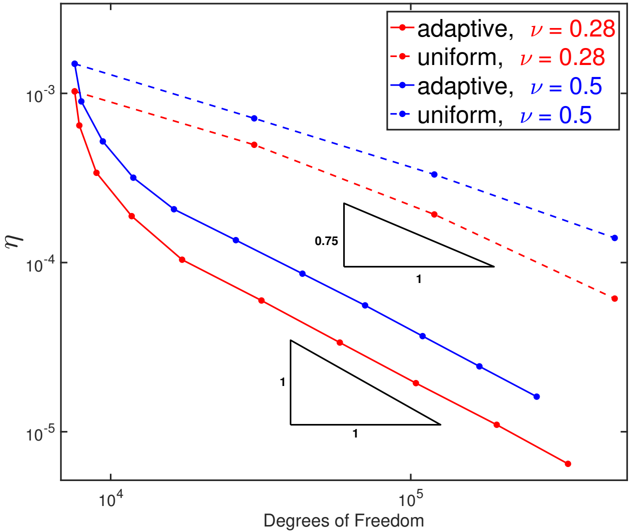
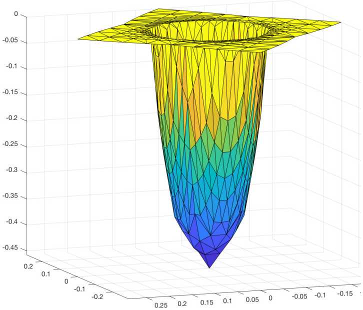
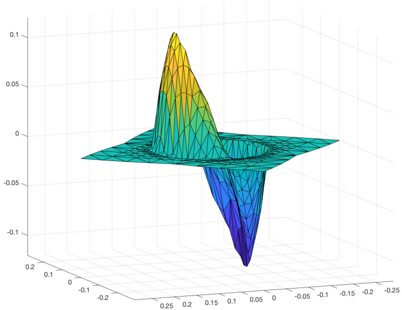


