Consistency of a phase field regularisation for an inverse problem governed by a quasilinear Maxwell system

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Abstract

An inverse problem of reconstructing the magnetic reluctivity in a quasilinear magnetostatic Maxwell system is studied. To overcome the ill-posedness of the inverse problem, we propose and investigate two regularisations posed as constrained minimisation problems. The first uses the total variation (perimeter) regularisation, and the second makes use of the phase field regularisation. Existence of minimisers, sequential stability with respect to data perturbation, and consistency as the regularisation parameters tending to zero are rigorously analysed. Under some regularity assumption, we infer a relation between the regularisation parameters that allows one to recover a solution to the original inverse problem from the phase field regularised problem. The second focus of the paper is set on the first-order analysis of both regularisation approaches. For the phase field approach, two types of optimality systems are derived through a weak directional differentiability result and the domain variation technique of shape calculus. As a final result, we show the convergence of the optimality conditions obtained from shape calculus, leading to a necessary optimality system for the total variation inverse problem.

Key words. Inverse problem, quasilinear magnetostatic Maxwell equations, total variation, phase field regularisation, optimality conditions, Γ-convergence.

AMS subject classification. 35R35, 35Q60, 35J62, 35R30

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a connected Lipschitz boundary containing two different types of materials. The first is a ferromagnetic material (iron, nickel, cobalt, etc.) which responses strongly to an external magnetic field, and the second is a nonmagnetic material (copper, aluminium, silver etc.) which is significantly less responsive to the magnetic field. Denoting the region occupied by the nonmagnetic material as Ω_0 and the complement region occupied by the ferromagnetic material as Ω_1, so that $\overline{\Omega} = \overline{\Omega}_0 \cup \overline{\Omega}_1$ and $\Omega_0 \cap \Omega_1 = \varnothing$, our interest lies in solving the inverse problem of determining the location of Ω_1 based on measurements of certain quantities.

Electromagnetic phenomena can be well-described with the help of Maxwell's equations. In this paper, we focus on the magnetostatic setting, for which the governing partial differential equations (PDEs) derived from Maxwell's equations reduce to

$$\operatorname{curl} \boldsymbol{H} = \boldsymbol{J}, \quad \boldsymbol{H} = \nu \boldsymbol{B}, \quad \operatorname{div} \boldsymbol{B} = 0. \tag{1.1}$$

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Here, $\boldsymbol{H}$, $\boldsymbol{B}$ and $\boldsymbol{J}$ denote the magnetic field, the magnetic induction, and the applied current density, respectively. The magnetic reluctivity ν is the reciprocal of the magnetic permeability μ, so that the second equation in (1.1) can be equivalently expressed as the more standard form $\boldsymbol{B} = \mu\boldsymbol{H}$.

Denoting by ν_0 as the vacuum magnetic reluctivity, for nonmagnetic materials under consideration, we assume that the corresponding magnetic reluctivity is equal to ν_0. This is reasonable as many nonmagnetic materials exhibit a permeability close to the vacuum permeability $\mu_0 = 1/\nu_0$. On the other hand, for the ferromagnetic material, the reluctivity ν highly depends on the magnitude of the magnetic induction $\boldsymbol{B}$, i.e., $\nu = \nu_1(|\boldsymbol{B}|)$ for some strictly positive function $\nu_1 \in C^1(\mathbb{R})$. We refer to Kaltenbacher et al. [28] concerning the identification of the nonlinear reluctivity $\nu_1(|\boldsymbol{B}|)$ based on magnetic induction measurements and its numerical realisation. Introducing the overall reluctivity as the function $\nu : [0,1] \times \mathbb{R} \to [0,\infty)$ defined by

$$\nu(u,s) = \nu_0(1-u) + \nu_1(s)u,$$

and using the classical magnetic vector potential formulation allows us to reformulate (1.1) into:

$$\begin{cases} \operatorname{curl}\left(\nu(u(x), |\operatorname{curl}\boldsymbol{y}(x)|)\operatorname{curl}\boldsymbol{y}(x)\right) = \boldsymbol{J}(x) & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{y} = 0 & \text{in } \Omega, \\ \boldsymbol{y} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Notice that the forward system (1.2) makes use of the perfectly conducting electric (PEC) boundary condition with $\boldsymbol{n}$ denoting the outer unit normal on $\partial\Omega$. Moreover, the magnetic vector potential $\boldsymbol{y}$ satisfies

$$\operatorname{curl}\boldsymbol{y} = \boldsymbol{B} \text{ in } \Omega, \tag{1.3}$$

and if $u(x) \in \{0,1\}$ a.e. in Ω, then

$$\nu(u, |\boldsymbol{B}|) = \begin{cases} \nu_0 & \text{in } \{u = 0\}, \\ \nu_1(|\boldsymbol{B}|) & \text{in } \{u = 1\}. \end{cases} \tag{1.4}$$

In essence, complete knowledge of u in Ω allows us to determine the location of Ω_0 and Ω_1 as the sets $\{u = 0\}$ and $\{u = 1\}$, respectively. Therefore, the function u plays the role of the solution to the inverse problem we introduce below.

The equation (1.2) gives rise to a quasilinear saddle point structure. Under appropriate assumptions for the magnetic reluctivity, the well-posedness of (1.2) can be deduced. This leads to the solution operator $\boldsymbol{S} : u \mapsto \boldsymbol{y}$ assigning every function u taking values in $[0,1]$ to the magnetic vector potential $\boldsymbol{y}$ posed on Ω. For the moment, let $\boldsymbol{Z}$ denote the solution space in which $\boldsymbol{y}$ belongs to (see (2.3) for its precise definition). Furthermore, let $\boldsymbol{y}_m \in \mathcal{O}$ denote a measurement of the vector potential $\boldsymbol{y}$, where $\mathcal{O}$ is a Hilbert space with norm $\|\cdot\|_{\mathcal{O}}$, and $\boldsymbol{G} : \boldsymbol{Z} \to \mathcal{O}$ a bounded continuous operator. Then, our inverse problem read as:

$$\text{Find } u \in L^1(\Omega),\ u(x) \in \{0,1\} \text{ a.e. in } \Omega \text{ s.t. } \boldsymbol{G} \circ \boldsymbol{S}(u) = \boldsymbol{y}_m \text{ in } \mathcal{O}. \tag{1.5}$$

The prototype setting for this paper is $\mathcal{O} = \boldsymbol{L}^2(D)$ and $\boldsymbol{G}(\boldsymbol{y}) = \boldsymbol{y}|_D$ where $D \subset \Omega$ is an open subset, and $\boldsymbol{y}_m$ is a measurement of the magnetic vector potential in D induced by the applied current density $\boldsymbol{J}$. Both $\boldsymbol{y}_m \in \mathcal{O}$ and $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$ are given data for the inverse problem (1.5). Under stronger assumptions such as $\partial\Omega$ is of class $C^{1,1}$ or $\partial\Omega$ is convex, one may even consider $\boldsymbol{y}_m$ to be a boundary measurement with $\mathcal{O} = \boldsymbol{L}^2(\Sigma)$ and $\boldsymbol{G}(\boldsymbol{y}) = \boldsymbol{y}|_\Sigma$, where Σ can be a part of the boundary ($\Sigma \subset \partial\Omega$) or the entire boundary ($\Sigma = \partial\Omega$).

Due to the compactness of the embedding $\boldsymbol{Z} \subset\subset \boldsymbol{L}^2(\Omega)$ (see (2.5)), the inverse problem (1.5) is ill-posed. The standard approach to tackle the ill-posedness is to employ the Tikhonov regularisation [16, 39]. From a theoretical point of view, perhaps the most intuitive is to penalise the perimeter of $\Omega_1 = \{u = 1\}$, leading to the total variation inverse problem (TVIP):

$$\text{Find } u^\alpha = \underset{v \in BV(\Omega,\{0,1\})}{\operatorname{argmin}} \Big(\frac{1}{2} \|\boldsymbol{G} \circ \boldsymbol{S}(v) - \boldsymbol{y}_m\|_{\mathcal{O}}^2 + \alpha TV(v) \Big).$$

Here, $\alpha > 0$ is a regularisation parameter, $TV(\cdot)$ denotes the total variation functional and $BV(\Omega, \{0,1\})$ is the set of functions of bounded variations with values in $\{0,1\}$ (see Section 2.2 for more details). On the other hand, for numerical purposes, the non-convexity of $BV(\Omega, \{0,1\})$ introduces challenging issues for implementation. One remedy is to rephrase TVIP as a shape optimization problem and derive optimality conditions using shape calculus [24, 25, 27]. The second is to consider a further regularisation in which the total variation functional $TV(\cdot)$ is approximated by the Ginzburg–Landau functional $E_\varepsilon(\cdot)$, where $\varepsilon > 0$ is a small parameter. This has been employed to great effect in various shape and topology optimisation problems, see [4–7, 21, 22, 25] and the references cited therein. In our present setting, it leads to the phase field inverse problem (PFIP):

$$\text{Find } u_\varepsilon^\alpha = \underset{v \in \mathcal{K}}{\operatorname{argmin}} \Big(\frac{1}{2} \|\boldsymbol{G} \circ \boldsymbol{S}(v) - \boldsymbol{y}_m\|_{\mathcal{O}}^2 + \alpha E_\varepsilon(v) \Big), \quad E_\varepsilon(v) := \int_\Omega \frac{\varepsilon}{2} |\nabla v|^2 + \frac{1}{\varepsilon} \Psi(v),$$

with a double well potential $\Psi(s)$ and a nonempty closed convex set $\mathcal{K} \subset H^1(\Omega)$ (see (5.2)–(5.3) for their definitions).

The goal of this paper is to examine the mathematical analysis of both regularisation approaches TVIP and PFIP. More precisely, we show the existence of minimisers and investigate desirable properties such as the sequential stability with respect to perturbations in the data $\boldsymbol{y}_m$, and asymptotic limits $\alpha \to 0$ and $\varepsilon \to 0$. To the best of the authors' knowledge, this paper is the first contribution towards TVIP and PFIP for the quasilinear Maxwell system (1.2) with two main novelties: The first is the convergence analysis for PFIP involving their respective minimisers and optimality conditions (Theorems 5.2 and 6.11). Here, two types of optimality systems for PFIP (Theorems 6.6 and 6.10) are established based on a weak directional differentiability result (Lemma 6.3) and the domain variation technique of shape calculus (Lemma 6.9). Based on the latter technique, we prove the convergence of the optimality conditions for PFIP as $\varepsilon \to 0$, leading to an optimality system for TVIP. The second main novelty, which seems not to have received much attention in the literature, is the convergence of solutions for PFIP to a solution of the original inverse problem (Theorem 5.3), and inferring (under ideal conditions) a relation of the form $\alpha = O(\varepsilon^d)$ with $d < 2$ (see (5.11)) between parameters α and ε that could serve as a useful guideline for numerical implementations. Let us mention that there are several possible numerical strategies to solve the phase field optimality conditions to numerically realise our solution, namely a parabolic variational inequality approach [4–6, 14] or the VMPT method [7, 8]. The analysis of these strategies, together with a finite element approximation of the forward model and adjoint system in a fashion similar to [43], would be the next step of our investigation. However, due to the length of the paper, we choose to defer the numerical investigations of the inverse problem (1.5) and its regularisations to future work.

Identification problems of material parameters in linear Maxwell's equations have been extensively investigated in many contributions, including [19, 29, 33, 35, 41]. On the other hand, the mathematical analysis for inverse problems governed by nonlinear Maxwell's equations is still in the early stages of development. In the context of the optimal control, stationary and evolutionary nonlinear Maxwell's equations were investigated in [32,42–44]. More recently, [12] analysed ill-posed backward nonlinear Maxwell's equations and derived a variational source condition for the convergence rate of the corresponding Tikhonov regularisation. We believe that our present results may lead to further progresses in the mathematical analysis of nonlinear electromagnetic inverse problems.

The paper is structured as follows: In Section 2 we introduce several useful auxiliary results and preliminaries. In Section 3 we prove the well-posedness of (1.2) and continuity properties of the solution operator $\boldsymbol{S}$. The existence of minimisers, sequential stability with respect to data perturbation, and asymptotic behaviour as $\alpha \to 0$ and $\varepsilon \to 0$ are discussed in Sections 4 and 5, respectively. In Section 6 the first-order necessary optimality conditions are derived, and we discuss the convergence of the phase field optimality conditions to the total variation optimality conditions. In the appendix we state a useful result involving the Γ-convergence of functionals.

2 Preliminaries

2.1 Function spaces for electromagnetic problems

For any open set $\Omega \subset \mathbb{R}^3$, the Hilbert spaces

$$\boldsymbol{H}(\mathrm{curl}) := \Big\{ \boldsymbol{f} \in \boldsymbol{L}^2(\Omega) \,:\, \mathrm{curl}\, \boldsymbol{f} \in \boldsymbol{L}^2(\Omega) \Big\}, \quad \boldsymbol{H}(\mathrm{div}) := \Big\{ \boldsymbol{f} \in \boldsymbol{L}^2(\Omega) \,:\, \mathrm{div}\, \boldsymbol{f} \in L^2(\Omega) \Big\}$$

are equipped with the norms

$$\|\boldsymbol{f}\|_{\boldsymbol{H}(\mathrm{curl})} := \big(\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{curl}\, \boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} \big)^{\frac{1}{2}}, \; \|\boldsymbol{f}\|_{\boldsymbol{H}(\mathrm{div})} := \big(\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\, \boldsymbol{f}\|^2_{L^2(\Omega)} \big)^{\frac{1}{2}}, \tag{2.1}$$

where $\mathrm{curl}\, \boldsymbol{f}$ and $\mathrm{div} \boldsymbol{f}$ are to be understood as the weak curl and weak divergence of $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, respectively. We define the subspace $\boldsymbol{H}_0(\mathrm{curl}) \subset \boldsymbol{H}(\mathrm{curl})$ as the completion of $C_c^\infty(\Omega;\mathbb{R}^3)$ with respect to the $\boldsymbol{H}(\mathrm{curl})$-topology, which admits the following characterisation (cf. e.g. [45, Appendix A])

$$\boldsymbol{H}_0(\mathrm{curl}) = \Big\{ \boldsymbol{f} \in \boldsymbol{H}(\mathrm{curl}) \,:\, (\mathrm{curl}\, \boldsymbol{f}, \boldsymbol{g})_{\boldsymbol{L}^2(\Omega)} = (\boldsymbol{f}, \mathrm{curl}\, \boldsymbol{g})_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{g} \in \boldsymbol{H}(\mathrm{curl}) \Big\}. \tag{2.2}$$

Our solution space, consisting of divergence-free $\boldsymbol{H}_0(\mathrm{curl})$ functions, is denoted as

$$\boldsymbol{Z} := \Big\{ \boldsymbol{f} \in \boldsymbol{H}_0(\mathrm{curl}) \,:\, (\boldsymbol{f}, \nabla \psi)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \psi \in H_0^1(\Omega) \Big\}, \tag{2.3}$$

and is equipped with the $\boldsymbol{H}(\mathrm{curl})$-norm (2.1). We now state several well-known results:

- The continuous embedding

$$\boldsymbol{H}_0(\mathrm{curl}) \cap \boldsymbol{H}(\mathrm{div}) \subset \boldsymbol{H}^{\frac{1}{2}+\sigma_c}(\Omega), \quad \sigma_c = \begin{cases} 0 & \text{if } \Omega \text{ is Lipschitz}, \\ 1/2 & \text{if } \Omega \text{ is convex/of class } C^{1,1}. \end{cases} \tag{2.4}$$

  We refer to [13, Thm. 2] for the first case and to [2, Thms. 2.12 and 2.17] for the second case.

- An immediate consequence is the Maxwell compactness property first attributed to Weck [40]

$$\boldsymbol{H}_0(\mathrm{curl}) \cap \boldsymbol{H}(\mathrm{div}) \subset\subset \boldsymbol{L}^2(\Omega). \tag{2.5}$$

- The Poincaré–Friedrich-type inequality: There exists a positive constant C such that

$$\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} \leq C\Big(\|\mathrm{div}\boldsymbol{f}\|^2_{\boldsymbol{L}^2} + \|\mathrm{curl}\boldsymbol{f}\|^2_{\boldsymbol{L}^2} \Big) \quad \forall \boldsymbol{f} \in \boldsymbol{H}_0(\mathrm{curl}) \cap \boldsymbol{H}(\mathrm{div}), \tag{2.6}$$

which also implies there exists a positive constant $\tilde{c}$ such that

$$\|\boldsymbol{f}\|_{\boldsymbol{H}(\mathrm{curl})} \leq \tilde{c}\,\|\mathrm{curl}\,\boldsymbol{f}\|_{\boldsymbol{L}^2} \quad \forall \boldsymbol{f} \in \boldsymbol{Z}. \tag{2.7}$$

For a domain Ω with connected boundary $\partial\Omega$, it holds that

$$\Big\{ \boldsymbol{y} \in \boldsymbol{H}_0(\mathrm{curl}) \cap \boldsymbol{H}(\mathrm{div}) \,:\, \mathrm{curl}\,\boldsymbol{y} = \boldsymbol{0},\, \mathrm{div}\,\boldsymbol{y} = 0 \text{ in } \Omega \Big\} = \{\boldsymbol{0}\}, \tag{2.8}$$

see for example [34, Thm. 1] or [2, equ. (3.23) and Prop. 3.18]. Then, (2.6) can be established with (2.5) and (2.8) using a contradiction argument akin to the usual proof of the Poincaré inequality, cf. [17, §5.8, Thm. 1].

2.2 Functions of bounded variations

We review basic properties for functions of bounded variations that are sufficient for our analysis. For a more detailed introduction we point to [1,18,23].

We say that $u \in L^1(\Omega)$ is a function of bounded variation in Ω if its distributional gradient $\mathrm{D}u$ is a finite Radon measure. The space of all such functions is denoted as $BV(\Omega)$ and is endowed with the norm $\|\cdot\|_{BV(\Omega)} = \|\cdot\|_{L^1(\Omega)} + TV(\cdot)$, where for $u \in BV(\Omega)$, the total variation $TV(u)$ is defined as

$$TV(u) := |\mathrm{D}u|\,(\Omega) := \sup\Big\{ \int_\Omega u \,\mathrm{div}\,\phi \,dx \;\text{ s.t. }\; \phi \in C^1_0(\Omega;\mathbb{R}^3),\; \|\phi\|_\infty \leq 1 \Big\}.$$

The space $BV(\Omega, \{0,1\})$ denotes the space of all $BV(\Omega)$ functions taking values in $\{0,1\}$. We say that a set $E \subset \Omega$ is a set of finite perimeter if $\chi_E \in BV(\Omega, \{0,1\})$, where for a set A, $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. Furthermore, if $u \in BV(\Omega, \{0,1\})$ is not constant, then there exists a measurable set of finite perimeter E^u defined as

$$E^u := \Big\{ x \in \Omega \,:\, \lim_{\delta \to 0} \frac{1}{|B_\delta(x)|} \int_{B_\delta(x)} u(y)\,dy \; = 1 \Big\},$$

such that $\chi_{E^u}(x) = u(x)$ a.e. in Ω, where $B_\delta(x)$ denotes the ball centred at x with radius δ, and $|B_\delta(x)|$ its Lebesgue measure. The perimeter of a subset $E \subset \Omega$ of finite perimeter is defined as $P_\Omega(E) = |\mathrm{D}\chi_E|(\Omega)$. We say that a sequence $(u_k)_{k\in\mathbb{N}} \subset BV(\Omega)$ converges to $u \in BV(\Omega)$ in the sense of intermediate convergence (or strict convergence) if

$$u_k \to u \text{ in } L^1(\Omega) \text{ and } TV(u_k) \to TV(u) \text{ in } \mathbb{R}. \tag{2.9}$$

Furthermore, if $(u_k)_{k\in\mathbb{N}} \subset BV(\Omega)$ is a bounded sequence, then there exists a subsequence $(k_n)_{n\in\mathbb{N}}$ and a limit $u \in BV(\Omega)$ such that $u_{k_n} \to u$ in $L^1(\Omega)$ and $TV(u) \leq \liminf_{n\to\infty} TV(u_{k_n})$.

2.3 Saddle point problems

The following is a simplified version of the result due to Scheurer [36, Props. 2.3 and 2.4] for nonlinear saddle point problems.

Lemma 2.1. *Let V and W be two reflexive Banach spaces with dual spaces V^* and W^*, respectively. Let $A : V \to V^*$ be a nonlinear operator, $b : V \times W \to \mathbb{R}$ be a bilinear form with a continuous and linear operator $B : V \to W^*$ defined as $\langle Bv, w\rangle_W = b(v,w)$, with B' denoting the adjoint of B. Furthermore, suppose the following hold:*

(i) $A : V \to V^*$ *is hemicontinuous, i.e.,* $\lim_{t\to 0}\langle A(x+ty), z\rangle_V = \langle A(x), z\rangle \ \forall x, y, z \in V$.

(ii) $\exists \beta > 0$ *s.t.* $\langle A(u) - A(v), u - v\rangle \geq \beta \|u - v\|_V^2 \ \forall u, v \in V_g = \{v \in V : Bv = g\}$.

