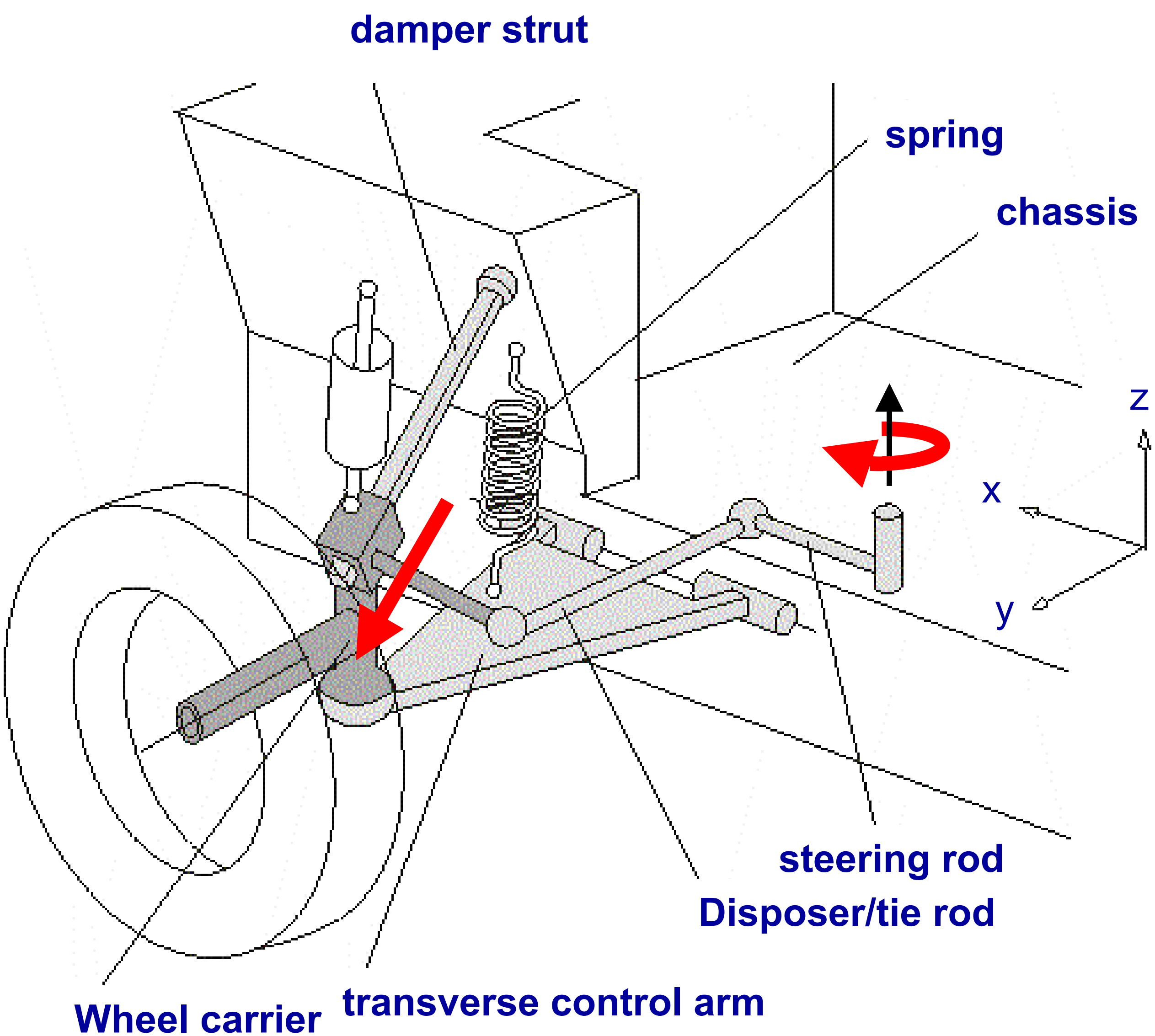
Chapter 3: Kinematics and Dynamics of Multibody Systems