3D example

$\mathbf{u}_D = (0, -0.025)^T$, $E = 2.5$, $\nu = 0.25$, $\mu_F = 0.4$



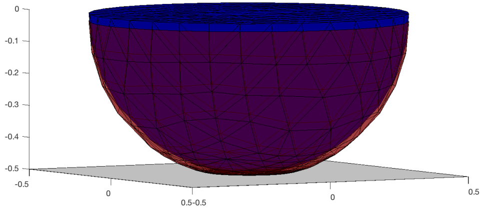
contact pressure

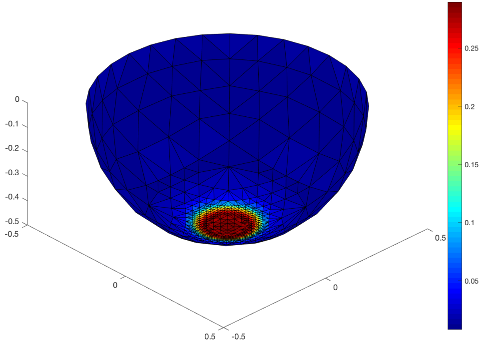


shearstress

3D example

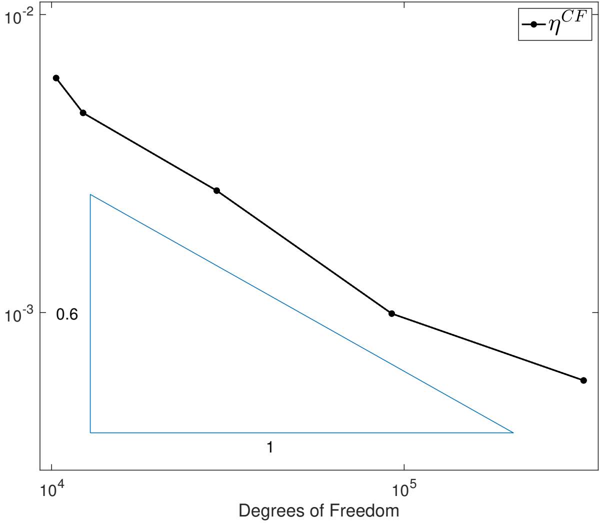
Deformation and Stress





3D example

Convergence



Reconstruction-Based A Posteriori Error Estimator

Problem in the Proof of Reliability

$$\eta^{CF}(\boldsymbol{\sigma}_h, \mathbf{u}_h^R)^2 := \eta(\boldsymbol{\sigma}_h, \mathbf{u}_h^R) + \langle \mathbf{n} \cdot \mathbf{u}_h^R - g, \mathbf{n} \cdot (\boldsymbol{\sigma}_h \cdot \mathbf{n}) \rangle_{\Gamma_C}$$
$$+ \langle \mathbf{n} \times \mathbf{u}_h^R, \mathbf{n} \times (\boldsymbol{\sigma}_h \cdot \mathbf{n}) \rangle_{\Gamma_C} + \langle |\mathbf{n} \times \mathbf{u}_h^R|, \mu_F |\mathbf{n} \cdot (\boldsymbol{\sigma}_h \cdot \mathbf{n})| \rangle_{\Gamma_C}$$

needs to be bounded from below by the error $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$!

Reconstruction-Based A Posteriori Error Estimator

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needs to be bounded from below by the error $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$!

For the 1st complementarity term:

$$\langle \mathbf{n} \cdot \mathbf{u}_h^R - g, \mathbf{n} \cdot (\boldsymbol{\sigma}_h \cdot \mathbf{n}) \rangle_{\Gamma_C} \geq \langle \mathbf{n} \cdot \mathbf{u}_h^R - \mathbf{n} \cdot \mathbf{u}, \mathbf{n} \cdot (\boldsymbol{\sigma}_h \cdot \mathbf{n}) - \mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}) \rangle_{\Gamma_C}$$

since either $\mathbf{n} \cdot \mathbf{u} = g$ (and $\mathbf{n} \cdot \mathbf{u}_h - g \leq 0$, $\mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}) \leq 0$)
or $\mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}) = 0$ (and $\mathbf{n} \cdot (\boldsymbol{\sigma}_h \cdot \mathbf{n}) \leq 0$, $\mathbf{n} \cdot \mathbf{u} - g \leq 0$)

and then trace theorems can be used (cf. Krause/Müller/S., 2017)

Reconstruction-Based A Posteriori Error Estimator

Problem in the Proof of Reliability

For the 2nd complementarity term, similarly:

$$\begin{aligned}
&\langle \mathbf{n} \times \mathbf{u}_h^R, \mathbf{n} \times (\boldsymbol{\sigma}_h \cdot \mathbf{n})\rangle_{\Gamma_C} + \langle |\mathbf{n} \times \mathbf{u}_h^R|, \mu_F |\mathbf{n} \cdot (\boldsymbol{\sigma}_h \cdot \mathbf{n})|\rangle_{\Gamma_C} \\
=&\langle \mathbf{n} \times \mathbf{u}_h^R, \mathbf{n} \times (\boldsymbol{\sigma}_h \cdot \mathbf{n})\rangle_{\Gamma_C} - \langle |\mathbf{n} \times \mathbf{u}_h^R|, \mu_F (\mathbf{n} \cdot (\boldsymbol{\sigma}_h \cdot \mathbf{n}))\rangle_{\Gamma_C} \\
\geq&\langle |\mathbf{n} \times \mathbf{u}_h^R| - |\mathbf{n} \times \mathbf{u}|, \mu_F (\mathbf{n} \cdot ((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \mathbf{n}))\rangle_{\Gamma_C}
\end{aligned}$$

Under which conditions does the inequality

$$\| |\mathbf{n} \times \mathbf{u}| - |\mathbf{n} \times \mathbf{u}_h^R| \|_{1/2,\Gamma_C} \lesssim \| \mathbf{n} \times \mathbf{u} - \mathbf{n} \times \mathbf{u}_h^R \|_{1/2,\Gamma_C}$$

hold?