(iii) $\exists \gamma > 0$ *s.t.* $\|A(u) - A(v)\|_{V^*} \leq \gamma \|u - v\|_V \ \forall u, v \in V$.

(iv) $\exists k > 0$ *s.t.* $\|B'q\|_{V^*} \geq k \|q\|_W$ *and* V *admits a direct decomposition with* $V_0 := \mathrm{Ker}B$ *and* $V_0^\perp$.

Then, for any $(f, g) \in V^* \times W^*$*, the nonlinear saddle point problem*

$$\begin{cases} \langle A(u), v\rangle_V + b(v, \phi) = \langle f, v\rangle_V & \forall v \in V, \\ b(u, \psi) = \langle g, \psi\rangle_W & \forall \psi \in W \end{cases} \tag{2.10}$$

admits a unique solution $(u, \phi) \in V \times W$ *such that*

$$\|u\|_V + \|\phi\|_W \leq C\big(\|f\|_{V^*} + \|g\|_{W^*} \big)$$

for some positive constant $C = C(\gamma, \beta, k)$*. Furthermore, let* $(\bar{u}, \bar{\phi}) \in V \times W$ *denote the unique solution to* (2.10) *with data* $(\bar{f}, \bar{g}) \in V^* \times W^*$*. Then, it holds that*

$$\|u - \bar{u}\|_V + \|\phi - \bar{\phi}\|_W \leq C\big(\|f - \bar{f}\|_{V^*} + \|g - \bar{g}\|_{W^*} \big)$$

for some positive constant $C = C(\gamma, \beta, k)$.

3 Analysis of the forward model

For a fixed function $u : \Omega \to [0, 1]$, we define the operator $A_u : \boldsymbol{H}_0(\mathrm{curl}) \to \boldsymbol{H}_0(\mathrm{curl})^*$ and the bilinear form $b : \boldsymbol{H}_0(\mathrm{curl}) \times H_0^1(\Omega) \to \mathbb{R}$ as

$$\begin{aligned} \langle A_u(\boldsymbol{y}), \boldsymbol{v}\rangle &:= \int_\Omega (\nu_0(1 - u) + \nu_1(|\mathrm{curl}\,\boldsymbol{y}|)u)\mathrm{curl}\,\boldsymbol{y} \cdot \mathrm{curl}\,\boldsymbol{v}\, dx \quad \forall \boldsymbol{y}, \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl}), \\ b(\boldsymbol{y}, \psi) &:= \int_\Omega \boldsymbol{y} \cdot \nabla\psi\, dx \quad \forall \boldsymbol{y} \in \boldsymbol{H}_0(\mathrm{curl}),\ \psi \in H_0^1(\Omega). \end{aligned} \tag{3.1}$$

Then, a mixed formulation of (1.2) reads as

$$\begin{cases} \langle A_u(\boldsymbol{y}), \boldsymbol{v}\rangle + b(\boldsymbol{v}, \phi) = (\boldsymbol{J}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl}), \\ b(\boldsymbol{y}, \psi) = 0 & \forall \psi \in H_0^1(\Omega). \end{cases} \tag{3.2}$$

The function ϕ is referred to as the Lagrange multiplier associated with (1.2). If

$$\mathrm{div}\boldsymbol{J} = 0 \text{ in } \Omega \quad \Leftrightarrow \quad (\boldsymbol{J}, \nabla\psi)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \psi \in H_0^1(\Omega) \tag{3.3}$$

holds, then choosing $\boldsymbol{v} = \nabla\psi$ in the first equality yields $b(\nabla\psi, \phi) = 0$ for all $\psi \in \boldsymbol{H}_0^1(\Omega)$, i.e., ϕ is a weak solution to the homogeneous Dirichlet Laplace problem, hence $\phi = 0$.

To analyse the forward model (3.1), we make the following assumptions (cf. [28] for their physical justification), which we assume to hold throughout the rest of the paper.

Assumption 3.1. *Let* $\nu_0 > 0$ *denote the vacuum magnetic reluctivity. We assume that*

(A1) $\nu_1 : [0, \infty) \to [0, \infty)$ *is a continuous function.*

(A2) *There exists a constant* $\underline{\nu} \in (0, \nu_0)$ *such that for all* $s \in [0, \infty)$,

$$\underline{\nu} \leq \nu_1(s) \leq \nu_0, \quad \lim_{s\to\infty} \nu_1(s) = \nu_0.$$

(A3) *There exists a constant* $\overline{\nu} \in [\nu_0, \infty)$ *such that the mapping* $s \mapsto \nu_1(s)s$ *satisfies*

$$\begin{aligned} (\nu_1(s)s - \nu_1(\hat{s})\hat{s})(s - \hat{s}) &\geq \underline{\nu}\,|s - \hat{s}|^2\,, \\ |\nu_1(s)s - \nu_1(\hat{s})\hat{s}| &\leq \overline{\nu}\,|s - \hat{s}|\,, \end{aligned} \tag{3.4}$$

for all $s, \hat{s} \in [0, \infty)$.

(A4) *The observation operator* $\boldsymbol{G} : \boldsymbol{Z} \to \mathcal{O}$ *is bounded and continuous.*

By [43, Lem. 2.2] we obtain the following inequalities: For all $\boldsymbol{v}, \hat{\boldsymbol{v}} \in \mathbb{R}^3$, it holds that

$$(\nu_1(|\boldsymbol{v}|)\boldsymbol{v} - \nu_1(|\hat{\boldsymbol{v}}|)\hat{\boldsymbol{v}}) \cdot (\boldsymbol{v} - \hat{\boldsymbol{v}}) \geq \underline{\nu}\,|\boldsymbol{v} - \hat{\boldsymbol{v}}|^2\,, \tag{3.5}$$

$$|\nu_1(|\boldsymbol{v}|)\boldsymbol{v} - \nu_1(|\hat{\boldsymbol{v}}|)\hat{\boldsymbol{v}}| \leq (2\nu_0 + \overline{\nu})\,|\boldsymbol{v} - \hat{\boldsymbol{v}}|\,. \tag{3.6}$$

As a consequence of the definition of A_u, the fact $\underline{\nu} \leq \nu_0$ and (2.7), we can verify

$$\langle A_u(\boldsymbol{y}) - A_u(\hat{\boldsymbol{y}}), \boldsymbol{y} - \hat{\boldsymbol{y}}\rangle \geq \hat{c}\,\|\boldsymbol{y} - \hat{\boldsymbol{y}}\|^2_{\boldsymbol{H}(\mathrm{curl})} \quad \forall \boldsymbol{y}, \hat{\boldsymbol{y}} \in \boldsymbol{Z}, \tag{3.7}$$

$$|\langle A_u(\boldsymbol{y}) - A_u(\hat{\boldsymbol{y}}), \boldsymbol{v}\rangle| \leq C_*\,\|\boldsymbol{y} - \hat{\boldsymbol{y}}\|_{\boldsymbol{H}(\mathrm{curl})}\,\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{curl})} \quad \forall \boldsymbol{y}, \hat{\boldsymbol{y}}, \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl}), \tag{3.8}$$

where $C_* = 2\nu_0 + \bar{\nu}$ and $\hat{c}$ depends only on $\underline{\nu}$ and the constant $\tilde{c}$ in (2.7). In particular, conditions (i), (ii) and (iii) of Lemma 2.1 are fulfilled with $\beta = \hat{c}$ and $\gamma = 2\nu_0 + \bar{\nu}$ for the choice of function spaces $V = \boldsymbol{H}_0(\mathrm{curl})$, $W = H^1_0(\Omega)$ and $V_0 = \mathrm{Ker}B = \boldsymbol{Z}$.

Meanwhile, following [43, equ. (3.11)], the bilinear from $b : \boldsymbol{H}_0(\mathrm{curl}) \times H^1_0(\Omega) \to \mathbb{R}$ satisfies the Ladyzhenskaya–Babuška–Brezzi (LBB) condition:

$$\sup_{\boldsymbol{y} \in \boldsymbol{H}_0(\mathrm{curl})} \frac{|b(\boldsymbol{y}, \psi)|}{\|\boldsymbol{y}\|_{\boldsymbol{H}(\mathrm{curl})}} \geq \frac{|b(\nabla\psi, \psi)|}{\|\nabla\psi\|_{L^2(\Omega)}} = \|\nabla\psi\|_{L^2(\Omega)} \geq \tilde{c}\,\|\psi\|_{H^1(\Omega)} \quad \forall \psi \in H^1_0(\Omega), \tag{3.9}$$

with a positive constant $\tilde{c}$ depending only on Ω. Standard results, e.g. [11, Thm. 0.1] yields that the LBB condition is equivalent to condition (iv) in Lemma 2.1. Hence, the well-posedness of (3.2) follows from Lemma 2.1.

Theorem 3.1. *For* $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$ *and* $u \in L^1(\Omega; [0,1]) := \{f \in L^1(\Omega) : 0 \leq f(x) \leq 1 \text{ a.e. in } \Omega\}$, *there exists a unique solution pair* $(\boldsymbol{y}, \phi) \in \boldsymbol{Z} \times H^1_0(\Omega)$ *to* (3.2). *If* $\boldsymbol{J}$ *satisfies* (3.3) *then* $\phi \equiv 0$. *Furthermore, there exists a positive constant* C *depending only on* ν_0, $\underline{\nu}$, $\bar{\nu}$ *and* Ω *such that*

$$\|\boldsymbol{y}\|_{\boldsymbol{H}(\mathrm{curl})} + \|\phi\|_{H^1_0(\Omega)} \leq C\,\|\boldsymbol{J}\|_{\boldsymbol{L}^2(\Omega)}\,. \tag{3.10}$$

We stress that the above estimate is independent of $u \in L^1(\Omega; [0,1])$ thanks to (3.7). The well-posedness of (3.2) allows us to define a solution operator

$$\boldsymbol{S} : L^1(\Omega; [0,1]) \to \boldsymbol{Z}, \quad u \mapsto \boldsymbol{y},$$

and the next result shows a continuity property.

Theorem 3.2. *Let* $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$ *and* $(u_k)_{k\in\mathbb{N}} \subset L^1(\Omega; [0,1])$ *denote a sequence converging strongly to some* $u \in L^1(\Omega; [0,1])$. *Let* $(\boldsymbol{y}_k, \phi_k)_{k\in\mathbb{N}} \subset \boldsymbol{Z} \times H^1_0(\Omega)$ *denote the corresponding solutions to* (3.2) *with data* $(u_k, \boldsymbol{J})_{k\in\mathbb{N}}$. *Then, it holds that*

$$\boldsymbol{y}_k \to \boldsymbol{y} \text{ in } \boldsymbol{Z}, \quad \phi_k \rightharpoonup \phi \text{ in } H^1_0(\Omega),$$

where the limiting pair $(y, \phi) \in \boldsymbol{Z} \times H^1_0(\Omega)$ *is the unique solution to* (3.2) *with data* $(u, \boldsymbol{J})$.

Proof. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence satisfying the hypothesis. From the estimate stated in Theorem 3.1, we can extract a non-relabelled subsequence and limit functions $(\boldsymbol{y},\phi) \in \boldsymbol{Z} \times H_0^1(\Omega)$ such that

$$\boldsymbol{y}_k \rightharpoonup \boldsymbol{y} \text{ in } \boldsymbol{H}(\text{curl}), \quad \phi_k \rightharpoonup \phi \text{ in } H^1(\Omega). \tag{3.11}$$

Setting $\hat{\boldsymbol{y}}_k = \boldsymbol{y}_k - \boldsymbol{y}$ as the difference, and upon subtracting the term $\int_\Omega [\nu_0(1-u_k) + \nu_1(|\text{curl}\,\boldsymbol{y}|)u_k]\text{curl}\,\boldsymbol{y}\cdot\text{curl}\,\boldsymbol{v}\,dx$ from both sides of (3.2) leads to

$$\begin{aligned}&\int_\Omega \nu_0(1-u_k)\text{curl}\,\hat{\boldsymbol{y}}_k\cdot\text{curl}\,\boldsymbol{v} + u_k[\nu_1(|\text{curl}\,\boldsymbol{y}_k|)\text{curl}\,\boldsymbol{y}_k - \nu_1(|\text{curl}\,\boldsymbol{y}|)\text{curl}\,\boldsymbol{y}]\cdot\text{curl}\,\boldsymbol{v}\,dx\\ &\quad= \int_\Omega \boldsymbol{J}\boldsymbol{v} - [\nu_0(1-u_k) + \nu_1(|\text{curl}\,\boldsymbol{y}|)u_k]\text{curl}\,\boldsymbol{y}\cdot\text{curl}\,\boldsymbol{v}\,dx - b(\boldsymbol{v},\phi_k) \quad \forall \boldsymbol{v}\in\boldsymbol{Z}.\end{aligned}$$

Substituting $\boldsymbol{v} = \hat{\boldsymbol{y}}_k \in \boldsymbol{Z}$ and employing the bounds $\underline{\nu} \le \nu_1(\cdot) \le \nu_0$, the strong monotonicity (3.5), the facts $0 \le u_k \le 1$ a.e. in Ω and $b(\hat{\boldsymbol{y}}_k,\phi_k) = 0$ gives

$$\underline{\nu}\,\|\text{curl}\,\hat{\boldsymbol{y}}_k\|_{\boldsymbol{L}^2}^2 \le \int_\Omega \boldsymbol{J}\hat{\boldsymbol{y}}_k - [\nu_0(1-u_k) + \nu_1(|\text{curl}\,\boldsymbol{y}|)u_k]\text{curl}\,\boldsymbol{y}\cdot\text{curl}\,\hat{\boldsymbol{y}}_k\,dx\,. \tag{3.12}$$

The right-hand side of (3.12) converges to zero thanks to the $\boldsymbol{L}^2(\Omega)$-weak convergences $\hat{\boldsymbol{y}}_k \rightharpoonup \boldsymbol{0}$, $\text{curl}\,\hat{\boldsymbol{y}}_k \rightharpoonup \boldsymbol{0}$ and the $\boldsymbol{L}^2(\Omega)$-strong convergence $\nu_i\,u_k\,\text{curl}\,\boldsymbol{y} \to \nu_i\,u\,\text{curl}\,\boldsymbol{y}$ for $i = 0,1$. Then, (2.5) and (2.7) imply

$$\boldsymbol{y}_k \to \boldsymbol{y} \text{ in } \boldsymbol{H}_0(\text{curl}). \tag{3.13}$$

Thus, after extracting a non-relabelled subsequence we obtain due to (A1)–(A2) that $\nu_1(|\text{curl}\,\boldsymbol{y}_k(x)|) \to \nu_1(|\text{curl}\,\boldsymbol{y}(x)|)$ a.e. in Ω and

$$\nu_1(|\text{curl}\,\boldsymbol{y}_k|)\text{curl}\,\boldsymbol{v} \to \nu_1(|\text{curl}\,\boldsymbol{y}|)\text{curl}\,\boldsymbol{v} \text{ in } \boldsymbol{L}^2(\Omega) \quad \forall \boldsymbol{v}\in\boldsymbol{H}_0(\text{curl}).$$

In conjunction with $u_k\,\text{curl}\,\boldsymbol{y}_k \to u\,\text{curl}\,\boldsymbol{y}$ in $\boldsymbol{L}^2(\Omega)$ derived from the generalized dominating convergence theorem and the facts $u_k\,\text{curl}\,\boldsymbol{y}_k \to u\,\text{curl}\,\boldsymbol{y}$ a.e. in Ω, $|u_k\,\text{curl}\,\boldsymbol{y}_k| \le |\text{curl}\,\boldsymbol{y}_k|$ and (3.13), it follows that

$$\int_\Omega u_k\,\nu_1(|\text{curl}\,\boldsymbol{y}_k|)\text{curl}\,\boldsymbol{y}_k\cdot\text{curl}\,\boldsymbol{v}\,dx \to \int_\Omega u\,\nu_1(|\text{curl}\,\boldsymbol{y}|)\text{curl}\,\boldsymbol{y}\cdot\text{curl}\,\boldsymbol{v}\,dx \quad \forall \boldsymbol{v}\in\boldsymbol{H}_0(\text{curl}).$$

Hence, passing to the limit $k\to\infty$ in (3.2) for $(\boldsymbol{y}_k,\phi_k)$ with data $(u_k,\boldsymbol{J})$ shows that the limiting pair $(\boldsymbol{y},\phi)$ satisfies (3.2) with data $(u,\boldsymbol{J})$. Now, since the unique solution of (3.2) with data $(u,\boldsymbol{J})$ is independent of the choice of the extracted subsequence $(\boldsymbol{y}_k,\phi_k)_{k\in\mathbb{N}}$, classical arguments yield that the convergence properties (3.11) and (3.13) hold true for the whole sequence. This completes the proof. □

Theorem 3.3. *Given $\boldsymbol{J}\in\boldsymbol{L}^2(\Omega)$ and $u,\bar{u}\in L^1(\Omega;[0,1])$, let $(\boldsymbol{y},\phi)$ and $(\bar{\boldsymbol{y}},\bar{\phi})$ denote the corresponding unique solutions to* (3.2) *with data $(u,\boldsymbol{J})$ and $(\bar{u},\boldsymbol{J})$, respectively. Then, there exists a positive constant C depending only on ν_0, $\underline{\nu}$ and Ω such that*

$$\|\boldsymbol{y}-\bar{\boldsymbol{y}}\|_{\boldsymbol{H}(\text{curl})} \le C\,\|(u-\bar{u})\text{curl}\,\bar{\boldsymbol{y}}\|_{\boldsymbol{L}^2(\Omega)}\,. \tag{3.14}$$

Proof. Let us write $\nu_1(\boldsymbol{y}) = \nu_1(|\text{curl}\,\boldsymbol{y}|)$ for convenience. Then, subtracting (3.2) for $(\boldsymbol{y},\phi)$ from the same equalities for $(\bar{\boldsymbol{y}},\bar{\phi})$ leads to

$$\begin{aligned}&\int_\Omega \boldsymbol{v}\cdot\nabla(\phi-\bar{\phi}) + [\nu_0(1-u)\text{curl}\,(\boldsymbol{y}-\bar{\boldsymbol{y}}) + u(\nu_1(\boldsymbol{y})\text{curl}\,\boldsymbol{y} - \nu_1(\bar{\boldsymbol{y}})\text{curl}\,\bar{\boldsymbol{y}})]\cdot\text{curl}\,\boldsymbol{v}\,dx\\ &\quad= \int_\Omega (u-\bar{u})[\nu_0 - \nu_1(\bar{\boldsymbol{y}})]\text{curl}\,\bar{\boldsymbol{y}}\cdot\text{curl}\,\boldsymbol{v}\,dx \quad \forall \boldsymbol{v}\in\boldsymbol{H}_0(\text{curl}),\\ &\int_\Omega (\boldsymbol{y}-\bar{\boldsymbol{y}})\cdot\nabla\psi\,dx = 0 \quad \forall\psi\in H_0^1(\Omega).\end{aligned}$$

Choosing $\boldsymbol{v} = \boldsymbol{y} - \bar{\boldsymbol{y}}$ and $\psi = \phi - \bar{\phi}$, keeping in mind that $\nu_1 \geq \underline{\nu}$, $u \in [0,1]$ and (3.5) yields the inequality

$$\underline{\nu} \, \|\operatorname{curl}(\boldsymbol{y} - \bar{\boldsymbol{y}})\|^2_{\boldsymbol{L}^2(\Omega)} \leq \int_\Omega [\nu_0 - \nu_1(\bar{\boldsymbol{y}})](u - \bar{u}) \operatorname{curl} \bar{\boldsymbol{y}} \cdot \operatorname{curl}(\boldsymbol{y} - \bar{\boldsymbol{y}}) \, dx \, .$$

Hence, in view of (A2),

$$\|\operatorname{curl}(\boldsymbol{y} - \bar{\boldsymbol{y}})\|_{\boldsymbol{L}^2(\Omega)} \leq \underline{\nu}^{-1} \nu_0 \, \|(u - \bar{u}) \operatorname{curl} \bar{\boldsymbol{y}}\|_{\boldsymbol{L}^2(\Omega)} \, .$$

In conclusion, the desired assertion comes from application of (2.7). □

4 Total variation inverse problem

Throughout this section let $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$, $\boldsymbol{y}_m \in \mathcal{O}$ and $\alpha > 0$ be fixed. The total variation regularised inverse problem (TVIP) reads as

$$\min_{v \in BV(\Omega,\{0,1\})} J(v) := J_f(v) + \alpha TV(v), \quad J_f(v) := \frac{1}{2} \|\boldsymbol{G} \circ \boldsymbol{S}(v) - \boldsymbol{y}_m\|^2_{\mathcal{O}} \, . \tag{4.1}$$

The following theorem shows that (4.1) exhibits desirable properties, such as existence of a solution and being (sequentially) stable with respect to data perturbations. Furthermore, under suitable conditions, a minimum-variation solution to the original inverse problem (1.5) can be obtained (provided the solution set is non-empty) from (4.1) as $\alpha \to 0$.