3.1 Fundamentals of Kinematics

3.2 Setting up the equations of motion
for multibody-systems II

3.1 Fundamentals of Kinematics



Topological basic principles

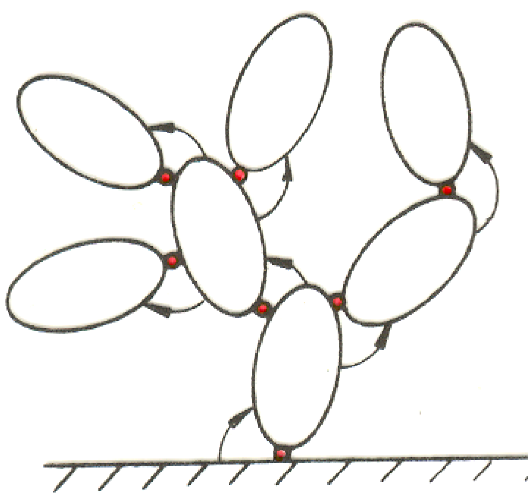
1. Open kinematical chain („tree-structure“)

- $\mathbf{n_B}$ bodies (without reference/inertial body)
- $\mathbf{n_G}$ joints
- The route between two bodies is distinct

  each body has exactly one predecessor

Hence it applies: $n_G = n_B$

Each joint has a dedicated body and connects the body with its predecessor



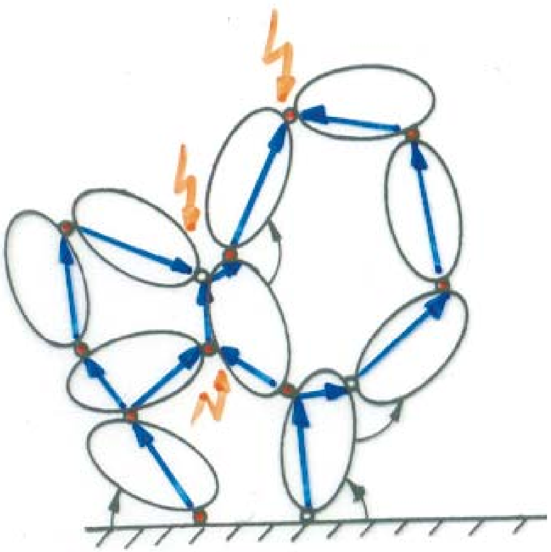
$n_B = n_G = 7$

Topological basic principles

2. Closed kinematical chain („loop-structure“)

- $\mathbf{n_B}$ bodies (without reference/inertial body)
- $\mathbf{n_G}$ joints
- each loop requires one extra joint

Hence : $n_G = n_B + n_L$



$n_B = 10$

$n_G = 13$

$n_L = 3$

Basic notation and coherences

f_{Gi} - degree of freedom of joint i

f_G - Sum of all degree of freedom of joints

f_{Li} - degree of freedom of loop i

Degree of freedom (Laufgrad) during spatial motion (Grübler; Kutzbach)

$$f = 6\,n_B - \sum_{i=1}^{n_G}(6 - f_{Gi})$$

$$n_L = n_G - n_B$$

$$f = \sum_{i=1}^{n_G} f_{Gi} - 6\,n_L$$
$$= f_G - 6\,n_L$$

Principles of assembling kinematical chains with closed loops

1. „sparse“ – method (cutting off at all bodies)

principle: „disconnecting“ all joints

constraints: concurrence of all joint parameters

2. „vector-loop“ – method (disconnecting loops)

principle: „disconnecting“ only one joint per loop

constraints: closure conditions of the loops

3. „topological“ – methods (loops as transference elements)