Reconstruction-Based A Posteriori Error Estimator

Problem in the Proof of Reliability

For the 2nd complementarity term, similarly:

$$
\begin{aligned}
&\langle \mathbf{n} \times \mathbf{u}_h^R, \mathbf{n} \times (\boldsymbol{\sigma}_h \cdot \mathbf{n}) \rangle_{\Gamma_C} + \langle |\mathbf{n} \times \mathbf{u}_h^R|, \mu_F |\mathbf{n} \cdot (\boldsymbol{\sigma}_h \cdot \mathbf{n})| \rangle_{\Gamma_C} \\
=&\langle \mathbf{n} \times \mathbf{u}_h^R, \mathbf{n} \times (\boldsymbol{\sigma}_h \cdot \mathbf{n}) \rangle_{\Gamma_C} - \langle |\mathbf{n} \times \mathbf{u}_h^R|, \mu_F (\mathbf{n} \cdot (\boldsymbol{\sigma}_h \cdot \mathbf{n})) \rangle_{\Gamma_C} \\
\geq&\langle |\mathbf{n} \times \mathbf{u}_h^R| - |\mathbf{n} \times \mathbf{u}|, \mu_F (\mathbf{n} \cdot ((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \mathbf{n})) \rangle_{\Gamma_C}
\end{aligned}
$$

Under which conditions does the inequality

$$
\| |\mathbf{n} \times \mathbf{u}| - |\mathbf{n} \times \mathbf{u}_h^R| \|_{1/2,\Gamma_C} \lesssim \| \mathbf{n} \times \mathbf{u} - \mathbf{n} \times \mathbf{u}_h^R \|_{1/2,\Gamma_C}
$$

hold?

Main assumption (in 2D):
The tangential displacement $\mathbf{n} \times \mathbf{u}$ is not allowed to be of opposite sign in arbitrarily small neighborhoods of any point on Γ_C.

Reconstruction-Based A Posteriori Error Estimator

Problem in the Proof of Reliability

For the 2nd complementarity term, similarly:

$$\begin{aligned}
&\langle \mathbf{n} \times \mathbf{u}_h^R, \mathbf{n} \times (\boldsymbol{\sigma}_h \cdot \mathbf{n}) \rangle_{\Gamma_C} + \langle |\mathbf{n} \times \mathbf{u}_h^R|, \mu_F |\mathbf{n} \cdot (\boldsymbol{\sigma}_h \cdot \mathbf{n})| \rangle_{\Gamma_C} \\
=&\langle \mathbf{n} \times \mathbf{u}_h^R, \mathbf{n} \times (\boldsymbol{\sigma}_h \cdot \mathbf{n}) \rangle_{\Gamma_C} - \langle |\mathbf{n} \times \mathbf{u}_h^R|, \mu_F (\mathbf{n} \cdot (\boldsymbol{\sigma}_h \cdot \mathbf{n})) \rangle_{\Gamma_C} \\
\geq&\langle |\mathbf{n} \times \mathbf{u}_h^R| - |\mathbf{n} \times \mathbf{u}|, \mu_F (\mathbf{n} \cdot ((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \mathbf{n})) \rangle_{\Gamma_C}
\end{aligned}$$

Under which conditions does the inequality

$$\| |\mathbf{n} \times \mathbf{u}| - |\mathbf{n} \times \mathbf{u}_h^R| \|_{1/2,\Gamma_C} \lesssim \| \mathbf{n} \times \mathbf{u} - \mathbf{n} \times \mathbf{u}_h^R \|_{1/2,\Gamma_C}$$

hold?

Main assumption (in 2D):

The tangential displacement $\mathbf{n} \times \mathbf{u}$ is not allowed to be of opposite sign in arbitrarily small neighborhoods of any point on Γ_C.

This is similar to the uniqueness criterion in

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