Theorem 4.1. *The following assertions hold:*

- *There exists at least one solution $u^\alpha \in BV(\Omega, \{0,1\})$ to (4.1).*
- *If $(\boldsymbol{y}_m^n)_{n \in \mathbb{N}} \subset \mathcal{O}$ is a sequence such that $\boldsymbol{y}_m^n \to \boldsymbol{y}_m$ in $\mathcal{O}$, and u_n^α denotes a solution to (4.1) with data $\boldsymbol{y}_m^n$, then along a non-relabelled subsequence it holds that u_n^α converges to a solution u^α to (4.1) with data $\boldsymbol{y}_m$ in the sense of intermediate convergence (2.9).*

Assume now that the inverse problem (1.5) has a solution in $BV(\Omega, \{0,1\})$. For any $\delta > 0$, let $(\alpha_\delta)_{\delta > 0}$ be a positive null sequence such that $\frac{\delta^2}{\alpha_\delta} \to 0$ as $\delta \to 0$. Let $u_\delta^{\alpha_\delta}$ be a solution to

$$\min_{v \in BV(\Omega,\{0,1\})} J_\delta(v) := \frac{1}{2} \left\|\boldsymbol{G} \circ \boldsymbol{S}(v) - \boldsymbol{y}_m^\delta\right\|^2_{\mathcal{O}} + \alpha_\delta TV(v),$$

where $\boldsymbol{y}_m^\delta$ satisfies $\left\|\boldsymbol{y}_m^\delta - \boldsymbol{y}_m\right\|_{\mathcal{O}} \leq \delta$. Then, there exists a non-relabelled subsequence of $(u_\delta^{\alpha_\delta})_{\delta > 0}$ and a solution $u \in BV(\Omega, \{0,1\})$ to the inverse problem (1.5) such that

$$u_\delta^{\alpha_\delta} \to u \text{ in } L^1(\Omega), \quad u_\delta^{\alpha_\delta} \rightharpoonup u \text{ in } BV(\Omega).$$

Furthermore, u satisfies $TV(u) \leq TV(w)$ for every solution $w \in BV(\Omega, \{0,1\})$ to the inverse problem (1.5), i.e., the limit u is a minimum-variation solution to (1.5).

The proof of Theorem 4.1 follows along similar lines of argument as in the proof of [5, Props. 2.2, 2.3, 2.4], compare also [16, Thms. 10.2, 10.3]. Although our present setting allows for a more abstract measurement space $\mathcal{O}$ and measurement operator $\boldsymbol{G}$, we mention that, for $u_n^\alpha \to u^\alpha$ in $L^1(\Omega)$, Theorem 3.2 together with the boundedness and continuity of $\boldsymbol{G}$ implies

$$\boldsymbol{G} \circ \boldsymbol{S}(u_n^\alpha) \to \boldsymbol{G} \circ \boldsymbol{S}(u^\alpha) \text{ in } \mathcal{O},$$

which yields $J_f(u_n^\alpha) \to J_f(u^\alpha)$. These observations, in conjunction with the arguments in [5] are sufficient to infer the assertions of Theorem 4.1. Hence, we omit the proof.

5 Phase field inverse problem

In all what follows, let $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$, $\boldsymbol{y}_m \in \mathcal{O}$ and $\alpha, \varepsilon > 0$ be fixed. The phase field regularised inverse problem (PFIP) reads as

$$\min_{v \in \mathcal{K}} J_\varepsilon(v) := J_f(v) + \alpha E_\varepsilon(v), \quad E_\varepsilon(v) := \int_\Omega \frac{\gamma\varepsilon}{2} |\nabla v|^2 + \frac{\gamma}{\varepsilon} \Psi(v)\, dx\,. \tag{5.1}$$

In the setting of (5.1), Ψ is a nonnegative double well potential with minima at 0 and 1, while γ is a constant depending only on Ψ. It is clear from the definition that for E_ε to be well-defined, we must expand the solution space $\mathcal{K}$ from $BV(\Omega, \{0,1\})$ to subsets of $H^1(\Omega)$. If Ψ is defined everywhere on $\mathbb{R}$, such as the smooth double well potential $\Psi(s) = s^2(1-s)^2$, we may choose $\mathcal{K}$ as the whole of $H^1(\Omega)$. Alternatively we can consider the double obstacle potential [9]

$$\Psi(s) = \begin{cases} \frac{1}{2}s(1-s) & \text{if } s \in [0,1], \\ +\infty & \text{otherwise,} \end{cases} \tag{5.2}$$

that is only finite over the interval $[0,1]$, so that the solution space for PFIP can be taken to be the following closed and convex set

$$\mathcal{K} := \{f \in H^1(\Omega) : 0 \le f \le 1 \text{ a.e. in } \Omega\}. \tag{5.3}$$

In this setting, $\gamma^{-1} = \int_0^1 \sqrt{2\Psi(y)}\, dy = \int_0^1 \sqrt{y(1-y)}\, dy = \frac{\pi}{8}$, then by minor modifications of [9, Thm. 3.7] detailed in Appendix A, we find that the following extended functionals

$$\mathcal{E}_\varepsilon(v) = \begin{cases} E_\varepsilon(v), & v \in \mathcal{K}, \\ +\infty, & v \notin \mathcal{K}, \end{cases} \quad \mathcal{E}_0(v) = \begin{cases} TV(v), & v \in BV(\Omega, \{0,1\}), \\ +\infty, & v \notin BV(\Omega, \{0,1\}) \end{cases}$$

satisfy $\mathcal{E}_\varepsilon(v) \xrightarrow{\Gamma} \mathcal{E}_0(v)$ in X, where $\xrightarrow{\Gamma}$ denotes Gamma convergence. Furthermore, from the continuity of $\boldsymbol{G} \circ \boldsymbol{S} : L^1(\Omega; [0,1]) \to \mathcal{O}$ and the property that Gamma convergence is stable under continuous perturbations [10, Rmk. 1.7], we have

$$J_\varepsilon(v) = J_f(v) + \alpha\mathcal{E}_\varepsilon(v) \xrightarrow{\Gamma} J_f(v) + \alpha\mathcal{E}_0(v) = J(v) \text{ as } \varepsilon \to 0, \tag{5.4}$$

which motivates the investigation of (5.1). For the rest of the paper, we consider $\mathcal{K}$ as defined in (5.3) and take Ψ as the double obstacle potential (5.2) with $\gamma = \frac{8}{\pi}$.

5.1 Properties of solutions

Theorem 5.1. *The following assertions hold:*

- *There exists at least one solution $u_\varepsilon^\alpha \in \mathcal{K}$ to* (5.1)*.*
- *If $(\boldsymbol{y}_m^n)_{n\in\mathbb{N}} \subset \mathcal{O}$ is a sequence such that $\boldsymbol{y}_m^n \to \boldsymbol{y}_m$ in $\mathcal{O}$, and $u_{\varepsilon,n}^\alpha \in \mathcal{K}$ denotes a solution to* (5.1) *with data $\boldsymbol{y}_m^n$, then along a non-relabelled subsequence it holds that $u_{\varepsilon,n}^\alpha \to u_\varepsilon^\alpha$ in $H^1(\Omega)$ where $u_\varepsilon^\alpha \in \mathcal{K}$ is a solution to* (5.1) *with data $\boldsymbol{y}_m$.*

Let us point out the analogue of intermediate convergence for $H^1(\Omega)$-functions would be the norm convergence of the gradient $\|\nabla u_{\varepsilon,n}^\alpha\|_{\boldsymbol{L}^2(\Omega)} \to \|\nabla u_\varepsilon^\alpha\|_{\boldsymbol{L}^2(\Omega)}$. Furthermore, since the arguments to prove Theorem 5.1 is somewhat standard in the literature, we will omit the proof of existence (which is shown via the direct method) and sketch the details for sequential stability.

Proof. We define $J_{\varepsilon,n}(v) := \frac{1}{2}\|\boldsymbol{G}\circ\boldsymbol{S}(v) - \boldsymbol{y}_m^n\|_{\mathcal{O}}^2 + \alpha E_\varepsilon(v)$ and by definition,

$$J_{\varepsilon,n}(u_{\varepsilon,n}^\alpha) \le J_{\varepsilon,n}(v) \quad \forall v \in \mathcal{K}. \tag{5.5}$$

Choosing, for instance, $v = 1$ yields the boundedness of $(u_{\varepsilon,n}^\alpha)_{n\in\mathbb{N}}$ in $\mathcal{K}$. Then, the compact embedding $H^1(\Omega) \subset L^2(\Omega)$ and Theorem 3.2 give

$$u_{\varepsilon,n}^\alpha \rightharpoonup u_\varepsilon^\alpha \text{ in } H^1(\Omega), \quad u_{\varepsilon,n}^\alpha \to u_\varepsilon^\alpha \text{ in } L^2(\Omega), \quad \boldsymbol{G}\circ\boldsymbol{S}(u_{\varepsilon,n}^\alpha) \to \boldsymbol{G}\circ\boldsymbol{S}(u_\varepsilon^\alpha) \text{ in } \mathcal{O}$$

along a non-relabelled subsequence with limit $u_\varepsilon^\alpha \in \mathcal{K}$ due to $\mathcal{K}$ being convex and closed. Passing to the limit $n\to\infty$ in (5.5) and employing weak lower semicontinuity, we arrive at $J_\varepsilon(u_\varepsilon^\alpha) \le J_\varepsilon(v)$ for all $v\in\mathcal{K}$, and so u_ε^α is a solution to (5.1). Meanwhile, passing to the limit $n\to\infty$ in the inequality (5.5) with the choice $v = u_\varepsilon^\alpha \in \mathcal{K}$ yields

$$J_\varepsilon(u_\varepsilon^\alpha) \le \liminf_{n\to\infty} J_{\varepsilon,n}(u_{\varepsilon,n}^\alpha) \le \lim_{n\to\infty} J_{\varepsilon,n}(u_{\varepsilon,n}^\alpha) \le \lim_{n\to\infty} J_{\varepsilon,n}(u_\varepsilon^\alpha) = J_\varepsilon(u_\varepsilon^\alpha),$$

from which we deduce that $\lim_{n\to\infty} E_\varepsilon(u_{\varepsilon,n}^\alpha) = E_\varepsilon(u_\varepsilon^\alpha)$. Together with $\|\Psi(u_{\varepsilon,n}^\alpha)\|_{L^1(\Omega)} \to \|\Psi(u_\varepsilon^\alpha)\|_{L^1(\Omega)}$, we then obtain $\|\nabla u_{\varepsilon,n}^\alpha\|_{\boldsymbol{L}^2(\Omega)}^2 \to \|\nabla u_\varepsilon^\alpha\|_{\boldsymbol{L}^2(\Omega)}^2$. □

5.2 Convergence of solutions

An immediate consequence of the Gamma convergence (5.4) is the following result concerning the asymptotic behaviour of minimisers $(u_\varepsilon^\alpha)_{\varepsilon>0}$ to (5.1) as $\varepsilon\to 0$.

Theorem 5.2. *Let $(u_\varepsilon^\alpha)_{\varepsilon>0} \subset \mathcal{K}$ denote a sequence of solutions to PFIP* (5.1)*. Then, there exist a non-relabelled subsequence and a limit $u^\alpha \in BV(\Omega,\{0,1\})$ such that $\lim_{\varepsilon\to0} u_\varepsilon^\alpha = u^\alpha$ in $L^1(\Omega)$, $\lim_{\varepsilon\to0} J_\varepsilon(u_\varepsilon^\alpha) = J(u^\alpha)$, and u^α is a solution to TVIP* (4.1)*.*

Proof. While the argument is somewhat standard, see for instance [21, Proof of Thm. 2], nevertheless we briefly sketch the details, as some of the elements of the proof will be used later. Let $w \in BV(\Omega,\{0,1\})$ be arbitrary, then by (3.10) it is clear that $J(w) < \infty$. We define the set $E^w := \{w = 1\}$ so that $w = \chi_{E^w}$. Our aim is to construct a sequence $(w_\varepsilon)_{\varepsilon>0} \subset \mathcal{K}$ such that $\|w_\varepsilon - w\|_{L^1(\Omega)} \to 0$ and $\limsup_{\varepsilon\to0} J_\varepsilon(w_\varepsilon) \le J(w)$. In the trivial case where $w \equiv 0$ (resp. $w\equiv 1$), which corresponds to $E^w = \emptyset$ (resp. $E^w = \Omega$), we can choose $w_\varepsilon = w$ for all $\varepsilon > 0$ so that $E_\varepsilon(w_\varepsilon) = 0$ and

$$J_\varepsilon(w_\varepsilon) = J_f(w_\varepsilon) = J_f(w) \le J(w) \quad \forall \varepsilon > 0.$$

In the non-trivial case where $0 < |E^w| < |\Omega|$, using [30, Lem. 1], we can approximate $E^w \subset \Omega$ by a sequence $(E_k)_{k\in N}$ of open bounded sets in $\mathbb{R}^3$ with smooth boundaries such that

$$\begin{aligned} &\lim_{k\to\infty} P_\Omega(E_k) = P_\Omega(E^w), \quad |(E_k\cap\Omega)\Delta E| \le \tfrac{1}{k}, \\ &\mathcal{H}_2(\partial E_k \cap \partial\Omega) = 0, \quad |E_k\cap\Omega| = |E^w| \text{ for } k \text{ sufficiently large,} \end{aligned}$$

where $A\Delta B$ denotes the symmetric difference between the sets A and B, and $\mathcal{H}_2$ denotes the two-dimensional Hausdorff measure. Then, setting $w_k = \chi_{E_k\cap\Omega}$ leads to

$$\|w_k - w\|_{L^1(\Omega)} = |(E_k\cap\Omega)\Delta E^w| \le \tfrac{1}{k}.$$

We apply item (ii) of Lemma A.1 with $v_0 = w_k$ and $A = E_k$, so that for each k there exists a sequence $(w_\varepsilon^k) \subset L^1(\Omega)$ with $\|w_\varepsilon^k - w_k\|_{L^1(\Omega)} = \mathcal{O}(\varepsilon)$ and $\limsup_{\varepsilon\to0} E_\varepsilon(w_\varepsilon^k) \le TV(w_k) = P_\Omega(E_k)$. Then, the diagonal sequence $(w_{\varepsilon_k}^k)_{k\in\mathbb{N}}$ fulfills per construction

$$\left\|w_{\varepsilon_k}^k - w\right\|_{L^1(\Omega)} = \mathcal{O}(k^{-1}), \quad \limsup_{k\to\infty} E_{\varepsilon_k}(w_{\varepsilon_k}^k) \le TV(w). \tag{5.6}$$

For this particular sequence $(\varepsilon_k)_{k\in\mathbb{N}}$, setting $\boldsymbol{y}_{\varepsilon_k} := \boldsymbol{S}(w^k_{\varepsilon_k})$ and $\boldsymbol{y}_w := \boldsymbol{S}(w)$, by Theorem 3.2 we have $\boldsymbol{y}_{\varepsilon_k} \to \boldsymbol{y}_w$ in $\boldsymbol{Z}$ as $k \to \infty$. Continuity of $\boldsymbol{G} : \boldsymbol{Z} \to \mathcal{O}$ implies $J_f(w^k_{\varepsilon_k}) \to J_f(w)$ as $k \to \infty$, and so $\limsup_{k\to\infty} J_{\varepsilon_k}(w^k_{\varepsilon_k}) \le J(w)$. Consequently, we infer $\sup_{k\in\mathbb{N}} J_{\varepsilon_k}(w^k_{\varepsilon_k}) < \infty$.

Next, by definition of u^α_ε as a solution to (5.1), it holds that

$$\alpha E_{\varepsilon_k}(u^\alpha_{\varepsilon_k}) \le J_{\varepsilon_k}(u^\alpha_{\varepsilon_k}) \le J_{\varepsilon_k}(w^k_{\varepsilon_k}) < \infty \quad \forall k \in \mathbb{N}. \tag{5.7}$$

This implies that $\sup_{k\in\mathbb{N}} E_{\varepsilon_k}(u^\alpha_{\varepsilon_k}) < \infty$ and by item (iii) of Lemma A.1, there exists a non-relabelled subsequence and a limit $u^\alpha \in BV(\Omega, \{0,1\})$ such that $u^\alpha_{\varepsilon_k} \to u^\alpha$ strongly in $L^1(\Omega)$. Furthermore, Theorem 3.2 asserts $\boldsymbol{S}(u^\alpha_{\varepsilon_k}) \to \boldsymbol{S}(u^\alpha)$ in $\boldsymbol{Z}$ and so $\lim_{k\to\infty} J_f(u^\alpha_{\varepsilon_k}) = J_f(u^\alpha)$, while when we invoke item (i) of Lemma A.1, (5.6) and (5.7), we obtain

$$\begin{aligned} J(u^\alpha) = J_f(u^\alpha) + \alpha TV(u^\alpha) &\le \liminf_{k\to\infty}(J_f(u^\alpha_{\varepsilon_k}) + \alpha E_{\varepsilon_k}(u^\alpha_{\varepsilon_k})) \\ &\le \limsup_{k\to\infty} J_{\varepsilon_k}(w^k_{\varepsilon_k}) \le J(w) = J_f(w) + \alpha TV(w). \end{aligned}$$

As $w \in BV(\Omega, \{0,1\})$ is arbitrary this implies that u^α is a solution to (4.1). Now, following the start of the proof, we construct a sequence $(v^k_{\varepsilon_k})_{k\in\mathbb{N}}$ satisfying (5.6) with u^α in place of w, and observe that

$$J(u^\alpha) \le \liminf_{k\to\infty} J_{\varepsilon_k}(u^\alpha_{\varepsilon_k}) \le \limsup_{k\to\infty} J_{\varepsilon_k}(u^\alpha_{\varepsilon_k}) \le \limsup_{k\to\infty} J_{\varepsilon_k}(v^k_{\varepsilon_k}) \le J(u^\alpha),$$

which implies $\lim_{k\to\infty} J_{\varepsilon_k}(u^\alpha_{\varepsilon_k}) = J(u^\alpha)$. □

Let us now address the convergence as $\alpha \to 0$ and $\varepsilon \to 0$.

Theorem 5.3. *Suppose that*

(a) $\boldsymbol{G} : \boldsymbol{Z} \to \mathcal{O}$ *is Lispchitz continuous.*

(b) *The inverse problem* (1.5) *has a solution* $u_* \in BV(\Omega, \{0,1\})$, *and there exist a positive null sequence* $(\varepsilon_k)_{k\in\mathbb{N}}$ *and a sequence of functions* $(w_k)_{k\in\mathbb{N}} \subset \mathcal{K}$ *such that* $w_k \to u_*$ *strongly in* $L^1(\Omega)$ *with* $\limsup_{k\to\infty} E_{\varepsilon_k}(w_k) \le TV(u_*)$.

(c) $(\alpha_k)_{k\in\mathbb{N}}$ *is a positive null sequence subordinate to* $(\varepsilon_k)_{k\in\mathbb{N}}$ *and* u_* *in the following sense*

$$\limsup_{k\to\infty} \frac{1}{\alpha_k} \|(w_k - u_*)\operatorname{curl} \boldsymbol{S}(u_*)\|^2_{\boldsymbol{L}^2(\Omega)} = 0. \tag{5.8}$$

Then, for a sequence of solutions $(u^{\alpha_k}_{\varepsilon_k})_{k\in\mathbb{N}} \subset \mathcal{K}$ *to PFIP* (5.1), *there exists a non-relabelled subsequence* $(k \to \infty)$ *and a solution* $u \in BV(\Omega, \{0,1\})$ *to the inverse problem* (1.5) *such that*

$$u^{\alpha_k}_{\varepsilon_k} \to u \text{ in } L^1(\Omega) \quad \text{and} \quad TV(u) \le TV(u_*). \tag{5.9}$$

Remark 5.1.