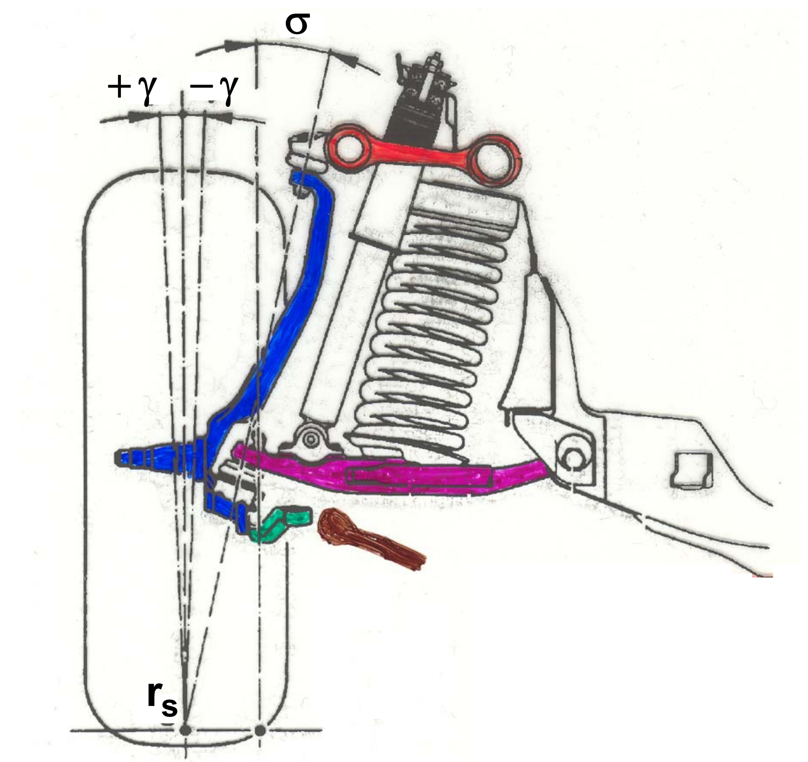
principle: kinematical loops are prepared as kinematical transfer elements (kinematical transformators)

constraints:
- transfer equations in the loop (nonlinear and local)
- coupler equations between the loops (linear and global)

3.1 Fundamental Procedures of Kinematics II

Illustration with the example of a double wishbone suspension (front wheel suspension):

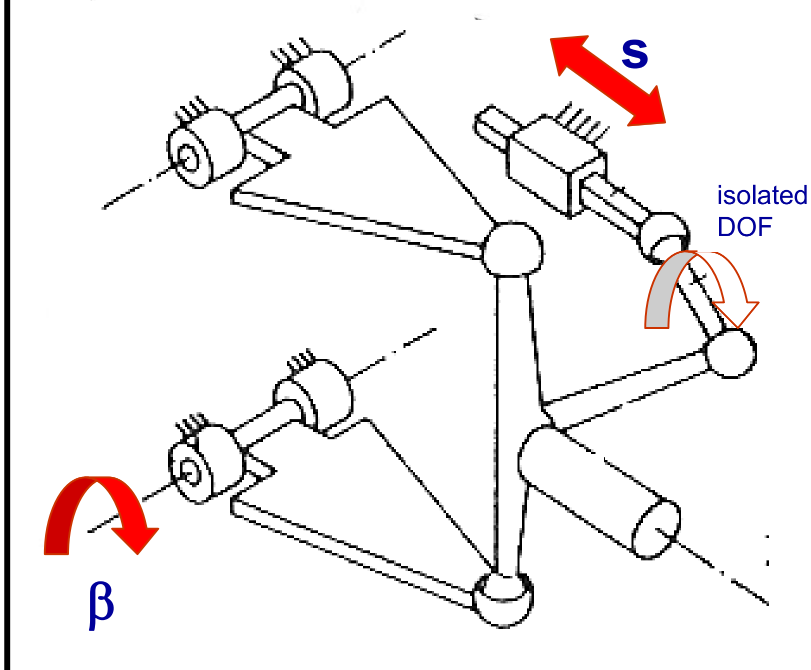
double wishbone wheel suspension:



Multi-body system

- 5 bodies
- 7 joints
- 2 kinematical loops

Rigid body model



System degrees of freedom

- compression (angle β)
- steering (displacement s)
- isolated degree of freedom ignored

