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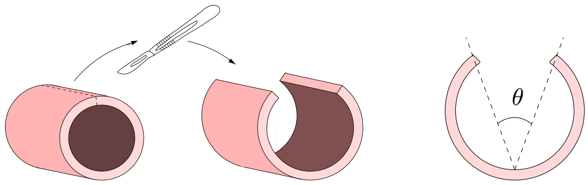
A numerical scheme for the incorporation of residual stresses in arteries

Motivation

An important phenomenon occurring in arteries is the presence of *residual stresses*, which are assumed to prevent large stress gradients in the artery. As a consequence of a radial cut arterial rings open up into a horseshoe leading to the conclusion that the artery is residually stressed in circumferential direction when it is intact but free from external forces. We present a novel scheme for the incorporation of residual stresses in order to enforce a *reduction of the stress gradients in radial direction*, and to achieve an *application to patient-specific arterial geometries*, see [1], [3].



Suitable measure for residual stresses

In order to arrive at a realistic stress distribution across the arterial wall we enforce a reduction of the gradients of suitable stress measures in radial direction. These are identified by decomposing the total stresses into ground stresses $\boldsymbol{\sigma}^*$ and reaction stresses $\boldsymbol{\sigma}^{\mathrm{r}}$, see for example [4] and [2], i.e.

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^* + \boldsymbol{\sigma}^{\mathrm{r}} \ ,$$

The reaction stresses result from the equilibrium conditions in consideration of the side conditions (incompressibility and (fictive) inextensibility of the fibers), i.e.

$$\boldsymbol{\sigma}^{\mathrm{r}} = -p\,\mathbf{1} + T_{(1)}\,\tilde{\boldsymbol{m}}_{(1)} + T_{(2)}\,\tilde{\boldsymbol{m}}_{(2)} \ , \qquad (1)$$

with $\tilde{\boldsymbol{m}}_{(a)} = \tilde{\boldsymbol{a}}_{(a)} \otimes \tilde{\boldsymbol{a}}_{(a)}$, $a = 1,2$ associated to the current preferred directions $\tilde{\boldsymbol{a}}_{(a)}$. Following eq. (1) the expressions $\mathrm{tr}\,\boldsymbol{\sigma}^*$ and $\boldsymbol{\sigma}^* : \tilde{\boldsymbol{m}}_{(a)}$ are absorbed into the pressure p and the fiber stresses $T_{(a)}$, respectively, leading to the side conditions on $\boldsymbol{\sigma}^*$

$$\boldsymbol{\sigma} : \tilde{\boldsymbol{m}}_{(a)} = \boldsymbol{\sigma}^{\mathrm{r}} : \tilde{\boldsymbol{m}}_{(a)} \quad \text{and} \quad \mathrm{tr}\boldsymbol{\sigma} = \mathrm{tr}\boldsymbol{\sigma}^{\mathrm{r}} \ ,$$

from which the pressure p and the fiber stresses $T_{(a)}$ follow:

$$T_{(1)} = [\,(1-\xi)\,\mathbf{1} : \boldsymbol{\sigma} - 2\,\boldsymbol{\sigma} : \tilde{\boldsymbol{m}}_{(1)} + (3\,\xi - 1)\,\boldsymbol{\sigma} : \tilde{\boldsymbol{m}}_{(2)}\,] \,/\, \Xi \ ,$$

$$T_{(2)} = [\,(1-\xi)\,\mathbf{1} : \boldsymbol{\sigma} + (3\,\xi - 1)\,\boldsymbol{\sigma} : \tilde{\boldsymbol{m}}_{(1)} - 2\,\boldsymbol{\sigma} : \tilde{\boldsymbol{m}}_{(2)}\,] \,/\, \Xi \ .$$

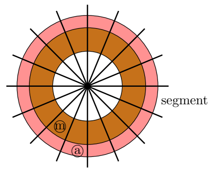
Here, we used the abbreviations $\xi = \tilde{\boldsymbol{m}}_{(1)} : \tilde{\boldsymbol{m}}_{(2)} = \cos^2\phi$, $\Xi = 3\,\xi^2 - 2\,\xi - 1$ and $\phi \sphericalangle \tilde{\boldsymbol{a}}_{(1)}, \tilde{\boldsymbol{a}}_{(2)}$.