(i) *We mention that the obvious choice for* $(w_k)_{k\in\mathbb{N}}$ *is the sequence constructed in the proof of Theorem 5.2 which satisfies* (5.6) *(with* u_* *in place of* w*). This fixes the null sequence* $(\varepsilon_k)_{k\in\mathbb{N}}$ *subordinate to* u_*. *In particular, since* $(w_k)_{k\in\mathbb{N}}$ *always exists, the statement of condition* (b) *can always be reduced to "the inverse problem* (1.5) *has a solution* $u_* \in BV(\Omega, \{0,1\})$*". However, in order to define the null sequence* $(\alpha_k)_{k\in\mathbb{N}}$ *subordinate to* $(\varepsilon_k)_{k\in\mathbb{N}}$ *and* u_*, *it is necessary to state condition* (b) *as it is presented.*

(ii) *If* u_* *is a minimum-variation solution to the inverse problem* (1.5)*, then the inequality* (5.9) *implies that* u *is a minimum-variation solution to* (1.5) *as well.*

Proof. For each $k \in \mathbb{N}$, let $u_{\varepsilon_k}^{\alpha_k} \in \mathcal{K}$ denote a solution to $\min_{v\in\mathcal{K}} J_k(v)$ where

$$J_k(v) \coloneqq J_f(v) + \alpha_k E_{\varepsilon_k}(v).$$

For the sequence $(w_k)_{k\in\mathbb{N}}$ in the hypothesis, we set $\boldsymbol{z}_k = \boldsymbol{S}(w_k)$. Then, from the inequality $J_k(u_{\varepsilon_k}^{\alpha_k}) \le J_k(w_k)$, we have

$$E_{\varepsilon_k}(u_{\varepsilon_k}^{\alpha_k}) \le \frac{1}{\alpha_k} J_f(w_k) + E_{\varepsilon_k}(w_k) = \frac{1}{2\alpha_k} \|\boldsymbol{G}(\boldsymbol{z}_k) - \boldsymbol{G}(\boldsymbol{y}_*)\|_{\mathcal{O}}^2 + E_{\varepsilon_k}(w_k), \tag{5.10}$$

since for $\boldsymbol{y}_* = \boldsymbol{S}(u_*)$ it holds that $\boldsymbol{G}(\boldsymbol{y}_*) = \boldsymbol{y}_m$ in $\mathcal{O}$. Lipschitz continuity of $\boldsymbol{G} : \boldsymbol{Z} \to \mathcal{O}$ and the estimate (3.14) imply that

$$E_{\varepsilon_k}(u_{\varepsilon_k}^{\alpha_k}) \le \frac{C}{\alpha_k} \|(w_k - u_*)\operatorname{curl} \boldsymbol{y}_*\|_{\boldsymbol{L}^2(\Omega)}^2 + E_{\varepsilon_k}(w_k).$$

The right-hand side is non-negative and its limit superior as $k \to \infty$ is bounded by the hypothesis. Hence, it is clear that $\sup_{k\in\mathbb{N}} E_{\varepsilon_k}(u_{\varepsilon_k}^{\alpha_k}) < \infty$. Invoking the compactness property (iii) of Lemma A.1 leads to the existence of a non-relabelled subsequence ($k \to \infty$) and a limit $u \in BV(\Omega, \{0,1\})$ such that $u_{\varepsilon_k}^{\alpha_k} \to u$ in $L^1(\Omega)$, and subsequently $J_f(u_{\varepsilon_k}^{\alpha_k}) \to J_f(u)$ as $k \to \infty$. On the other hand, since $E_{\varepsilon_k}(u_{\varepsilon_k}^{\alpha_k}) \ge 0$ and $\alpha_k > 0$, we have

$$J_f(u_{\varepsilon_k}^{\alpha_k}) \le J_k(u_{\varepsilon_k}^{\alpha_k}) \le J_f(w_k) + \alpha_k E_{\varepsilon_k}(w_k) \le C \|(w_k - u_*)\operatorname{curl} \boldsymbol{y}_*\|_{\boldsymbol{L}^2(\Omega)}^2 + \alpha_k E_{\varepsilon_k}(w_k).$$

Taking limit superior on both sides and employing the hypothesis $\limsup_{k\to\infty} E_{\varepsilon_k}(w_k) \le TV(u_*)$ leads to

$$J_f(u) = \lim_{k\to\infty} J_f(u_{\varepsilon_k}^{\alpha_k}) \le C \limsup_{k\to\infty} \|(w_k - u_*)\operatorname{curl} \boldsymbol{y}_*\|_{\boldsymbol{L}^2(\Omega)}^2 + (\limsup_{k\to\infty} \alpha_k) TV(u_*) = 0.$$

From the equality $J_f(u) = 0$ we infer that $\boldsymbol{G} \circ \boldsymbol{S}(u) = \boldsymbol{y}_m$, i.e., u is a solution to the inverse problem (1.5). Moreover, as $u_{\varepsilon_k}^{\alpha_k} \to u$ in $L^1(\Omega)$, by the hypothesis, (5.10) and the item (i) of Lemma A.1,

$$\begin{aligned} TV(u) &\le \liminf_{k\to\infty} E_{\varepsilon_k}(u_{\varepsilon_k}^{\alpha_k}) \le \limsup_{k\to\infty} \Big(\frac{1}{\alpha_k} J_f(w_k) + E_{\varepsilon_k}(w_k)\Big) \\ &\le C \limsup_{k\to\infty} \frac{1}{\alpha_k} \|(w_k - u_*)\operatorname{curl} \boldsymbol{y}_*\|_{\boldsymbol{L}^2(\Omega)}^2 + \limsup_{k\to\infty} E_{\varepsilon_k}(w_k) \le TV(u_*). \end{aligned}$$

This completes the proof. □

Under the existence of a $BV(\Omega, \{0,1\})$-solution to (1.5) satisfying some regularity assumption, we can simplify (5.8) to a relation between the null sequences $(\varepsilon_k)_{k\in\mathbb{N}}$ and $(\alpha_k)_{k\in\mathbb{N}}$ that is more practical for numerical implementations.

Corollary 5.4. *Suppose that*

(A) $\boldsymbol{G} : \boldsymbol{Z} \to \mathcal{O}$ *is Lipschitz continuous.*

(B) *The inverse problem* (1.5) *has a solution* $u_* \in BV(\Omega, \{0,1\})$ *such that* $\operatorname{curl} \boldsymbol{S}(u_*) \in \boldsymbol{L}^p(\Omega)$ *for some* $p > 2$ *and there exists an open bounded set* $A \subset \mathbb{R}^3$ *with a smooth boundary satisfying* $\{u_* = 1\} = A \cap \Omega$ *and* $\mathcal{H}_2(\partial A \cap \partial\Omega) = 0$.

Furthermore, for any positive null sequence $(\varepsilon_k)_{k\in\mathbb{N}}$, let $(\alpha_k)_{k\in\mathbb{N}}$ be a positive null sequence such that

$$\limsup_{k\to\infty} \frac{\varepsilon_k^2}{\alpha_k} = 0. \tag{5.11}$$

Then, the assertions of Theorem 5.3 are valid.

Proof. By the hypothesis, we can apply item (ii) of Lemma A.1 and obtain a family $(w_\varepsilon)_{\varepsilon>0} \subset \mathcal{K}$ such that $\|w_\varepsilon - u_*\|_{L^1(\Omega)} = \mathcal{O}(\varepsilon)$ and $\limsup_{\varepsilon\to 0} E_\varepsilon(w_\varepsilon) \le TV(u_*)$. Then, a short calculation shows

$$\frac{1}{\alpha_k}\|(w_{\varepsilon_k} - u_*)\operatorname{curl}\boldsymbol{y}_*\|^2_{\boldsymbol{L}^2(\Omega)} \le \frac{1}{\alpha_k}\|\operatorname{curl}\boldsymbol{y}_*\|^2_{\boldsymbol{L}^p(\Omega)}\|w_{\varepsilon_k} - u_*\|^2_{L^{2p/(p-2)}(\Omega)} \le \frac{C}{\alpha_k}\|w_{\varepsilon_k} - u_*\|^2_{L^1(\Omega)}$$

on account of the fact that $0 \le w_{\varepsilon_k}(x), u_*(x) \le 1$ for a.e. $x \in \Omega$. Hence, condition (5.8) is fulfilled if (5.11) holds. □

Let us point out that in Corollary 5.4 the null sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ does not need to be subordinate to the true solution u_* (or its approximating sequence) as in Theorem 5.3, which gives greater flexibility at a cost of assuming more regularity on the true solution. Furthermore, we note from the condition (5.11) and the estimate $\|w_{\varepsilon_k} - u_*\|_{L^1(\Omega)} = \mathcal{O}(\varepsilon_k)$ that ε_k plays a similar role to the parameter δ in Theorem 4.1.

6 First-order analysis

Following [43, Sec. 3.2] we introduce a vector function

$$\boldsymbol{\mathcal{F}} : [0,1]\times\mathbb{R}^3 \to \mathbb{R}^3, \quad \boldsymbol{\mathcal{F}}(u,\boldsymbol{s}) = [\nu_0(1-u) + \nu_1(|\boldsymbol{s}|)u]\boldsymbol{s}, \tag{6.1}$$

so that the operator $A_u : \boldsymbol{H}_0(\operatorname{curl}) \to \boldsymbol{H}_0(\operatorname{curl})^*$ defined in (3.1) can be expressed as

$$\langle A_u(\boldsymbol{y}), \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{\mathcal{F}}(u, \operatorname{curl}\boldsymbol{y})\cdot\operatorname{curl}\boldsymbol{v}\,dx \quad \forall \boldsymbol{y},\boldsymbol{v}\in\boldsymbol{H}_0(\operatorname{curl}). \tag{6.2}$$

Assumption 6.1. *In addition to Assumption 3.1, we further assume that*

(A5) $\nu_1 \in C^1(\mathbb{R})$ *and there exists a positive constant $C_{\boldsymbol{\mathcal{F}}}$ such that*

$$\left|\frac{\partial\mathcal{F}_i(u,\boldsymbol{s})}{\partial s_j}\right| \le C_{\boldsymbol{\mathcal{F}}} \quad \textit{for all } u\in[0,1],\ \boldsymbol{s}\in\mathbb{R}^3,\ i,j\in\{1,2,3\}.$$

(A6) $\mathcal{O} = L^2(D)$ *where $D\subset\Omega$ is an open subset, and $\boldsymbol{G} : \boldsymbol{H}(\operatorname{curl}) \to \mathcal{O}$ is the restriction operator $\boldsymbol{G}(\boldsymbol{y}) = \boldsymbol{y}|_D$.*

One example of ν_1 that satisfies (A5) is (see also [43, Example 3.5])

$$\nu_1(s) = \nu_0 - \theta\exp(-\beta s^2) \tag{6.3}$$

with constants $\beta \ge 0$ and $0 \le \theta < \nu_0$. Denoting the Jacobian matrix function of $\boldsymbol{\mathcal{F}}$ by $\nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}} : \Omega\times[0,1]\times\mathbb{R}^3 \to \mathbb{R}^{3\times 3}$, where

$$\nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}}(u,\boldsymbol{s}) = \left(\frac{\partial\mathcal{F}_i}{\partial s_j}(u,\boldsymbol{s})\right)_{1\le i,j\le 3} = [\nu_0(1-u)+\nu_1(|\boldsymbol{s}|)u]\mathbb{I} + u\frac{\nu_1'(|\boldsymbol{s}|)}{|\boldsymbol{s}|}\boldsymbol{s}\otimes\boldsymbol{s},$$

we see that (A5) implies $\nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}}$ is bounded for all $u\in[0,1]$ and $\boldsymbol{s}\in\mathbb{R}^3$. An immediate consequence of (A5) and (3.4) is the following, which follows from a minor modification of [43, Prop. 3.7].

Lemma 6.1. *For all $u \in [0,1]$ and all $\boldsymbol{s}, \boldsymbol{a} \in \mathbb{R}^3$, it holds that*

$$\nu_1(|\boldsymbol{s}|) + \nu_1'(|\boldsymbol{s}|)\,|\boldsymbol{s}| \geq \underline{\nu}, \quad \nabla_{\boldsymbol{s}} \boldsymbol{\mathcal{F}}(u, \boldsymbol{s})\boldsymbol{a} \cdot \boldsymbol{a} \geq \underline{\nu}\,|\boldsymbol{a}|^2 .$$

Consequently,

$$\int_\Omega \nabla_{\boldsymbol{s}} \boldsymbol{\mathcal{F}}(u, \operatorname{curl} \boldsymbol{y})\boldsymbol{v} \cdot \boldsymbol{v}\, dx \geq \underline{\nu}\, \|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \quad \forall u \in [0,1],\ \boldsymbol{y} \in \boldsymbol{H}_0(\operatorname{curl})\,, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega). \tag{6.4}$$

6.1 Optimality system via directional differentiability

In this section, we establish a necessary optimality system for (5.1) through the use of the weak directional differentiability of the solution operator $\boldsymbol{S} : \mathcal{K} \to \boldsymbol{Z}$. To show this property, let us first discuss the linearised equation associated with the saddle point system (3.2). In the following, let $\bar{u} \in \mathcal{K}$, $h \in L^\infty(\Omega)$ and $\bar{\boldsymbol{y}} = \boldsymbol{S}(\bar{u})$. We seek a unique solution $(\boldsymbol{z}, \theta) \in \boldsymbol{Z} \times H_0^1(\Omega)$ to the linear saddle point problem

$$\begin{cases} a_{\bar{\boldsymbol{y}}}(\boldsymbol{z}, \boldsymbol{v}) + b(\boldsymbol{v}, \theta) = f_{h,\bar{\boldsymbol{y}}}(\boldsymbol{v}) & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{curl}), \\ b(\boldsymbol{z}, \psi) = 0 & \forall \psi \in H_0^1(\Omega), \end{cases} \tag{6.5}$$

where the bilinear form $a_{\bar{\boldsymbol{y}}} : \boldsymbol{H}_0(\operatorname{curl}) \times \boldsymbol{H}_0(\operatorname{curl}) \to \mathbb{R}$ and the linear form $f_{h,\bar{\boldsymbol{y}}} : \boldsymbol{H}_0(\operatorname{curl}) \to \mathbb{R}$ are given as

$$a_{\bar{\boldsymbol{y}}}(\boldsymbol{z}, \boldsymbol{v}) := \int_\Omega \nabla_{\boldsymbol{s}} \boldsymbol{\mathcal{F}}(\bar{u}, \operatorname{curl} \bar{\boldsymbol{y}}) \operatorname{curl} \boldsymbol{z} \cdot \operatorname{curl} \boldsymbol{v}\, dx\,, \tag{6.6}$$

$$f_{h,\bar{\boldsymbol{y}}}(\boldsymbol{v}) := \int_\Omega h(\nu_0 - \nu_1(|\operatorname{curl} \bar{\boldsymbol{y}}|)) \operatorname{curl} \bar{\boldsymbol{y}} \cdot \operatorname{curl} \boldsymbol{v}\, dx$$

for all $\boldsymbol{z}, \boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{curl})$. Strictly speaking, the Lagrange multiplier θ is zero as $f_{h,\bar{\boldsymbol{y}}}(\nabla \psi) = 0$ for all $\psi \in H_0^1(\Omega)$. But we include it to retain the saddle point structure.

Lemma 6.2. *Let $\bar{u} \in \mathcal{K}$, $h \in L^\infty(\Omega)$ and $\bar{\boldsymbol{y}} = \boldsymbol{S}(\bar{u})$. Then, there exists a unique solution $\boldsymbol{z} = \boldsymbol{z}(\bar{u}, h) \in \boldsymbol{Z}$, $\theta = 0 \in H_0^1(\Omega)$ to* (6.5) *satisfying*

$$\|\boldsymbol{z}\|_{\boldsymbol{H}(\operatorname{curl})} \leq C\, \|h\|_{L^\infty(\Omega)}\, \|\bar{\boldsymbol{y}}\|_{\boldsymbol{H}(\operatorname{curl})} \tag{6.7}$$

for a positive constant C depending only on ν_0, $\underline{\nu}$ and Ω.

Proof. Since (6.5) is a linear saddle point problem, to apply standard results [11, Thm. 1.1, Cor. 1.1] it suffices to note that

$$\begin{cases} |a_{\bar{\boldsymbol{y}}}(\boldsymbol{z}, \boldsymbol{v})| \leq C_{\boldsymbol{\mathcal{F}}}\, \|\boldsymbol{z}\|_{\boldsymbol{H}(\operatorname{curl})}\, \|\boldsymbol{v}\|_{\boldsymbol{H}(\operatorname{curl})} & \forall \boldsymbol{z}, \boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{curl}), \\ a_{\bar{\boldsymbol{y}}}(\boldsymbol{z}, \boldsymbol{z}) \geq \underline{\nu}\, \|\operatorname{curl} \boldsymbol{z}\|^2_{\boldsymbol{L}^2(\Omega)} \geq \underline{\nu} \tilde{c}^{-2}\, \|\boldsymbol{z}\|^2_{\boldsymbol{H}(\operatorname{curl})} & \forall \boldsymbol{z} \in \boldsymbol{Z}, \end{cases} \tag{6.8}$$

$$\left|\boldsymbol{f}_{h,\bar{\boldsymbol{y}}}(\boldsymbol{v})\right| \leq 2\nu_0\, \|h\|_{L^\infty(\Omega)}\, \|\bar{\boldsymbol{y}}\|_{\boldsymbol{H}(\operatorname{curl})}\, \|\boldsymbol{v}\|_{\boldsymbol{H}(\operatorname{curl})} \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{curl}),$$

thanks to (A2), (A5), (2.7) and (6.4). □

Lemma 6.3. *Let $\bar{u}, u \in \mathcal{K}$ and $\bar{\boldsymbol{y}} = \boldsymbol{S}(\bar{u})$. Furthermore, let $\boldsymbol{z} = \boldsymbol{z}(\bar{u}, h) \in \boldsymbol{Z}$ denote the unique solution to* (6.5) *with $h = u - \bar{u} \in L^\infty(\Omega)$. Then, the solution operator $\boldsymbol{S} : \mathcal{K} \to \boldsymbol{Z}$ is weakly directionally differentiable at $\bar{u}$ in the direction $u - \bar{u}$, i.e.,*

$$\frac{\boldsymbol{S}(\bar{u} + \tau(u - \bar{u})) - \boldsymbol{S}(\bar{u})}{\tau} \rightharpoonup \boldsymbol{z} \text{ in } \boldsymbol{Z} \text{ as } \tau \downarrow 0.$$

Proof. Let $(\tau_k)_{k\in\mathbb{N}} \subset (0,1]$ be a null sequence. Since $\mathcal{K} \subset H^1(\Omega)$ is convex,

$$u_{\tau_k} := \bar{u} + \tau_k(u - \bar{u}) = \tau_k u + (1-\tau_k)\bar{u} \in \mathcal{K} \quad \forall k \in \mathbb{N}.$$

Thus, for all $k \in \mathbb{N}$, the forward problem (3.2) corresponding to data $(u_{\tau_k}, \boldsymbol{J})$ admits a unique solution $(\boldsymbol{y}_{\tau_k}, \phi_{\tau_k}) \in \boldsymbol{Z} \times H_0^1(\Omega)$. By definition,

$$\begin{cases} \langle A_{u_{\tau_k}}(\boldsymbol{y}_{\tau_k}) - A_{\bar{u}}(\bar{\boldsymbol{y}}), \boldsymbol{v}\rangle + b(\boldsymbol{v}, \phi_{\tau_k} - \bar{\phi}) = 0 & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl}), \\ b(\boldsymbol{y}_{\tau_k} - \bar{\boldsymbol{y}}, \psi) = 0 & \forall \psi \in H_0^1(\Omega). \end{cases} \tag{6.9}$$

Choosing $\boldsymbol{v} = \nabla(\phi_{\tau_k} - \bar{\phi})$ yields that

$$\phi_{\tau_k} - \bar{\phi} = 0 \quad \forall k \in \mathbb{N}. \tag{6.10}$$

For this reason, inserting $\boldsymbol{v} = \boldsymbol{y}_{\tau_k} - \bar{\boldsymbol{y}} \in \boldsymbol{Z}$ in (6.9) and using the abbreviation $\nu_1(\boldsymbol{y}) = \nu_1(|\mathrm{curl}\,\boldsymbol{y}|)$, we obtain

$$\begin{aligned} 0 &= \langle A_{u_{\tau_k}}(\boldsymbol{y}_{\tau_k}) - A_{\bar{u}}(\bar{\boldsymbol{y}}), \boldsymbol{y}_{\tau_k} - \bar{\boldsymbol{y}}\rangle \\ &= \int_\Omega \Big(\nu_0(1-\bar{u})\mathrm{curl}\,(\boldsymbol{y}_{\tau_k} - \bar{\boldsymbol{y}}) + \bar{u}[\nu_1(\boldsymbol{y}_{\tau_k})\mathrm{curl}\,\boldsymbol{y}_{\tau_k} - \nu_1(\bar{\boldsymbol{y}})\mathrm{curl}\,\bar{\boldsymbol{y}}]\Big)\cdot \mathrm{curl}\,(\boldsymbol{y}_{\tau_k} - \bar{\boldsymbol{y}})\,dx \\ &\quad + \tau_k \int_\Omega h(\nu_1(\boldsymbol{y}_{\tau_k}) - \nu_0)\mathrm{curl}\,\boldsymbol{y}_{\tau_k}\cdot \mathrm{curl}(\boldsymbol{y}_{\tau_k} - \bar{\boldsymbol{y}})\,dx\,. \end{aligned}$$

From (2.7), (3.5) and (3.10) it follows that

$$\|\boldsymbol{y}_{\tau_k} - \bar{\boldsymbol{y}}\|_{\boldsymbol{H}(\mathrm{curl})} \le C\,\|\mathrm{curl}\,(\boldsymbol{y}_{\tau_k} - \bar{\boldsymbol{y}})\|_{\boldsymbol{L}^2(\Omega)} \le C\tau_k\,\|h\|_{L^\infty(\Omega)} \quad \forall k \in \mathbb{N}.$$

Consequently, we can extract a non-relabelled subsequence of $(\tau_k)_{k\in\mathbb{N}}$ such that

$$\boldsymbol{y}_{\tau_k} \to \bar{\boldsymbol{y}} \text{ in } \boldsymbol{H}_0(\mathrm{curl}), \quad \mathrm{curl}\,\boldsymbol{y}_{\tau_k}(x) \to \mathrm{curl}\,\bar{\boldsymbol{y}}(x) \text{ a.e. in } \Omega, \quad \frac{(\boldsymbol{y}_{\tau_k} - \bar{\boldsymbol{y}})}{\tau_k} \rightharpoonup \boldsymbol{Y} \text{ in } \boldsymbol{Z}, \tag{6.11}$$

as $k \to \infty$. To identify the equation for $\boldsymbol{Y}$, we notice that (6.1)–(6.2) and (6.9)–(6.10) yield

$$(\boldsymbol{\mathcal{F}}(u_{\tau_k}, \mathrm{curl}\,\boldsymbol{y}_{\tau_k}) - \boldsymbol{\mathcal{F}}(\bar{u}, \mathrm{curl}\,\bar{\boldsymbol{y}}), \mathrm{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl}). \tag{6.12}$$

Invoking the integral form of the mean value theorem, a short calculation shows that

$$\begin{aligned} &\boldsymbol{\mathcal{F}}(u_{\tau_k}, \mathrm{curl}\,\boldsymbol{y}_{\tau_k}) - \boldsymbol{\mathcal{F}}(\bar{u}, \mathrm{curl}\,\boldsymbol{y}_{\tau_k}) + \boldsymbol{\mathcal{F}}(\bar{u}, \mathrm{curl}\,\boldsymbol{y}_{\tau_k}) - \boldsymbol{\mathcal{F}}(\bar{u}, \mathrm{curl}\,\bar{\boldsymbol{y}}) \\ &\quad = \tau_k h(\nu_1(|\mathrm{curl}\,\boldsymbol{y}_{\tau_k}|) - \nu_0)\mathrm{curl}\,\boldsymbol{y}_{\tau_k} + \int_0^1 \nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}}(\bar{u}, \mathrm{curl}\,((1-\theta)\boldsymbol{y}_{\tau_k} + \theta\bar{\boldsymbol{y}}))\mathrm{curl}\,(\boldsymbol{y}_{\tau_k} - \bar{\boldsymbol{y}})\,d\theta. \end{aligned}$$

From (6.11) and the dominated convergence theorem together with (A2) and (A5), we infer that

$$\begin{aligned} &\lim_{k\to\infty} \int_0^1 \nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}}(\bar{u}, \mathrm{curl}\,((1-\theta)\boldsymbol{y}_{\tau_k} + \theta\bar{\boldsymbol{y}}))^\top \mathrm{curl}\,\boldsymbol{v}\,d\theta = \nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}}(\bar{u}, \mathrm{curl}\,\bar{\boldsymbol{y}})^\top \mathrm{curl}\,\boldsymbol{v} && \text{in } \boldsymbol{L}^2(\Omega), \\ &\lim_{k\to\infty} h(\nu_1(|\mathrm{curl}\,\boldsymbol{y}_{\tau_k}|) - \nu_0)\mathrm{curl}\,\boldsymbol{v} = h(\nu_1(|\mathrm{curl}\,\bar{\boldsymbol{y}}|) - \nu_0)\mathrm{curl}\,\boldsymbol{v} && \text{in } \boldsymbol{L}^2(\Omega), \end{aligned}$$

for any $\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl})$. Hence, dividing (6.12) by τ_k and then passing to the limit $k \to \infty$ shows that $\boldsymbol{Y} \in \boldsymbol{Z}$ satisfies

$$\begin{cases} \Big(h(\nu_1(|\mathrm{curl}\,\bar{\boldsymbol{y}}|) - \nu_0)\mathrm{curl}\,\bar{\boldsymbol{y}} + \nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}}(\bar{u}, \mathrm{curl}\,\bar{\boldsymbol{y}})\mathrm{curl}\,\boldsymbol{Y}, \mathrm{curl}\,\boldsymbol{v}\Big)_{\boldsymbol{L}^2(\Omega)} = 0 & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl}), \\ b(\boldsymbol{Y}, \psi) = 0 & \forall \psi \in H_0^1(\Omega). \end{cases}$$

It follows therefore that $\boldsymbol{Y} = \boldsymbol{z}$ is the unique solution to (6.5). Since $\boldsymbol{Y} = \boldsymbol{z}$ is independent of the choice of the subsequence of $(\tau_k)_{k\in\mathbb{N}}$, we conclude that the assertion is valid. □

Corollary 6.4. *Let $\bar{u}, u \in \mathcal{K}$ and $\bar{\boldsymbol{y}} = \boldsymbol{S}(\bar{u})$. Furthermore, let $\boldsymbol{z} = \boldsymbol{z}(\bar{u}, h) \in \boldsymbol{Z}$ denote the unique solution to* (6.5) *with $h = u - \bar{u} \in L^\infty(\Omega)$. Then, the objective functional $J_\varepsilon : \mathcal{K} \to \mathbb{R}$ is directionally differentiable at $\bar{u}$ in the direction $u - \bar{u}$ with the directional derivative*

$$\delta J_\varepsilon(\bar{u}, u - \bar{u}) = (\bar{\boldsymbol{y}} - \boldsymbol{y}_m, \boldsymbol{z})_{\boldsymbol{L}^2(D)} + \alpha \int_\Omega \frac{8\varepsilon}{\pi} \nabla \bar{u} \cdot \nabla (u - \bar{u}) + \frac{8}{\pi\varepsilon}\Big(\frac{1}{2} - \bar{u}\Big)(u - \bar{u})\, dx\,. \tag{6.13}$$

Proof. According to (A6), we have that $J_\varepsilon(v) = J_f(v) + \alpha E_\varepsilon(v)$ with

$$J_f(v) = \frac{1}{2}\, \|\boldsymbol{S}(v) - \boldsymbol{y}_m\|^2_{\boldsymbol{L}(D)}\,, \quad E_\varepsilon(v) = \int_\Omega \frac{8\varepsilon}{2\pi} |\nabla v|^2 + \frac{8}{\pi\varepsilon}\Psi(v)\, dx \quad \forall v \in \mathcal{K}.$$

Thanks to Lemma 6.3, standard arguments imply that

$$\lim_{\tau \downarrow 0} \frac{J_f(\bar{u} + \tau(u - \bar{u})) - J_f(\bar{u})}{\tau} = (\bar{\boldsymbol{y}} - \boldsymbol{y}_m, \boldsymbol{z})_{\boldsymbol{L}^2(D)} \text{ in } \mathbb{R}.$$

Also, in view of (5.2) and since $\bar{u} + \tau(u - \bar{u}) \in \mathcal{K}$ holds for all $\tau \in [0, 1]$, a straightforward computation yields

$$\lim_{\tau \downarrow 0} \frac{E_\varepsilon(\bar{u} + \tau(u - \bar{u})) - E_\varepsilon(\bar{u})}{\tau} = \int_\Omega \frac{8\varepsilon}{\pi} \nabla \bar{u} \cdot \nabla (u - \bar{u}) + \frac{8}{\pi\varepsilon}\Big(\frac{1}{2} - \bar{u}\Big)(u - \bar{u})\, dx \text{ in } \mathbb{R}.$$

In conclusion, the assertion is valid. ☐

For every $\bar{u} \in \mathcal{K}$ with the corresponding state $\bar{\boldsymbol{y}} = \boldsymbol{S}(\bar{u})$, let us now introduce the adjoint system associated with (5.1) as follows:

$$\begin{cases} a^*_{\bar{\boldsymbol{y}}}(\boldsymbol{q}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\bar{\boldsymbol{y}} - \boldsymbol{y}_m, \boldsymbol{v})_{\boldsymbol{L}^2(D)} & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl}), \\ b(\boldsymbol{q}, \psi) = 0 & \forall \psi \in H^1_0(\Omega), \end{cases} \tag{6.14}$$

where the bilinear form $a^*_{\bar{\boldsymbol{y}}} : \boldsymbol{H}_0(\mathrm{curl}) \times \boldsymbol{H}_0(\mathrm{curl}) \to \mathbb{R}$ is defined as

$$a^*_{\bar{\boldsymbol{y}}}(\boldsymbol{q}, \boldsymbol{v}) := (\nabla_{\boldsymbol{s}} \boldsymbol{\mathcal{F}}(\bar{u}, \mathrm{curl}\, \bar{\boldsymbol{y}})^\top \mathrm{curl}\, \boldsymbol{q}, \mathrm{curl}\, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{q}, \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl}).$$

It is easy to see that analogous properties to (6.8) can be shown for $a^*_{\bar{\boldsymbol{y}}}(\cdot, \cdot)$. Hence, using standard results for linear saddle point problems [11, Thm. 1.1, Cor. 1.1], we obtain the following well-posedness result for (6.14).

Lemma 6.5. *Let $\bar{u} \in \mathcal{K}$ with the corresponding state $\bar{\boldsymbol{y}} = \boldsymbol{S}(\bar{u})$. Then, there exists a unique pair $(\boldsymbol{q}, p) \in \boldsymbol{Z} \times H^1_0(\Omega)$ to the adjoint system* (6.14) *associated with $\bar{u}$ and satisfies the estimate*

$$\|\boldsymbol{q}\|_{\boldsymbol{H}(\mathrm{curl})} + \|p\|_{H^1_0(\Omega)} \le C\, \|\bar{\boldsymbol{y}} - \boldsymbol{y}_m\|_{\boldsymbol{L}^2(D)}$$

for a positive constant C depending only on ν_0, $\underline{\nu}$ and Ω.

Employing the adjoint system (6.14) and the established directional differentiability result, let us now prove an optimality system for PFIP (5.1) in form of a variational inequality:

Theorem 6.6. *Let $u^\alpha_\varepsilon \in \mathcal{K}$ be a solution to the PFIP* (5.1) *with the corresponding state $\boldsymbol{y}^\alpha_\varepsilon = \boldsymbol{S}(u^\alpha_\varepsilon)$. Furthermore, let $(\boldsymbol{q}, p) \in \boldsymbol{Z} \times H^1_0(\Omega)$ denote the unique solution to* (6.14) *associated with u^α_ε. Then, it holds that*

$$\begin{aligned} &\int_\Omega \Big((\nu_0 - \nu_1(|\mathrm{curl}\, \boldsymbol{y}^\alpha_\varepsilon|))\, \mathrm{curl}\, \boldsymbol{y}^\alpha_\varepsilon \cdot \mathrm{curl}\, \boldsymbol{q} + \frac{8\alpha}{\pi\varepsilon}\Big(\frac{1}{2} - u^\alpha_\varepsilon\Big)\Big)(u - u^\alpha_\varepsilon)\, dx \\ &\quad + \int_\Omega \frac{8\alpha\varepsilon}{\pi} \nabla u^\alpha_\varepsilon \cdot \nabla (u - u^\alpha_\varepsilon)\, dx \ge 0 \quad \forall u \in \mathcal{K}. \end{aligned} \tag{6.15}$$

Proof. In view of Corollary 6.4 and the optimality of u_ε^α,

$$(\boldsymbol{y}_\varepsilon^\alpha - \boldsymbol{y}_m, \boldsymbol{z})_{\boldsymbol{L}^2(D)} + \alpha \int_\Omega \frac{8\varepsilon}{\pi} \nabla u_\varepsilon^\alpha \cdot \nabla (u - u_\varepsilon^\alpha) + \frac{8}{\pi\varepsilon}\Big(\frac{1}{2} - u_\varepsilon^\alpha\Big)(u - u_\varepsilon^\alpha)\, dx \geq 0 \quad \forall u \in \mathcal{K}, \tag{6.16}$$

where $\boldsymbol{z} \in \boldsymbol{Z}$ denote the unique solution to (6.5) with $\bar{\boldsymbol{y}} = \boldsymbol{y}_\varepsilon^\alpha$ and $h = u - u_\varepsilon^\alpha \in L^\infty(\Omega)$. Inserting $\boldsymbol{v} = \boldsymbol{z}$ in the adjoint system (6.14) leads to

$$(\nabla_{\boldsymbol{s}} \boldsymbol{\mathcal{F}}(u_\varepsilon^\alpha, \operatorname{curl} \boldsymbol{y}_\varepsilon^\alpha)^\top \operatorname{curl} \boldsymbol{q}, \operatorname{curl} \boldsymbol{z})_{\boldsymbol{L}^2(\Omega)} + b(\boldsymbol{z}, p) = (\bar{\boldsymbol{y}} - \boldsymbol{y}_m, \boldsymbol{z})_{\boldsymbol{L}^2(D)}.$$

Meanwhile, substituting $(\boldsymbol{v}, \psi) = (\boldsymbol{q}, p)$ in (6.5) (with $\bar{\boldsymbol{y}} = \boldsymbol{y}_\varepsilon^\alpha$ and $h = u - u_\varepsilon^\alpha$) yields

$$(\nabla_{\boldsymbol{s}} \boldsymbol{\mathcal{F}}(u_\varepsilon^\alpha, \operatorname{curl} \boldsymbol{y}_\varepsilon^\alpha) \operatorname{curl} \boldsymbol{z}, \operatorname{curl} \boldsymbol{q})_{\boldsymbol{L}^2(\Omega)} = f_{u - u_\varepsilon^\alpha, \boldsymbol{y}_\varepsilon^\alpha}(\boldsymbol{q}), \quad b(\boldsymbol{z}, p) = 0.$$

Hence, the above equalities give

$$(\boldsymbol{y}_\varepsilon^\alpha - \boldsymbol{y}_m, \boldsymbol{z})_{\boldsymbol{L}^2(D)} = \int_\Omega (u - u_\varepsilon^\alpha)(\nu_0 - \nu_1(|\operatorname{curl} \boldsymbol{y}_\varepsilon^\alpha|)) \operatorname{curl} \boldsymbol{y}_\varepsilon^\alpha \cdot \operatorname{curl} \boldsymbol{q}\, dx\,, \tag{6.17}$$

and (6.15) is an immediate consequence of (6.16)–(6.17). □

6.2 Optimality system via shape calculus and its convergence

In this section, we derive an alternative optimality system for PFIP (5.1) via the domain variation technique of shape calculus. Our main result is the convergence of this system as $\varepsilon \to 0$, which leads to an optimality system for TVIP (4.1).

Assumption 6.2. *In addition to Assumption 6.1, we further assume*

(A7) *The domain Ω is either a convex domain or a domain with $C^{1,1}$-boundary.*

(A8) *The prescribed current density $\boldsymbol{J}$ satisfies $\boldsymbol{J} \in \boldsymbol{H}^1(\Omega)$ and the measurement vector potential $\boldsymbol{y}_m$ additionally satisfies $\boldsymbol{y}_m \in \boldsymbol{H}^1(\Omega)$.*

We remark that (A8) is required due to the domain variation methodology we employ to derive optimality conditions. Moreover, (A7) together with (2.4) and (2.6) implies that

$$\|\boldsymbol{f}\|_{\boldsymbol{H}^1(\Omega)} \leq C\Big(\|\operatorname{curl} \boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\operatorname{div} \boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \Big) \quad \forall \boldsymbol{f} \in \boldsymbol{H}_0(\operatorname{curl}) \cap \boldsymbol{H}(\operatorname{div}),$$

and hence $\boldsymbol{S}(u) \in \boldsymbol{H}^1(\Omega)$ for any $u \in L^1(\Omega; [0,1])$. This improved regularity is needed to prove the differentiability of certain transformed solutions (see Lemma 6.9) in preparation for the main result (Theorem 6.11).

The optimality conditions derived in this section involves domain variations, which is performed with admissible transformations and their corresponding velocity fields.

Definition 6.1. *The space $\mathcal{V}_{\mathrm{ad}}$ of admissible velocity fields is defined as the set of all $V \in C^0([-\tau, \tau] \times \overline{\Omega}, \mathbb{R}^3)$ where $\tau > 0$ is a fixed small constant such that for all $t \in [-\tau, \tau]$,*

(V1) *$V(t, \cdot) : \overline{\Omega} \to \mathbb{R}^3$, $V(t, \cdot) \in C_c^2(\overline{\Omega}, \mathbb{R}^3)$ and there exists $C > 0$ such that*

$$\|V(\cdot, y) - V(\cdot, z)\|_{C^0([-\tau,\tau],\mathbb{R}^3)} \leq C\,|y - z| \quad \forall y, z \in \overline{\Omega}.$$

(V2) *$V(t, x) \cdot \boldsymbol{n}(x) = 0$ for all $x \in \partial\Omega$.*

Then, the space $\mathcal{T}_{\text{ad}}$ of admissible transformations is defined as the set of corresponding solutions to the ordinary differential equation

$$\partial_t T_t(x) = V(T_t(x)), \quad T_0(x) = x,$$

for $V \in \mathcal{V}_{ad}$, which yields a mapping $T : [-\tau_0, \tau_0] \times \overline{\Omega} \to \overline{\Omega}$, $T_t(x) := T(t,x)$, for some $\tau_0 \in (0,\tau)$ sufficiently small.

Remark 6.1. *By [38, Thm. 2.18] the transformation $T : [-\tau_0, \tau_0] \times \overline{\Omega} \to \overline{\Omega}$ corresponding to an admissible velocity field $V \in \mathcal{V}_{ad}$ is bijective and satisfies the properties*

$$T \text{ and } T^{-1} \in C^0([-\tau_0, \tau_0]; C^2(\overline{\Omega}; \mathbb{R}^3)),$$

which implies

$$\|T_t - \mathrm{I}\|_{\boldsymbol{L}^\infty(\Omega)} \to 0, \quad \left\|T_t^{-1} - \mathrm{I}\right\|_{\boldsymbol{L}^\infty(\Omega)} \to 0 \text{ as } t \to 0,$$

where $\mathrm{I}(x) = x$ is the identity map. Moreover, by [38, Lem. 2.31 and Prop. 2.35], for any compact set $U \subset \Omega$, it holds that

$$\left\|t^{-1}(\mathrm{D}T_t - \mathbb{I}) - \mathrm{D}V(0)\right\|_{\boldsymbol{W}^{1,\infty}(U)} \to 0, \tag{6.18}$$

$$\left\|t^{-1}(\xi_t - 1) - \operatorname{div} V(0)\right\|_{W^{1,\infty}(U)} \to 0, \tag{6.19}$$

$$\left\|t^{-1}(T_t - \mathrm{I}) - V(0)\right\|_{\boldsymbol{L}^\infty(U)} \to 0 \tag{6.20}$$

as $t \to 0$, where $\mathbb{I}$ is the identity matrix, $\mathrm{D}T_t$ is the Jacobian matrix of T_t and $\xi(t,x) := \xi_t(x) := \det \mathrm{D}T_t(x)$.