Computation of residual stresses

The incorporation of residual stresses is based on a decomposition of the domain, which is done in two steps:

1. Decomposition into $\mathrm{n_{SG}}$ segments, and
2. subdivision of the segments into $\mathrm{n_{MAT}}$ sectors, depending on the number of materials present in each segment.

Considering a two-dimensional cross-section, the number of sectors can be specified as $\mathrm{n_{SC}} = \mathrm{n_{SG}} \cdot \mathrm{n_{MAT}}$. The following figure shows the decomposition into sectors for a circular ring.



amount of layers: $\mathrm{n_{MAT}} = 2$

amount of segments: $\mathrm{n_{SG}} = 16$

↪ **sectors:** $\mathrm{n_{SC}} = 32$

When the artery is loaded with a physiological internal pressure, the local volume average values of the fiber stresses are computed

$$\overline{T}^{\,i}_{(a)} = \frac{1}{V_i} \int_{\mathcal{B}_i} T_{(a)}(\boldsymbol{x})\ \mathrm{d}v \ , \quad \text{for} \quad a = 1,2$$

with $i = 1, \ldots, \mathrm{n_{SC}}$ and $\boldsymbol{x} \in \mathcal{B}_i$. The difference between this mean value and the fiber stresses yields the increments $\Delta T_{(a)} = T_{(a)} - \overline{T}^{\,i}_{(a)}$ in $\mathcal{B}_i$. These stresses in turn are used for an estimation of the residual stresses

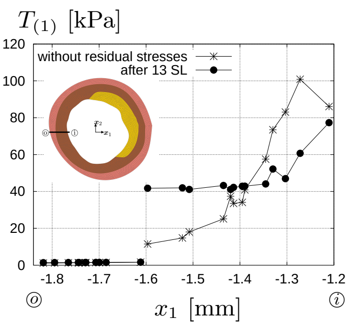
$$\boldsymbol{\sigma}^{\mathrm{res}} = -\Delta p\ \mathbf{1} + \Delta T_{(1)}\tilde{\boldsymbol{m}}_{(1)} + \Delta T_{(2)}\tilde{\boldsymbol{m}}_{(2)} \ ,$$

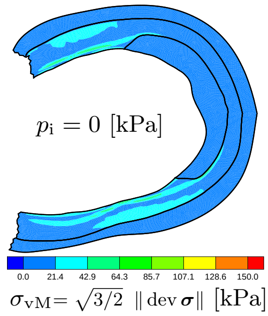
where Δp can be computed from the condition $\mathrm{tr}\Delta\boldsymbol{\sigma} = 0$. Within an iterative process so-called smoothing-loops (SL) are applied with a certain amount of the computed residual stresses.

Numerical simulation

We consider a patient-specific arterial cross-section and use

- a discretization with 6 015 quadratic triangular elements,
- a decomposition into $\mathrm{n_{SC}} = 32 \cdot 2 = 72$ sectors,
- an inner pressure of $p_i = 16$ kPa, and
- 13 smoothing-loops each with an amount of 10%.





As a result of the application of the residual stresses we observe:

- The stress-gradient in the media is smoothed.
- The artery opens after a horizontal cut (while $p_\mathrm{i} = 0$).
- The von-Mises stresses are nearly zero in the opened state.

References

[1] S. Brinkhues, Dissertation, University Duisburg-Essen, Institute of Mechanics, Report No.: 11, 2012.

[2] J. Schröder, Dissertation, University of Hannover, Institute of Mechanics (Civil Engineering), Chair I, Report No.: I-1, 1996.

[3] J. Schröder & S. Brinkhues, 2012, in preparation.

[4] A.J.M. Spencer, Oxford University Press, 1972.

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