While we recognise $\overline{T_t(\Omega)} = \overline{\Omega}$ for all $t \in [-\tau_0, \tau_0]$ due to (V2), in some of the calculations below we use the notation $\Omega_t = T_t(\Omega)$ for better clarity. Through the relation

$$\nabla(f \circ T_t) = [(\nabla f) \circ T_t]\mathrm{D}T_t = \mathrm{D}T_t^\top (\nabla f) \circ T_t, \tag{6.21}$$

it is clear that $f \in H^1(\Omega_t)$ iff $f \circ T_t \in H^1(\Omega)$. In contrast, composition with the mapping T_t or T_t^{-1} modifies the curl operator in an undesirable way (see [26, §3] or [27, §4.4])

$$\widehat{\operatorname{curl}}\, \boldsymbol{y} = \xi_t \mathrm{D}T_t^{-1}\Big(\mathrm{D}T_t^{-\top} \times \mathrm{D}(\boldsymbol{y} \circ T_t^{-1}) \circ T_t\Big) \quad \text{for } \boldsymbol{y} : \Omega_t \to \mathbb{R}^3, \tag{6.22}$$

where $\widehat{\operatorname{curl}}$ is the curl operator with respect to the transformed variables $\widehat{x} = T_t(x)$ and the tensor product $\boldsymbol{A} \times \boldsymbol{B}$ for two second order tensors $\boldsymbol{A} = (a_{kl})_{1 \le k,l \le 3}$ and $\boldsymbol{B} = (b_{kl})_{1 \le k,l \le 3}$ is defined as $\boldsymbol{A} \times \boldsymbol{B} = \sum_{j,k=1}^3 \epsilon_{ijk} \sum_{l=1}^3 a_{kl} b_{lj}$ with the antisymmetric tensor ϵ_{ijk}. In particular, $\boldsymbol{y} \in \boldsymbol{H}(\text{curl};\Omega)$ may not imply $\boldsymbol{y} \circ T_t \in \boldsymbol{H}(\text{curl};\Omega_t)$. Although via (A7) the solution to the state equation belongs to $\boldsymbol{H}^1(\Omega)$, the expression (6.22) for the curl operator is still difficult to work with. On the other hand, the transformation

$$\widehat{\boldsymbol{y}} := \mathrm{D}T_t^{-\top} \boldsymbol{y} \circ T_t^{-1} \quad \text{for } \boldsymbol{y} : \Omega \to \mathbb{R}^3 \tag{6.23}$$

is curl preserving, i.e., it holds that (see [31, Cor. 3.58])

$$(\widehat{\operatorname{curl}}\, \widehat{\boldsymbol{y}}) \circ T_t = \xi_t^{-1} \mathrm{D}T_t \operatorname{curl} \boldsymbol{y} \quad \text{for } \boldsymbol{y} : \Omega \to \mathbb{R}^3, \tag{6.24}$$

and so $\operatorname{curl} \boldsymbol{y} \in \boldsymbol{L}^2(\Omega)$ if and only if $\widehat{\operatorname{curl}}\, \widehat{\boldsymbol{y}} \in \boldsymbol{L}^2(\Omega_t)$. Another advantage of (6.23) is that, for given $\boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{H}(\text{curl})$ with $\widehat{\boldsymbol{y}} := \mathrm{D}T_t^{-\top} \boldsymbol{y} \circ T_t^{-1}$ and $\widehat{\boldsymbol{z}} := \mathrm{D}T_t^{-\top} \boldsymbol{z} \circ T_t^{-1}$, we have

$$\int_{\Omega_t} \widehat{\operatorname{curl}}\, \widehat{\boldsymbol{y}} \cdot \widehat{\boldsymbol{z}} \, d\widehat{x} = \int_\Omega \xi_t \big(\xi_t^{-1} \mathrm{D}T_t \operatorname{curl} \boldsymbol{y} \cdot \mathrm{D}T_t^{-\top} \boldsymbol{z}\big)\, dx = (\operatorname{curl} \boldsymbol{y}, \boldsymbol{z})_{\boldsymbol{L}^2(\Omega)}.$$

Hence, $\boldsymbol{y} \in \boldsymbol{H}_0(\text{curl};\Omega)$ iff $\widehat{\boldsymbol{y}} = \mathrm{D}T_t^{-\top}\boldsymbol{y} \circ T_t^{-1} \in \boldsymbol{H}_0(\text{curl};\Omega_t)$ and $\boldsymbol{y} \in \boldsymbol{H}_0(\text{curl};\Omega_t)$ iff $\mathrm{D}T_t^{\top}\boldsymbol{y} \circ T_t \in \boldsymbol{H}_0(\text{curl};\Omega)$. Therefore, in the sequel we focus mainly on the transformation (6.23) for the vector potential $\boldsymbol{y}$. Furthermore, we use the following notations

$$A(t) := \frac{1}{\xi_t}\mathrm{D}T_t^{\top}\mathrm{D}T_t, \quad A^{-1}(t) = \xi_t\mathrm{D}T_t^{-1}\mathrm{D}T_t^{-\top}, \quad B(t) := \frac{1}{\xi_t}\mathrm{D}T_t, \quad B^{-1}(t) = \xi_t\mathrm{D}T_t^{-1}.$$

It is well-known that $A(t)$, $A^{-1}(t)$, $B(t)$ and $B^{-1}(t)$ are differentiable at $t = 0$ with derivatives [15, Chap. 9, Thm. 4.1 and equ. (4.26)]

$$\begin{aligned} A'(0) &= -\mathrm{div}V(0)\mathbb{I} + \mathrm{D}V(0) + (\mathrm{D}V(0))^{\top} = -(A^{-1})'(0), \\ B'(0) &= -\mathrm{div}V(0)\mathbb{I} + \mathrm{D}V(0) = -(B^{-1})'(0). \end{aligned}$$

Then, by Remark 6.1, there exists a positive constant C such that for all $t \in (-\tau_0, \tau_0)$ with τ_0 sufficiently small,

$$\begin{aligned} &\left\|t^{-1}(A(t) - \mathbb{I})\right\|_{\boldsymbol{L}^\infty(\Omega)} + \left\|t^{-1}\mathrm{div}(A^{-1}(t) - \mathbb{I})\right\|_{\boldsymbol{L}^\infty(\Omega)} + \left\|t^{-1}(\xi_t - 1)\right\|_{\boldsymbol{L}^\infty(\Omega)} \\ &\quad + \left\|t^{-1}(B(t) - \mathbb{I})\right\|_{\boldsymbol{L}^\infty(\Omega)} + \left\|t^{-1}(B^{-1}(t) - \mathbb{I})\right\|_{\boldsymbol{L}^\infty(\Omega)} + \left\|t^{-1}(T_t - \mathrm{I})\right\|_{\boldsymbol{L}^\infty(\Omega)} \le C. \end{aligned} \tag{6.25}$$

Lemma 6.7. *Let $V \in \mathcal{V}_{ad}$ be given and $T \in \mathcal{T}_{ad}$ be the corresponding transformation. For any $t \in [-\tau_0, \tau_0]$ and $u \in BV(\Omega, \{0,1\})$, let $u^t := u \circ T_t^{-1}$. Then, it holds that*

$$\left\|u^t - u\right\|_{L^1(\Omega)} \to 0 \text{ as } t \to 0.$$

Consequently, by Theorem 3.2 and (2.4), if $(\boldsymbol{y}, \phi), (\boldsymbol{y}^t, \phi^t) \in \boldsymbol{H}^1(\Omega) \times H_0^1(\Omega)$ are unique solutions of (3.2) corresponding to data $(u, \boldsymbol{J})$ and $(u^t, \boldsymbol{J})$, respectively, then

$$\boldsymbol{y}^t \to \boldsymbol{y} \text{ in } \boldsymbol{H}^1(\Omega), \quad \phi^t \rightharpoonup \phi \text{ in } H_0^1(\Omega).$$

Proof. Suppose for the moment that $u \in W^{1,p}(\Omega)$ for $p \ge 1$. For $x \in \Omega$, the relation

$$u^t(x) - u(x) = \int_0^1 \nabla u(x + s(T_t^{-1}(x) - x)) \cdot (T_t^{-1}(x) - x)\, ds$$

and the change of variable $z = [(1-s)\mathrm{I} + sT_t^{-1}](x)$ yields

$$\begin{aligned} \left\|u^t - u\right\|_{L^p(\Omega)}^p &\le \left\|T_t^{-1} - \mathrm{I}\right\|_{\boldsymbol{L}^\infty(\Omega)}^p \int_0^1 \int_\Omega \left|\nabla u(x + s(T_t^{-1}(x) - x))\right|^p dx\ ds \\ &= \left\|T_t^{-1} - \mathrm{I}\right\|_{\boldsymbol{L}^\infty(\Omega)}^p \int_0^1 \int_\Omega |\nabla u(z)|^p \det(\mathrm{D}((1-s)\mathrm{I} + sT_t^{-1})^{-1})\, dz\, ds \\ &\le c_* \left\|T_t^{-1} - \mathrm{I}\right\|_{\boldsymbol{L}^\infty(\Omega)}^p \left\|\nabla u\right\|_{\boldsymbol{L}^p(\Omega)}^p \to 0 \text{ as } t \to 0, \end{aligned} \tag{6.26}$$

where the regularity of the corresponding velocity field $V \in \mathcal{V}_{ad}$ implies the boundedness of the determinant, see [38, p. 69]. Now, for a fixed $u \in BV(\Omega, \{0,1\})$, by following the start of the proof of Theorem 5.2, we can construct a sequence $(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{K} \subset H^1(\Omega)$ such that $u_\varepsilon \to u$ strongly in $L^1(\Omega)$ as $\varepsilon \to 0$ and $\limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \le TV(u)$. Given an arbitrary $\zeta > 0$, there exists $\varepsilon_0 = \varepsilon_0(\zeta) > 0$ such that

$$\left\|u_{\varepsilon_0} - u\right\|_{L^1(\Omega)} < \frac{\zeta}{3} \quad \text{and} \quad E_{\varepsilon_0}(u_{\varepsilon_0}) \le TV(u) + 1.$$

Then, as $\left\|T_t^{-1} - \mathrm{I}\right\|_{\boldsymbol{L}^\infty(\Omega)} \to 0$ as $t \to 0$, there exists $\tau_1 = \tau_1(\zeta, \varepsilon_0) > 0$ such that

$$\left\|T_t^{-1} - \mathrm{I}\right\|_{\boldsymbol{L}^\infty(\Omega)} < \frac{\zeta}{3}\sqrt{\frac{\varepsilon_0}{2(TV(u)+1)\,|\Omega|\,c_*^2}} \quad \text{for all } t \in (-\tau_1, \tau_1),$$

where c_* is the constant in (6.26). Hence, for $t \in (-\tau_1, \tau_1)$ we find that

$$\|u^t - u\|_{L^1(\Omega)} \le \|u^t - u^t_{\varepsilon_0}\|_{L^1(\Omega)} + \|u_{\varepsilon_0} - u\|_{L^1(\Omega)} + \|u^t_{\varepsilon_0} - u_{\varepsilon_0}\|_{L^1(\Omega)}$$
$$\le \frac{2\zeta}{3} + c_* \|\nabla u_{\varepsilon_0}\|_{\boldsymbol{L}^1(\Omega)} \|T_t^{-1} - \mathrm{I}\|_{L^\infty(\Omega)} < \zeta,$$

since $\varepsilon_0 \|\nabla u_{\varepsilon_0}\|^2_{\boldsymbol{L}^2(\Omega)} \le 2(TV(u) + 1)$ and $\|\nabla u_{\varepsilon_0}\|_{\boldsymbol{L}^1(\Omega)} \le |\Omega|^{1/2} \|\nabla u_{\varepsilon_0}\|_{\boldsymbol{L}^2(\Omega)}$. Arbitrariness of ζ yields $u^t \to u$ in $L^1(\Omega)$. □

Lemma 6.8. *Let $V \in \mathcal{V}_{ad}$ be given and $T \in \mathcal{T}_{ad}$ be the corresponding transformation. For $\boldsymbol{J} \in \boldsymbol{H}^1(\Omega)$ it holds that*

$$\|t^{-1}(\boldsymbol{J} \circ T_t - \boldsymbol{J}) - (\nabla \boldsymbol{J})V(0)\|_{\boldsymbol{L}^2(\Omega)} \to 0 \text{ as } t \to 0.$$

Consequently, there exists a positive constant C such that for all $t \in (-\tau_0, \tau_0)$ with τ_0 sufficiently small,

$$\|t^{-1}(\boldsymbol{J} \circ T_t - \boldsymbol{J})\|_{\boldsymbol{L}^2(\Omega)} \le C.$$

Proof. Fix $i \in \{1, 2, 3\}$ and denote by J_i the ith component of $\boldsymbol{J}$. For $t \in (-\tau_0, \tau_0)$, and $x \in \Omega$, thanks to the relation

$$t^{-1}((J_i \circ T_t)(x) - J_i(x)) - \nabla J_i(x) \cdot V(0, x)$$
$$= \int_0^1 [\nabla J_i(x + s(T_t(x) - x)) - \nabla J_i(x)] \cdot \frac{T_t(x) - x}{t}\, ds + \nabla J_i(x) \cdot \Big(\frac{T_t(x) - x}{t} - V(0, x)\Big),$$

we infer, similar to the proof of Lemma 6.7, that

$$\|t^{-1}(J_i \circ T_t - J_i) - \nabla J_i \cdot V(0)\|^2_{L^2(\Omega)}$$
$$\le \|t^{-1}(T_t - \mathrm{I})\|^2_{\boldsymbol{L}^\infty(\Omega)} \int_0^1 \int_\Omega |\nabla J_i(x + s(T_t(x) - x)) - \nabla J_i(x)|^2 \, dx \, ds$$
$$+ \|\nabla J_i\|^2_{\boldsymbol{L}^2(\Omega)} \|t^{-1}(T_t - \mathrm{I}) - V(0)\|^2_{\boldsymbol{L}^\infty(\Omega)}.$$

The second term on the right-hand side tends to zero as $t \to 0$ by Remark 6.1, while the first term tends to zero as $t \to 0$ by (6.25) and the dominated convergence theorem. □

Lemma 6.9. *Let $V \in \mathcal{V}_{ad}$ be an admissible velocity with the corresponding transformation $T \in \mathcal{T}_{ad}$. Let $u \in L^1(\Omega; [0, 1])$ and define $u^t := u \circ T_t^{-1}$ with the unique solution $(\boldsymbol{y}^t, \phi^t) \in \boldsymbol{H}^1(\Omega_t) \times H^1_0(\Omega_t)$ to (3.2) corresponding to data $(u^t, \boldsymbol{J})$, and we denote $(u^0, \boldsymbol{y}^0, \phi^0)$ simply by $(u, \boldsymbol{y}, \phi)$. Then, there exists $\tau_0 > 0$ such that the mappings*

$$[-\tau_0, \tau_0] \ni t \mapsto \mathrm{D}T_t^\top(\boldsymbol{y}^t \circ T_t) \in \boldsymbol{H}^1(\Omega), \quad [-\tau_0, \tau_0] \ni t \mapsto (\phi^t \circ T_t) \in H^1_0(\Omega)$$

are Gâteaux differentiable at $t = 0$, with Gâteaux derivatives $\dot{\boldsymbol{Y}}[V] := \frac{d}{dt}(\mathrm{D}T_t^\top(\boldsymbol{y}^t \circ T_t))\big|_{t=0} \in \boldsymbol{H}^1(\Omega)$ and $\dot{\phi}[V] := \frac{d}{dt}(\phi^t \circ T_t)\big|_{t=0} \in H^1_0(\Omega)$ that are the unique solution to the linear saddle point problem

$$\begin{cases} a_{\boldsymbol{y}}(\dot{\boldsymbol{Y}}[V], \boldsymbol{v}) + b(\boldsymbol{v}, \dot{\phi}[V]) = G_{u,\boldsymbol{y},\phi}(\boldsymbol{v}) & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl}), \\ b(\dot{\boldsymbol{Y}}[V], \psi) = H_{\boldsymbol{y}}(\psi) & \forall \psi \in H^1_0(\Omega), \end{cases} \tag{6.27}$$

where $a_{\boldsymbol{y}}(\cdot,\cdot)$ *is defined as in* (6.6), *while* $G_{u,\boldsymbol{y},\phi} : \boldsymbol{H}_0(\mathrm{curl}) \to \mathbb{R}$ *and* $H_{\boldsymbol{y}} : H_0^1(\Omega) \to \mathbb{R}$ *are defined as*

$$\begin{aligned} G_{u,\boldsymbol{y},\phi}(\boldsymbol{v}) &:= \int_\Omega A'(0)\boldsymbol{v}\cdot\nabla\phi - (\nu_0(1-u)+\nu_1(|\mathrm{curl}\,\boldsymbol{y}|)u)(A'(0)\,\mathrm{curl}\,\boldsymbol{y})\cdot\mathrm{curl}\,\boldsymbol{v}\,dx \\ &\quad - \int_\Omega u\frac{\nu_1'(|\mathrm{curl}\,\boldsymbol{y}|)}{|\mathrm{curl}\,\boldsymbol{y}|}(\mathrm{curl}\,\boldsymbol{y}\cdot B'(0)\mathrm{curl}\,\boldsymbol{y})\mathrm{curl}\,\boldsymbol{y}\cdot\mathrm{curl}\,\boldsymbol{v}\,dx \qquad (6.28) \\ &\quad + \int_\Omega \boldsymbol{v}\cdot(\nabla\boldsymbol{J})V(0) - B'(0)\boldsymbol{v}\cdot\boldsymbol{J}\,dx\,, \\ H_{\boldsymbol{y}}(\psi) &:= \int_\Omega A'(0)\boldsymbol{y}\cdot\nabla\psi\,dx\,. \end{aligned}$$

Proof. Thanks to the smoothness of $V \in \mathcal{V}_{ad}$, we can choose τ_0 sufficiently small so that there exist positive constants $c_0,\dots,c_4$, independent of t such that

$$c_0 \le \xi_t \le c_1, \quad c_2|\zeta|^2 \le A(t)\zeta\cdot\zeta \le c_3|\zeta|^2, \quad |B(t)| + \left|B^{-1}(t)\right| \le c_4 \qquad (6.29)$$

hold for all $t \in [-\tau_0,\tau_0]$ and for all non-zero $\zeta \in \mathbb{R}^3$. We consider the functional

$$\begin{aligned} &F : [-\tau_0,\tau_0]\times\boldsymbol{H}_0(\mathrm{curl})\times H_0^1(\Omega) \to \boldsymbol{H}_0(\mathrm{curl})^*\times H^{-1}(\Omega), \\ &F(t,(\boldsymbol{y},\phi)) = (F_1(t,(\boldsymbol{y},\phi)),F_2(t,(\boldsymbol{y},\phi))) \in \boldsymbol{H}_0(\mathrm{curl})^*\times H^{-1}(\Omega), \end{aligned}$$

defined as

$$\begin{aligned} F_1(t,(\boldsymbol{y},\phi))[\boldsymbol{v}] &:= \int_\Omega (\nu_0(1-u)+u\nu_1(|B(t)\mathrm{curl}\,\boldsymbol{y}|))A(t)\mathrm{curl}\,\boldsymbol{y}\cdot\mathrm{curl}\,\boldsymbol{v}\,dx \\ &\quad + \int_\Omega A^{-1}(t)\boldsymbol{v}\cdot\nabla\phi - B^{-\top}(t)\boldsymbol{v}\cdot(\boldsymbol{J}\circ T_t)\,dx\,, \\ F_2(t,(\boldsymbol{y},\phi))[\psi] &:= \int_\Omega \boldsymbol{y}\cdot A^{-1}(t)\nabla\psi\,dx\,. \end{aligned}$$

Let $(\boldsymbol{y}^t,\phi^t) \in \boldsymbol{H}^1(\Omega_t)\times H_0^1(\Omega_t)$ denote the unique solution to (3.2) on Ω_t corresponding to data $(u^t,\boldsymbol{J})$, which by (3.10) satisfies

$$\|\boldsymbol{y}^t\|_{\boldsymbol{H}^1(\Omega_t)} + \|\phi^t\|_{H_0^1(\Omega_t)} \le C\,\|\boldsymbol{J}\|_{\boldsymbol{L}^2(\Omega)} \qquad (6.30)$$

for a positive constant C depending only on ν_0, $\underline{\nu}$, $\overline{\nu}$ and Ω. For convenience, we set $\widetilde{\boldsymbol{y}}^t := \mathrm{D}T_t^\top\boldsymbol{y}^t\circ T_t$, so that from (6.23) and (6.24) we have the relation $B(t)\mathrm{curl}\,\widetilde{\boldsymbol{y}}^t = (\widehat{\mathrm{curl}}\,\boldsymbol{y}^t)\circ T_t$. By the classical transformation theorem and (6.21), for every $\psi \in H_0^1(\Omega)$, it holds that

$$F_2(t,(\widetilde{\boldsymbol{y}}^t,\phi^t\circ T_t))[\psi] = \int_\Omega \xi_t(\boldsymbol{y}^t\circ T_t)\cdot\mathrm{D}T_t^{-\top}\nabla\psi\,dx = \int_{\Omega_t}\boldsymbol{y}^t\cdot\nabla(\psi\circ T_t^{-1})\,d\widehat{x} = 0.$$

Furthermore, for every $\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl};\Omega) \Leftrightarrow \widehat{\boldsymbol{v}} := \mathrm{D}T_t^{-\top}\boldsymbol{v}\circ T_t^{-1} \in \boldsymbol{H}_0(\mathrm{curl};\Omega_t)$, we obtain due to (6.24) that

$$\begin{aligned} &F_1(t,(\widetilde{\boldsymbol{y}}^t,\phi^t\circ T_t))[\boldsymbol{v}] \\ &\quad = \int_{\Omega_t}(\nu_0(1-u^t)+u^t\nu_1(|\widehat{\mathrm{curl}}\,\boldsymbol{y}^t|))\,\widehat{\mathrm{curl}}\,\boldsymbol{y}^t\cdot\widehat{\mathrm{curl}}\,\widehat{\boldsymbol{v}} + \widehat{\boldsymbol{v}}\cdot\nabla\phi^t - \widehat{\boldsymbol{v}}\cdot\boldsymbol{J}\,d\widehat{x} = 0. \end{aligned}$$

Hence,

$$F(t,(\widetilde{\boldsymbol{y}}^t,\phi^t\circ T_t)) = 0 \quad \forall t \in [-\tau_0,\tau_0].$$

Let $\Upsilon_t := \widetilde{\boldsymbol{y}}^t - \boldsymbol{y} \in \boldsymbol{H}_0(\mathrm{curl};\Omega) \cap \boldsymbol{H}^1(\Omega)$ and $\Phi_t := \phi^t \circ T_t - \phi \in H_0^1(\Omega)$. For arbitrary $\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl};\Omega)$ and $\psi \in H_0^1(\Omega)$, we find that the difference $0 = F(t,(\widetilde{\boldsymbol{y}}^t, \phi^t \circ T_t))[(\boldsymbol{v},\psi)] - F(0,(\boldsymbol{y},\phi))[(\boldsymbol{v},\psi)]$ reads as (suppressing the t-dependence in the matrices and in ξ_t)

$$0 = \int_\Omega (A^{-1}\widetilde{\boldsymbol{y}}^t - \boldsymbol{y}) \cdot \nabla\psi \, dx = \int_\Omega A^{-1}\Upsilon_t \cdot \nabla\psi + (A^{-1} - \mathbb{I})\boldsymbol{y} \cdot \nabla\psi \, dx\,, \tag{6.31}$$

$$\begin{aligned} 0 = &\int_\Omega -B^{-\top}\boldsymbol{v} \cdot (\boldsymbol{J} \circ T_t - \boldsymbol{J}) - \boldsymbol{J} \cdot (B^{-\top} - \mathbb{I})\boldsymbol{v} + A^{-1}\boldsymbol{v} \cdot \nabla\Phi_t + (A^{-1} - \mathbb{I})\boldsymbol{v} \cdot \nabla\phi \, dx \\ &+ \int_\Omega \nu_0(1-u)\Big(A\mathrm{curl}\,\Upsilon_t \cdot \mathrm{curl}\,\boldsymbol{v} + (A-\mathbb{I})\mathrm{curl}\,\boldsymbol{y} \cdot \mathrm{curl}\,\boldsymbol{v}\Big)\, dx \\ &+ \int_\Omega \xi u\Big(\big[\nu_1(|B\mathrm{curl}\,\widetilde{\boldsymbol{y}}^t|)B\mathrm{curl}\,\widetilde{\boldsymbol{y}}^t - \nu_1(|B\mathrm{curl}\,\boldsymbol{y}|)B\mathrm{curl}\,\boldsymbol{y}\big] \cdot B\mathrm{curl}\,\boldsymbol{v}\Big)\, dx \\ &+ \int_\Omega \xi u\Big(\nu_1(|B\mathrm{curl}\,\boldsymbol{y}|)B\mathrm{curl}\,\boldsymbol{y} - \nu_1(|\mathrm{curl}\,\boldsymbol{y}|)\mathrm{curl}\,\boldsymbol{y}\Big) \cdot B\mathrm{curl}\,\boldsymbol{v}\, dx \\ &+ \int_\Omega (\xi - 1)u\nu_1(|\mathrm{curl}\,\boldsymbol{y}|)\mathrm{curl}\,\boldsymbol{y} \cdot B\mathrm{curl}\,\boldsymbol{v} + u\nu_1(|\mathrm{curl}\,\boldsymbol{y}|)\mathrm{curl}\,\boldsymbol{y} \cdot (B - \mathbb{I})\mathrm{curl}\,\boldsymbol{v}\, dx \\ =:&\, I_1 + I_2 + I_3 + I_4 + I_5, \end{aligned} \tag{6.32}$$

where in the above we used the relation $A(t) = \xi_t B^\top(t)B(t)$. We first consider $\boldsymbol{v} = \nabla\Phi_t$ in (6.32) and obtain due to $\mathrm{curl}\,\nabla \equiv 0$,

$$\|\nabla\Phi_t\|_{\boldsymbol{L}^2(\Omega)} \le C\Big(\big\|B^{-\top} - \mathbb{I}\big\|_{\boldsymbol{L}^\infty(\Omega)} + \big\|A^{-1} - \mathbb{I}\big\|_{\boldsymbol{L}^\infty(\Omega)} + \|\boldsymbol{J} \circ T_t - \boldsymbol{J}\|_{\boldsymbol{L}^2(\Omega)} \Big), \tag{6.33}$$

for some positive constant C depending only on $\|\boldsymbol{J}\|_{\boldsymbol{L}^2(\Omega)}$, c_3 and c_4 thanks to (3.10) and (6.29). By virtue of the bounds (3.10), (6.29) and (6.30), as well as the property (3.6) we can infer the following:

$$\begin{aligned} I_1 &\ge -C\Big(\big\|B^{-\top} - \mathbb{I}\big\|_{\boldsymbol{L}^\infty(\Omega)} + \big\|A^{-1} - \mathbb{I}\big\|_{\boldsymbol{L}^\infty(\Omega)} + \|\boldsymbol{J} \circ T_t - \boldsymbol{J}\|_{\boldsymbol{L}^2(\Omega)} \Big) \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \int_\Omega A^{-1}\boldsymbol{v} \cdot \nabla\Phi_t \, dx \\ I_2 &\ge \int_\Omega (1-u)\nu_0 A\mathrm{curl}\,\Upsilon_t \cdot \mathrm{curl}\,\boldsymbol{v}\, dx - C\,\|A - \mathbb{I}\|_{\boldsymbol{L}^\infty(\Omega)} \|\mathrm{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}\,, \\ I_4 &\ge -C\,\|B - \mathbb{I}\|_{\boldsymbol{L}^\infty(\Omega)} \|\mathrm{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}\,, \\ I_5 &\ge -C\Big(\|\xi - 1\|_{L^\infty(\Omega)} + \|B - \mathbb{I}\|_{\boldsymbol{L}^\infty(\Omega)} \Big) \|\mathrm{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}\,, \end{aligned}$$

for positive constants C depending only on $c_0, \ldots, c_4$, ν_0, $\underline{\nu}$, $\overline{\nu}$, Ω and $\|\boldsymbol{J}\|_{\boldsymbol{L}^2(\Omega)}$. For the term I_3, when choosing $\boldsymbol{v} = \Upsilon_t = \widetilde{\boldsymbol{y}}^t - \boldsymbol{y}$, we employ (3.5) and the relation $A = \xi B^\top B$ to deduce that

$$I_3 \ge \int_\Omega \xi u \underline{\nu} |B\mathrm{curl}\,(\widetilde{\boldsymbol{y}}^t - \boldsymbol{y})|^2 = \int_\Omega u\underline{\nu} A\mathrm{curl}\,\Upsilon_t \cdot \mathrm{curl}\,\Upsilon_t \, dx$$

Hence, choosing $\boldsymbol{v} = \Upsilon_t$ in (6.32), and using the positivity of ξ_t, the positive-definiteness of $A(t)$ and the lower bound $\nu_0 \ge \underline{\nu}$, as well as the estimates above leads to

$$\begin{aligned} \|\mathrm{curl}\,\Upsilon_t\|^2_{\boldsymbol{L}^2(\Omega)} \le\, & C\Big(\|B - \mathbb{I}\|_{\boldsymbol{L}^\infty(\Omega)} + \big\|A^{-1} - \mathbb{I}\big\|_{\boldsymbol{L}^\infty(\Omega)} + \|\boldsymbol{J} \circ T_t - \boldsymbol{J}\|_{\boldsymbol{L}^2(\Omega)} \Big) \|\Upsilon_t\|_{\boldsymbol{L}^2(\Omega)} \\ &+ C\Big(\|A - \mathbb{I}\|_{\boldsymbol{L}^\infty(\Omega)} + \|B - \mathbb{I}\|_{\boldsymbol{L}^\infty(\Omega)} + \|\xi - 1\|_{L^\infty(\Omega)} \Big) \|\mathrm{curl}\,\Upsilon_t\|_{\boldsymbol{L}^2(\Omega)} \\ &+ C\left| \int_\Omega A^{-1}\Upsilon_t \cdot \nabla\Phi_t \, dx \right|. \end{aligned} \tag{6.34}$$

On the one hand, from (6.31) and the estimates (3.10) and (6.33), we see that

$$\begin{aligned} \left| \int_\Omega A^{-1}\Upsilon_t \cdot \nabla\Phi_t \, dx \right| &= \left| \int_\Omega (\mathbb{I} - A^{-1})\boldsymbol{y} \cdot \nabla\Phi_t \, dx \right| \\ &\le C\Big(\big\|\mathbb{I} - A^{-1}\big\|^2_{\boldsymbol{L}^\infty(\Omega)} + \big\|\mathbb{I} - B^{-\top}\big\|^2_{\boldsymbol{L}^\infty(\Omega)} + \|\boldsymbol{J} \circ T_t - \boldsymbol{J}\|^2_{\boldsymbol{L}^2(\Omega)} \Big). \end{aligned} \tag{6.35}$$

On the other hand, we infer from the first equality of (6.31) that

$$\int_\Omega \Upsilon_t \cdot \nabla\psi \, dx = \int_\Omega ((\mathbb{I} - A^{-1})\widetilde{\boldsymbol{y}}^t) \cdot \nabla\psi \, dx \quad \forall \psi \in H_0^1(\Omega).$$

Since $\mathrm{D}T_t^\top \boldsymbol{y}^t \circ T_t \in \boldsymbol{H}^1(\Omega)$, see the discussion after Remark 6.1, it holds that $(\mathbb{I} - A^{-1})\widetilde{\boldsymbol{y}}^t \in \boldsymbol{H}^1(\Omega)$, which implies the relation $\mathrm{div}\Upsilon_t = \mathrm{div}((\mathbb{I} - A^{-1})\widetilde{\boldsymbol{y}}^t) \in L^2(\Omega)$ holds with the estimate

$$\|\mathrm{div}\Upsilon_t\|^2_{L^2(\Omega)} \le C\Big(\big\|\mathbb{I} - A^{-1}\big\|^2_{\boldsymbol{L}^\infty(\Omega)} + \big\|\mathrm{div}(\mathbb{I} - A^{-1})^\top\big\|^2_{\boldsymbol{L}^\infty(\Omega)} \Big), \tag{6.36}$$

where $C > 0$ depends only on $\|\boldsymbol{y}^t\|_{\boldsymbol{H}^1(\Omega)}$ (which is bounded by $\|\boldsymbol{J}\|_{\boldsymbol{L}^2(\Omega)}$ due to (6.30)) and $\|\mathrm{D}T_t\|_{\boldsymbol{W}^{1,\infty}(\Omega)}$ (which is bounded in t due to Remark 6.1). Hence, substituting (6.35) into (6.34), and adding (6.36) to the resulting inequality, by Young's inequality applied to the terms involving the $\boldsymbol{L}^2(\Omega)$-norm of Υ_t, we arrive at

$$\begin{aligned}
&\|\mathrm{curl}\Upsilon_t\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\Upsilon_t\|^2_{L^2(\Omega)} - K\,\|\Upsilon_t\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\quad \le C\Big(\|A - \mathbb{I}\|^2_{\boldsymbol{L}^\infty(\Omega)} + \|B - \mathbb{I}\|^2_{\boldsymbol{L}^\infty(\Omega)} + \big\|\mathbb{I} - A^{-1}\big\|^2_{\boldsymbol{W}^{1,\infty}(\Omega)} + \big\|\mathbb{I} - B^{-1}\big\|^2_{\boldsymbol{L}^\infty(\Omega)} \Big) \\
&\qquad + C\Big(\|\xi - 1\|^2_{L^\infty(\Omega)} + \|\boldsymbol{J} \circ T_t - \boldsymbol{J}\|^2_{\boldsymbol{L}^2(\Omega)} \Big),
\end{aligned}$$

for some positive constant $K > 0$ that can be chosen as small as one desires, with the constant C appearing twice above depending on K. Invoking (2.6) and choosing K sufficiently small, together with (2.4), (6.25), (6.33), and Lemma 6.8 we then infer the uniform bounds

$$\big\|t^{-1}\Upsilon_t\big\|_{\boldsymbol{H}^1(\Omega)} + \big\|t^{-1}\Phi_t\big\|_{H_0^1(\Omega)} \le C, \tag{6.37}$$

so that along a non-relabelled subsequence

$$t^{-1}\Upsilon_t \rightharpoonup \dot{\boldsymbol{Y}}[V] \text{ in } \boldsymbol{H}^1(\Omega), \quad t^{-1}\Phi_t \rightharpoonup \dot{\phi}[V] \text{ in } H_0^1(\Omega) \tag{6.38}$$

for some functions $\dot{\boldsymbol{Y}}[V] \in \boldsymbol{H}^1(\Omega)$ and $\dot{\phi}[V] \in H_0^1(\Omega)$. Let us remark $\mathrm{div}\Upsilon_t \neq 0$ due to the definition of $\widetilde{\boldsymbol{y}}^t$, and so the Poincaré inequality (2.7) for $\boldsymbol{Z}$-valued functions cannot be used to control the $\boldsymbol{L}^2(\Omega)$-norm of Υ_t arising on the right-hand side of (6.34). Therefore, it is necessary to derive an estimate for $\mathrm{div}\Upsilon_t$ and in turn we require the regularity $\widetilde{\boldsymbol{y}}^t \in \boldsymbol{H}^1(\Omega_t)$ for all $t \in [-\tau_0, \tau_0]$, which is guaranteed with the assumption (A7) for the domain Ω and the continuous embedding (2.4).

Next, consider the vector function $\boldsymbol{\mathcal{I}}(\boldsymbol{s}) = \nu_1(|\boldsymbol{s}|)\boldsymbol{s}$. Then, similar to the proof of Lemma 6.3, the integral form of the mean value theorem gives

$$\begin{aligned}
\boldsymbol{\mathcal{I}}(B\mathrm{curl}\,\widetilde{\boldsymbol{y}}^t) - \boldsymbol{\mathcal{I}}(B\mathrm{curl}\,\boldsymbol{y}) &= \int_0^1 \nabla\boldsymbol{\mathcal{I}}(\theta B\mathrm{curl}\,\boldsymbol{y} + (1-\theta)B\mathrm{curl}\,\widetilde{\boldsymbol{y}}^t)B\mathrm{curl}\,\Upsilon_t \, d\theta, \\
\boldsymbol{\mathcal{I}}(B\mathrm{curl}\,\boldsymbol{y}) - \boldsymbol{\mathcal{I}}(\mathrm{curl}\,\boldsymbol{y}) &= \int_0^1 \nabla\boldsymbol{\mathcal{I}}(\zeta\mathrm{curl}\,\boldsymbol{y} + (1-\zeta)B\mathrm{curl}\,\boldsymbol{y})(B - \mathbb{I})\mathrm{curl}\,\boldsymbol{y} \, d\zeta,
\end{aligned}$$

From the definition (6.22), for fixed $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, it is clear that $\widehat{\mathrm{curl}}\,\boldsymbol{u} \to \mathrm{curl}\,\boldsymbol{u}$ as $t \to 0$, and since $B(t) \to \mathbb{I}$ and $\mathrm{D}\boldsymbol{y}^t \to \mathrm{D}\boldsymbol{y}$ a.e. in Ω as $t \to 0$ (the latter due to Lemma 6.7), we infer that $B\mathrm{curl}\,\widetilde{\boldsymbol{y}}^t = (\widehat{\mathrm{curl}}\,\boldsymbol{y}^t) \circ T_t \to \mathrm{curl}\,\boldsymbol{y}$ a.e. in Ω as $t \to 0$. Moreover, since $\nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}}(u, \boldsymbol{s}) = \nu_0(1-u)\mathbb{I} + u\nabla\boldsymbol{\mathcal{I}}(\boldsymbol{s})$, we deduce with the help of (A5) and the dominated convergence theorem that

$$\begin{aligned}
\frac{I_4}{t} &= \int_\Omega \int_0^1 \xi_t \nabla\boldsymbol{\mathcal{I}}(\zeta\mathrm{curl}\,\boldsymbol{y} + (1-\zeta)B\mathrm{curl}\,\boldsymbol{y})\frac{(B - \mathbb{I})}{t}\mathrm{curl}\,\boldsymbol{y} \cdot B\mathrm{curl}\,\boldsymbol{v} \, d\zeta \, dx \\
&\to \int_\Omega \nabla\boldsymbol{\mathcal{I}}(\mathrm{curl}\,\boldsymbol{y})B'(0)\mathrm{curl}\,\boldsymbol{y} \cdot \mathrm{curl}\,\boldsymbol{v} \, dx\,.
\end{aligned}$$

Similarly, since $\int_0^1 B^\top \nabla \boldsymbol{\mathcal{I}}((1-\theta)B\mathrm{curl}\,\boldsymbol{y} + \theta \mathrm{curl}\,\Upsilon_t)^\top B \mathrm{curl}\,\boldsymbol{v}\, d\theta \to \nabla \boldsymbol{\mathcal{I}}(\mathrm{curl}\,\boldsymbol{y})^\top \mathrm{curl}\,\boldsymbol{v}$ in $\boldsymbol{L}^2(\Omega)$ by the dominated convergence theorem, we infer that

$$\begin{aligned}\frac{I_3}{t} &= \int_\Omega \int_0^1 \nabla \boldsymbol{\mathcal{I}}(\theta B \mathrm{curl}\,\boldsymbol{y} + (1-\theta)B\mathrm{curl}\,\widetilde{\boldsymbol{y}}^t) B\mathrm{curl}\,\frac{\Upsilon_t}{t} \cdot B\mathrm{curl}\,\boldsymbol{v}\, d\theta\, dx \\ &\to \int_\Omega \nabla \boldsymbol{\mathcal{I}}(\mathrm{curl}\,\boldsymbol{y})\mathrm{curl}\,\dot{\boldsymbol{Y}}[V] \cdot \mathrm{curl}\,\boldsymbol{v}\, dx\,.\end{aligned}$$

Hence, dividing (6.31) and (6.32) by t and sending $t \to 0$, applying the convergences in Remark 6.1, Lemma 6.8 and (6.38), we see that $(\dot{\boldsymbol{Y}}[V], \dot{\phi}[V]) \in \boldsymbol{H}^1(\Omega) \times H_0^1(\Omega)$ satisfies

$$\begin{aligned} 0 &= \int_\Omega \dot{\boldsymbol{Y}}[V] \cdot \nabla \psi - A'(0)\boldsymbol{y} \cdot \nabla \psi \, dx\,, \\ 0 &= \int_\Omega -\boldsymbol{v} \cdot (\nabla \boldsymbol{J})V(0) + B'(0)\boldsymbol{J} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \dot{\phi}[V] - A'(0)\boldsymbol{v} \cdot \nabla \phi \, dx \\ &\quad + \int_\Omega \nu_0 (1-u)\mathrm{curl}\,\dot{\boldsymbol{Y}}[V] \cdot \mathrm{curl}\,\boldsymbol{v} + \nu_0(1-u)A'(0)\mathrm{curl}\,\boldsymbol{y} \cdot \mathrm{curl}\,\boldsymbol{v}\, dx \\ &\quad + \int_\Omega u\Big(\nabla \boldsymbol{\mathcal{I}}(\mathrm{curl}\,\boldsymbol{y})\mathrm{curl}\,\dot{\boldsymbol{Y}}[V] + \nabla \boldsymbol{\mathcal{I}}(\mathrm{curl}\,\boldsymbol{y})B'(0)\mathrm{curl}\,\boldsymbol{y}\Big) \cdot \mathrm{curl}\,\boldsymbol{v}\, dx \\ &\quad + \int_\Omega (u\nu_1(|\mathrm{curl}\,\boldsymbol{y}|)\mathrm{curl}\,\boldsymbol{y} \cdot \mathrm{curl}\,\boldsymbol{v})\mathrm{div} V(0) + (B^\top)'(0)\boldsymbol{\mathcal{I}}(\mathrm{curl}\,\boldsymbol{y}) \cdot \mathrm{curl}\,\boldsymbol{v}\, dx \end{aligned}$$

for arbitrary $\psi \in H_0^1(\Omega)$ and $\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl})$, which is the linear saddle point problem (6.27) after noting the relation $A'(0) = B'(0) + (B^\top)'(0) + \mathrm{div}V(0)$. Uniqueness of solutions to (6.27) follows easily from the coercivity estimate (6.4) and the linearity of the problem. □

Remark 6.2. *The function $\dot{\phi}[V]$ is the material derivative of ϕ, while the function $\dot{\boldsymbol{Y}}[V]$ is related to the material derivative $\dot{\boldsymbol{y}}[V] := \frac{d}{dt}(\boldsymbol{y}(t) \circ T_t)\big|_{t=0}$ of $\boldsymbol{y}$ via the formula*

$$\dot{\boldsymbol{y}}[V] = \dot{\boldsymbol{Y}}[V] - (\mathrm{D}V(0))^\top \boldsymbol{y}, \tag{6.39}$$

and the equations satisfied by $\dot{\boldsymbol{y}}[V]$ can be easily deduced from (6.27).

We prove now an alternate necessary optimality system for PFIP (5.1) derived by the method of domain variations.

Theorem 6.10. *For fixed $\alpha, \varepsilon > 0$, let $u_\varepsilon^\alpha \in \mathcal{K}$ be a solution to* (5.1) *with corresponding solution $(\boldsymbol{y}_\varepsilon^\alpha, \phi_\varepsilon^\alpha) \in \boldsymbol{Z} \times H_0^1(\Omega)$ to* (3.2)*. For every admissible velocity $V \in \mathcal{V}_{ad}$, let $(\dot{\boldsymbol{Y}}_\varepsilon^\alpha[V], \dot{\phi}_\varepsilon^\alpha[V]) \in \boldsymbol{H}^1(\Omega) \times H_0^1(\Omega)$ denote the unique solution to* (6.27) *corresponding to $(u_\varepsilon^\alpha, \boldsymbol{y}_\varepsilon^\alpha, \phi_\varepsilon^\alpha)$. Then, it holds that*

$$\begin{aligned} 0 &= \int_D (\boldsymbol{y}_\varepsilon^\alpha - \boldsymbol{y}_m) \cdot (\dot{\boldsymbol{Y}}_\varepsilon^\alpha[V] - (\mathrm{D}V(0))^\top \boldsymbol{y}_\varepsilon^\alpha - (\nabla \boldsymbol{y}_m)V(0)) + \frac{1}{2}|\boldsymbol{y}_\varepsilon^\alpha - \boldsymbol{y}_m|^2 \,\mathrm{div}V(0)\, dx \\ &\quad + \alpha \int_\Omega \Big(\frac{\gamma\varepsilon}{2}|\nabla u_\varepsilon^\alpha|^2 + \frac{\gamma}{\varepsilon}\Psi(u_\varepsilon^\alpha)\Big)\mathrm{div}V(0) - \gamma\varepsilon \nabla u_\varepsilon^\alpha \cdot (\nabla V(0))\nabla u_\varepsilon^\alpha \, dx \quad \forall V \in \mathcal{V}_{\mathrm{ad}}. \end{aligned} \tag{6.40}$$

Proof. Let $V \in \mathcal{V}_{ad}$ be an admissible velocity with the corresponding transformation $T \in \mathcal{T}_{ad}$. We introduce the functional

$$g : (-\tau_0, \tau_0) \to \mathbb{R}, \quad g(t) := J_f(u_\varepsilon^\alpha \circ T_t^{-1}) + \alpha E_\varepsilon(u_\varepsilon^\alpha \circ T_t^{-1}).$$

Since $u_\varepsilon^\alpha \in \mathcal{K}$, by (6.21) we have that

$$u_\varepsilon^\alpha \circ T_t^{-1} \in \mathcal{K} \quad \forall t \in (-\tau_0, \tau_0).$$

For this reason, as $u_\varepsilon^\alpha \in \mathcal{K}$ is a solution to (5.1), we have

$$g(0) \le g(t) \quad \forall t \in (-\tau_0, \tau_0),$$

from which it follows that

$$0 = g'(0) = \frac{d}{dt} J_f(u_\varepsilon^\alpha \circ T_t^{-1})\big|_{t=0} + \frac{d}{dt} E_\varepsilon(u_\varepsilon^\alpha \circ T_t^{-1})\big|_{t=0}. \tag{6.41}$$

By a change of variables, the derivative of the first term is obtained as follows:

$$\begin{aligned} \frac{d}{dt} J_f(u^{\alpha,t})\big|_{t=0} &= \frac{d}{dt}\frac{1}{2}\int_D \left|\boldsymbol{S}(u_\varepsilon^\alpha \circ T_t^{-1}) \circ T_t - \boldsymbol{y}_m \circ T_t\right|^2 \xi_t \, dx \,\big|_{t=0} \qquad (6.42) \\ &\underbrace{=}_{(6.19)} \int_D (\boldsymbol{y}_\varepsilon^\alpha - \boldsymbol{y}_m) \cdot \frac{d}{dt}(\boldsymbol{y}_\varepsilon^\alpha(t) \circ T_t - \boldsymbol{y}_m \circ T_t)\big|_{t=0} + \frac{1}{2}|\boldsymbol{y}_\varepsilon^\alpha - \boldsymbol{y}_m|^2 \operatorname{div} V(0) \, dx \\ &= \int_D (\boldsymbol{y}_\varepsilon^\alpha - \boldsymbol{y}_m) \cdot (\dot{\boldsymbol{Y}}_\varepsilon^\alpha[V] - (\mathrm{D}V(0))^\top \boldsymbol{y}_\varepsilon^\alpha - (\nabla \boldsymbol{y}_m) V(0)) + \frac{1}{2}|\boldsymbol{y}_\varepsilon^\alpha - \boldsymbol{y}_m|^2 \operatorname{div} V(0) \, dx \,, \end{aligned}$$

where we have used (6.39) and the formula $\frac{d}{dt}\boldsymbol{y}_m \circ T_t\big|_{t=0} = (\nabla \boldsymbol{y}_m)V(0)$ for the last equality. Meanwhile, the directional derivative $\frac{d}{dt}E_\varepsilon(u_\varepsilon^\alpha \circ T_t^{-1})\big|_{t=0}$ is obtained (cf. [25, Lem. 7.5]) as follows:

$$\frac{d}{dt} E_\varepsilon(u_\varepsilon^\alpha \circ T_t^{-1})\big|_{t=0} = \alpha \int_\Omega \Big(\frac{\gamma\varepsilon}{2}|\nabla u_\varepsilon^\alpha|^2 + \frac{\gamma}{\varepsilon}\Psi(u_\varepsilon^\alpha)\Big) \operatorname{div} V(0) - \gamma\varepsilon \nabla u_\varepsilon^\alpha \cdot (\nabla V(0)) \nabla u_\varepsilon^\alpha \, dx \,. \tag{6.43}$$

In conclusion, the assertion follows from (6.41)–(6.43). □

Our main result on the convergence of the optimality system (6.40) as $\varepsilon \to 0$ is formulated as follows.

Theorem 6.11. *For fixed $\alpha > 0$, let $(u_\varepsilon^\alpha)_{\varepsilon>0} \subset \mathcal{K}$ be solutions to PFIP (5.1). For any $T \in \mathcal{T}_{\mathrm{ad}}$ with corresponding velocity field $V \in \mathcal{V}_{\mathrm{ad}}$, there exists a non-relabelled subsequence such that*

$$u_\varepsilon^\alpha \to u^\alpha \text{ in } L^1(\Omega), \qquad J_\varepsilon(u_\varepsilon^\alpha) \to J(u^\alpha) \text{ in } \mathbb{R}, \tag{6.44}$$

$$\dot{\boldsymbol{Y}}_\varepsilon^\alpha[V] \rightharpoonup \dot{\boldsymbol{Y}}^\alpha[V] \text{ in } \boldsymbol{H}^1(\Omega), \qquad \dot{\phi}_\varepsilon^\alpha[V] \rightharpoonup \dot{\phi}^\alpha[V] \text{ in } H_0^1(\Omega), \tag{6.45}$$

where $u^\alpha \in BV(\Omega, \{0,1\})$ is a solution to TVIP (4.1), and the pair $(\dot{\boldsymbol{Y}}^\alpha[V], \dot{\phi}^\alpha[V]) \in \boldsymbol{H}_0(\mathrm{curl}) \cap \boldsymbol{H}^1(\Omega) \times H_0^1(\Omega)$ satisfies (6.27) corresponding to $(u^\alpha, \boldsymbol{y}^\alpha, \phi^\alpha)$. Furthermore, it holds that

$$\mathrm{D}J_\varepsilon(u_\varepsilon^\alpha)[V] \to \mathrm{D}J(u^\alpha)[V] \text{ in } \mathbb{R}, \tag{6.46}$$

where

$$\begin{aligned} \mathrm{D}J(u^\alpha)[V] = &\int_D (\boldsymbol{y}^\alpha - \boldsymbol{y}_m) \cdot (\dot{\boldsymbol{Y}}^\alpha[V] - (\mathrm{D}V(0))^\top \boldsymbol{y}^\alpha - (\nabla \boldsymbol{y}_m)V(0)) \, dx \\ &+ \int_D \frac{1}{2}|\boldsymbol{y}^\alpha - \boldsymbol{y}_m|^2 \operatorname{div} V(0) \, dx + \alpha \int_\Omega (\operatorname{div} V(0) - \mu \cdot (\nabla V(0))\mu) \, d|\mathrm{D}\chi_{\{u^\alpha = 1\}}|, \end{aligned}$$

with $\mu = \frac{\mathrm{D}\chi_{\{u^\alpha=1\}}}{|\mathrm{D}\chi_{\{u^\alpha=1\}}|}$ as the generalised unit normal on the set $\{u^\alpha = 1\}$.

Remark 6.3. *In general, $u^\alpha \circ T_t^{-1}$ may not belong to $BV(\Omega,\{0,1\})$ for $T \in \mathcal{T}_{\rm ad}$, and thus we cannot use $u^\alpha \circ T_t^{-1}$ as a comparison function to deduce analogous statements in Theorem 6.10 for TVIP. However, if we assume $\partial\{u^\alpha = 1\}$ is Lipschitz, then for any $\phi \in C_0^1(\Omega_t;\mathbb{R}^3)$ with $|\phi| \le 1$ a.e. in $\Omega_t = T_t(\Omega)$, a short calculation shows*

$$\begin{aligned}\int_{\Omega_t}(u^\alpha \circ T_t^{-1})\widehat{\operatorname{div}}\phi\, d\widehat{x} &= \int_\Omega \xi(t)u^\alpha \mathrm{D}T_t^{-\top} : \nabla(\phi\circ T_t)\,dx\\ &= -\int_{\{u^\alpha=1\}}\operatorname{div}(\xi(t)\mathrm{D}T_t^{-1})\cdot(\phi\circ T_t)\,dx + \int_{\partial\{u^\alpha=1\}}[\xi(t)\mathrm{D}T_t^{-\top}\cdot(\phi\circ T_t)]\boldsymbol{n}\,d\mathcal{H}^2,\end{aligned}$$

where the right-hand side can be bounded independently of ϕ, which implies $u^\alpha\circ T_t^{-1} \in BV(\Omega_t,\{0,1\})$. Denoting by $(\boldsymbol{y}^{\alpha,t},\phi^{\alpha,t})$ the unique solution to (3.2) corresponding to data $(u^\alpha\circ T_t^{-1},\boldsymbol{J})$, from the proofs of Lemma 6.9 and Theorem 6.10 we can identify $\dot{\boldsymbol{Y}}^\alpha[V]$ and $\dot{\phi}^\alpha[V]$ as the Gâteaux derivatives $\frac{d}{dt}(\mathrm{D}T_t^\top(\boldsymbol{y}^{\alpha,t}\circ T_t))|_{t=0}$ and $\frac{d}{dt}(\phi^{\alpha,t}\circ T_t)|_{t=0}$, and $\mathrm{D}J(u^\alpha)[V]$ as the shape gradient for TVIP.

Proof. The first assertion (6.44) is the conclusion of Theorem 5.2. Let $(\boldsymbol{y}_\varepsilon^\alpha,\phi_\varepsilon^\alpha)$ and $(\boldsymbol{y}^\alpha,\phi^\alpha)$ denote the unique solution of (3.2) corresponding to data $(u_\varepsilon^\alpha,\boldsymbol{J})$ and $(u^\alpha,\boldsymbol{J})$, respectively. Then, the assertions of Theorem 3.2 and assumption (A5) imply that

$$G_{u_\varepsilon^\alpha,\boldsymbol{y}_\varepsilon^\alpha,\phi_\varepsilon^\alpha}(\boldsymbol{v}) \to G_{u^\alpha,\boldsymbol{y}^\alpha,\phi^\alpha}(\boldsymbol{v}),\quad H_{\boldsymbol{y}_\varepsilon^\alpha}(\psi)\to H_{\boldsymbol{y}^\alpha}(\psi)\quad \forall \boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{curl}),\ \psi\in H_0^1(\Omega).$$

In particular, (A5) ensures that

$$\left|u_\varepsilon^\alpha\frac{\nu_1'(|\mathrm{curl}\,\boldsymbol{y}_\varepsilon^\alpha|)}{|\mathrm{curl}\,\boldsymbol{y}_\varepsilon^\alpha|}(\mathrm{curl}\,\boldsymbol{y}_\varepsilon^\alpha\otimes\mathrm{curl}\,\boldsymbol{y}_\varepsilon^\alpha)\mathrm{curl}\,\boldsymbol{v}\right| \le (C_{\boldsymbol{\mathcal{F}}}+\nu_0)\,|\mathrm{curl}\,\boldsymbol{v}|\quad \forall\boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{curl}),$$

which allows us to pass to the limit with the help of the dominated convergence theorem. Moreover, (3.10) and the smoothness of $V\in\mathcal{V}_{ad}$ imply

$$\left\|G_{u_\varepsilon^\alpha,\boldsymbol{y}_\varepsilon^\alpha,\phi_\varepsilon^\alpha}\right\|_{\boldsymbol{H}(\mathrm{curl})^*} + \left\|H_{\boldsymbol{y}_\varepsilon^\alpha}\right\|_{H^{-1}(\Omega)} \le C\,\|\boldsymbol{J}\|_{\boldsymbol{H}^1(\Omega)}$$

for a positive constant C independent of ε. Since $(\dot{\boldsymbol{Y}}_\varepsilon^\alpha[V],\dot{\phi}_\varepsilon^\alpha[V])\in\boldsymbol{H}^1(\Omega)\times H_0^1(\Omega)$ is a solution to the linear saddle point problem (6.27) with right-hand side $(G_{u_\varepsilon^\alpha,\boldsymbol{y}_\varepsilon^\alpha,\phi_\varepsilon^\alpha},H_{\boldsymbol{y}_\varepsilon^\alpha})$, by Lemma 2.1 and (2.4), we see that

$$\left\|\dot{\boldsymbol{Y}}_\varepsilon^\alpha[V]\right\|_{\boldsymbol{H}^1(\Omega)} + \left\|\dot{\phi}_\varepsilon^\alpha[V]\right\|_{H_0^1(\Omega)} \le C.$$

Hence, along a non-relabelled subsequence, we have the weak convergence $\dot{\boldsymbol{Y}}_\varepsilon^\alpha \rightharpoonup \dot{\boldsymbol{Y}}^\alpha[V]\in\boldsymbol{H}_0(\mathrm{curl})\cap\boldsymbol{H}^1(\Omega)$, and $\dot{\phi}_\varepsilon^\alpha[V]\rightharpoonup\dot{\phi}^\alpha[V]\in H_0^1(\Omega)$. One then finds from passing to the limit in equations for $(\dot{\boldsymbol{Y}}_\varepsilon^\alpha[V],\dot{\phi}_\varepsilon^\alpha[V])$ that $(\dot{\boldsymbol{Y}}^\alpha[V],\dot{\phi}^\alpha[V])$ satisfies (6.27) corresponding to $(u^\alpha,\boldsymbol{y}^\alpha,\phi^\alpha)$. This shows the second assertion (6.45).

For the last assertion (6.46) concerning the convergence of the optimality conditions, by the strong convergence of $\boldsymbol{y}_\varepsilon^\alpha$ and $\dot{\boldsymbol{Y}}_\varepsilon^\alpha[V]$ in $\boldsymbol{L}^2(\Omega)$, it is easy to see that $\mathrm{D}J_f(u_\varepsilon^\alpha)[V]\to\mathrm{D}J_f(u^\alpha)[V]$. Meanwhile, the convergence

$$\begin{aligned}\mathrm{D}E_\varepsilon(u_\varepsilon^\alpha)[V] &= \int_\Omega\Big(\frac{\gamma\varepsilon}{2}|\nabla u_\varepsilon^\alpha|^2+\frac{\gamma}{\varepsilon}\Psi(u_\varepsilon^\alpha)\Big)\mathrm{div}V(0) - \gamma\varepsilon\nabla u_\varepsilon^\alpha\cdot(\nabla V(0))\nabla u_\varepsilon^\alpha\,dx\\ &\to \int_\Omega(\mathrm{div}V(0)-\mu\cdot(\nabla V(0))\mu)\,d|\mathrm{D}\chi_{\{u^\alpha=1\}}| = \mathrm{D}TV(u^\alpha)[V]\end{aligned}$$

can be deduced as a consequence of (6.44) and the calculations in the proof of [20, Thm. 4.2]. This completes the proof. □

A Gamma convergence of the Ginzburg–Landau functional

Lemma A.1. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with a Lipschitz boundary. For $\varepsilon > 0$, let $E_\varepsilon(v) = \frac{8}{\pi}\int_\Omega \frac{\varepsilon}{2}|\nabla v|^2 + \frac{1}{2\varepsilon}v(1-v)\,dx$ denote the Ginzburg–Landau functional.*

(i) *If $(v_\varepsilon)_{\varepsilon>0} \subset \mathcal{K}$ is a sequence such that $v_\varepsilon \to v_0$ strongly in $L^1(\Omega)$ and $\liminf_{\varepsilon\to 0} E_\varepsilon(v_\varepsilon) < \infty$, then $v_0 \in BV(\Omega,\{0,1\})$ with $TV(v_0) \le \liminf_{\varepsilon\to 0} E_\varepsilon(v_\varepsilon)$.*

(ii) *Let $A \subset \mathbb{R}^d$ be an open bounded set with a smooth boundary ∂A satisfying $\mathcal{H}_2(\partial A \cap \partial\Omega) = 0$, and let $v_0 = \chi_{A\cap\Omega}$. Then, there is a family $(v_\varepsilon)_{\varepsilon>0}$ of Lipschitz continuous functions on Ω such that $\|v_\varepsilon - v_0\|_{L^1(\Omega)} = \mathcal{O}(\varepsilon)$, $0 \le v_\varepsilon(x) \le 1$ a.e. in Ω and satisfies the properties $\int_\Omega v_\varepsilon\,dx = \int_\Omega v_0\,dx = |A\cap\Omega|$ and $\limsup_{\varepsilon\to 0} E_\varepsilon(v_\varepsilon) \le TV(v_0)$.*

(iii) *Let $(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{K}$ be a sequence satisfying $\sup_{\varepsilon>0} E_\varepsilon(u_\varepsilon) < \infty$. Then, there exists a non-relabelled subsequence $\varepsilon \to 0$ and a limit function u such that $u_\varepsilon \to u$ strongly in $L^1(\Omega)$ with $u \in BV(\Omega,\{0,1\})$.*

Items (i) and (ii) are also known as the liminf and limsup inequalities of Gamma convergence, while item (iii) is called the compactness property. In the following we only outline the modifications necessary to adapt [9, Props. 3.8, 3.11] for our consideration of the potential $\Psi(s) = \frac{1}{2}s(1-s)$. We point out that in the constant γ and functional $\mathcal{E}_\gamma$ used in [9] correspond in our present notation to $\gamma = \varepsilon^2$ and $\mathcal{E}_\gamma = \varepsilon E_\varepsilon$. For further details, we also refer to [25, Sec. 6.2] and [37, Sec. 1B].

Proof. Item (i) follows directly from the proof of [9, Prop. 3.8] by replacing [9, equ. (3.58)] with $\phi(t) = \int_0^t \frac{8}{\pi}\sqrt{2\Psi(s)}\,ds$ which satisfies $\phi(0) = 0$ and $\phi(1) = 1$. Notice that $\phi'(t) > 0$ for all $t \in (0,1)$, and hence $\phi : [0,1] \to [0,1]$ is strictly increasing and bijective. We denote by $\varphi : [0,1] \to [0,1]$ the inverse of ϕ which is also strictly increasing.

Item (ii) follows directly from the proof of [9, Prop. 3.11] by replacing [9, equ. (3.65), (3.66)] with

$$\chi^*(t) := \begin{cases} 1 & t \ge 0, \\ 0 & t < 0, \end{cases} \qquad \zeta_\varepsilon(t) := \begin{cases} 1 & t > \eta_\varepsilon := \pi\varepsilon, \\ \frac{1}{2} + \frac{1}{2}\sin\left(\frac{t}{\varepsilon} - \frac{\pi}{2}\right) & 0 \le t \le \eta_\varepsilon, \\ 0 & t < 0. \end{cases}$$

A short calculation shows that [9, equ. (3.67)] is satisfied, i.e., $\varepsilon\zeta_\varepsilon'(t) = \sqrt{2\Psi(\zeta_\varepsilon(t))}$ for all $t \in [0,\eta_\varepsilon]$. We mention that the convergence rate $\|v_\varepsilon - v\|_{L^1(\Omega)} = \mathcal{O}(\varepsilon)$ is in fact a consequence of the calculations in [9, p. 271].

For item (iii), we adopt some ideas from the proofs of [3, Prop. 4.1] and [30, Prop. 3]. Let $(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{K}$ be a sequence satisfying $\sup_{\varepsilon>0} E_\varepsilon(u_\varepsilon) < \infty$. We set $w_\varepsilon(x) := \phi(u_\varepsilon(x))$ for a.e. $x \in \Omega$, where ϕ is the function defined in the proof of assertion (i). Then, Lipschitz continuity of ϕ implies that $\nabla w_\varepsilon = \phi'(u_\varepsilon)\nabla u_\varepsilon = \frac{8}{\pi}\sqrt{2\Psi(u_\varepsilon)}\nabla u_\varepsilon$. Young's inequality now shows that

$$\|\nabla w_\varepsilon\|_{\boldsymbol{L}^1(\Omega)} \le \frac{8}{\pi}\int_\Omega \frac{1}{\varepsilon}\Psi(u_\varepsilon) + \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2\,dx = E_\varepsilon(u_\varepsilon).$$

Together with the fact $\|w_\varepsilon\|_{L^1(\Omega)} \le |\Omega|$ we find that $(w_\varepsilon)_{\varepsilon>0}$ is a bounded sequence in $W^{1,1}(\Omega) \subset BV(\Omega)$. BV compactness leads to the existence of a non-relabelled subsequence and a limit $w \in BV(\Omega)$ such that $w_\varepsilon \rightharpoonup w$ in $BV(\Omega)$, $w_\varepsilon \to w$ strongly in $L^1(\Omega)$ and a.e. in

Ω. The candidate limit of $(u_\varepsilon)_{\varepsilon>0}$ along that subsequence is $u(x) := \varphi(w(x))$. Since φ has bounded derivative on $[0,1]$ we infer

$$\|u_\varepsilon - u\|_{L^1(\Omega)} = \|\varphi(w_\varepsilon) - \varphi(w)\|_{L^1(\Omega)} \le C\,\|w_\varepsilon - w\|_{L^1(\Omega)} \to 0.$$

To deduce that the limit u belongs to $BV(\Omega,\{0,1\})$ we simply apply item (i) since $\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \le \limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \le \sup_{\varepsilon>0} E_\varepsilon(u_\varepsilon) < \infty$. □

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