System overview

$n_G = 7$

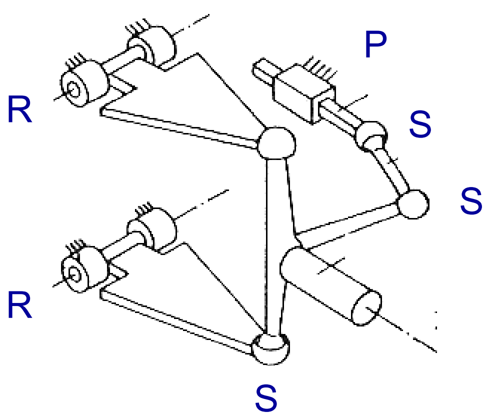
$n_B = 5$

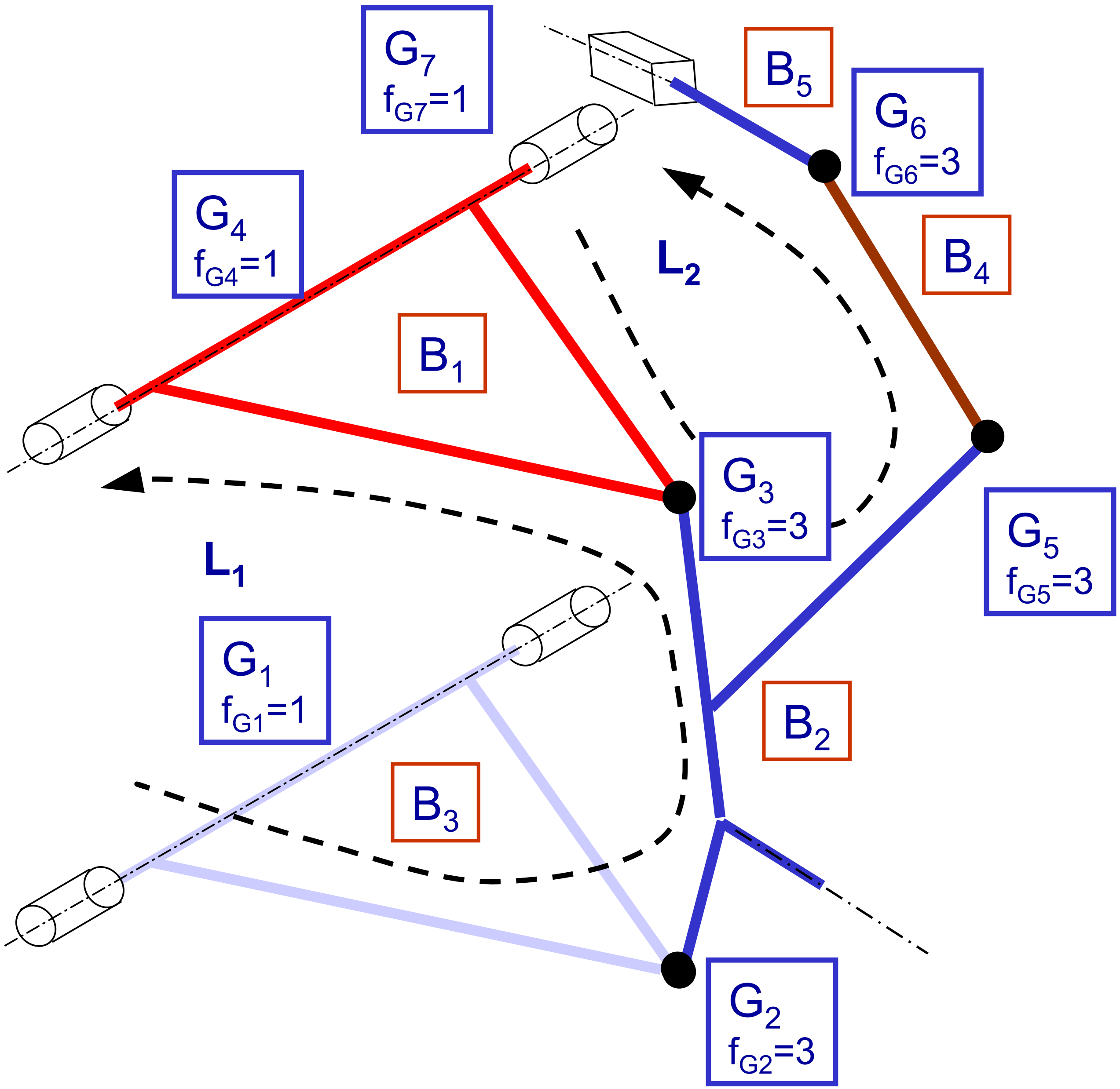
$n_L = n_G - n_B = 2$

$$f = 6\, n_B - \sum_{i=1}^{n_G} (6 - f_{Gi})$$

$$= 6 \cdot 5 - (3 \cdot 5 + 4 \cdot 3)$$

$$= 30 - 27 = 3$$





1. „sparse“ – method (cutting of all bodies)

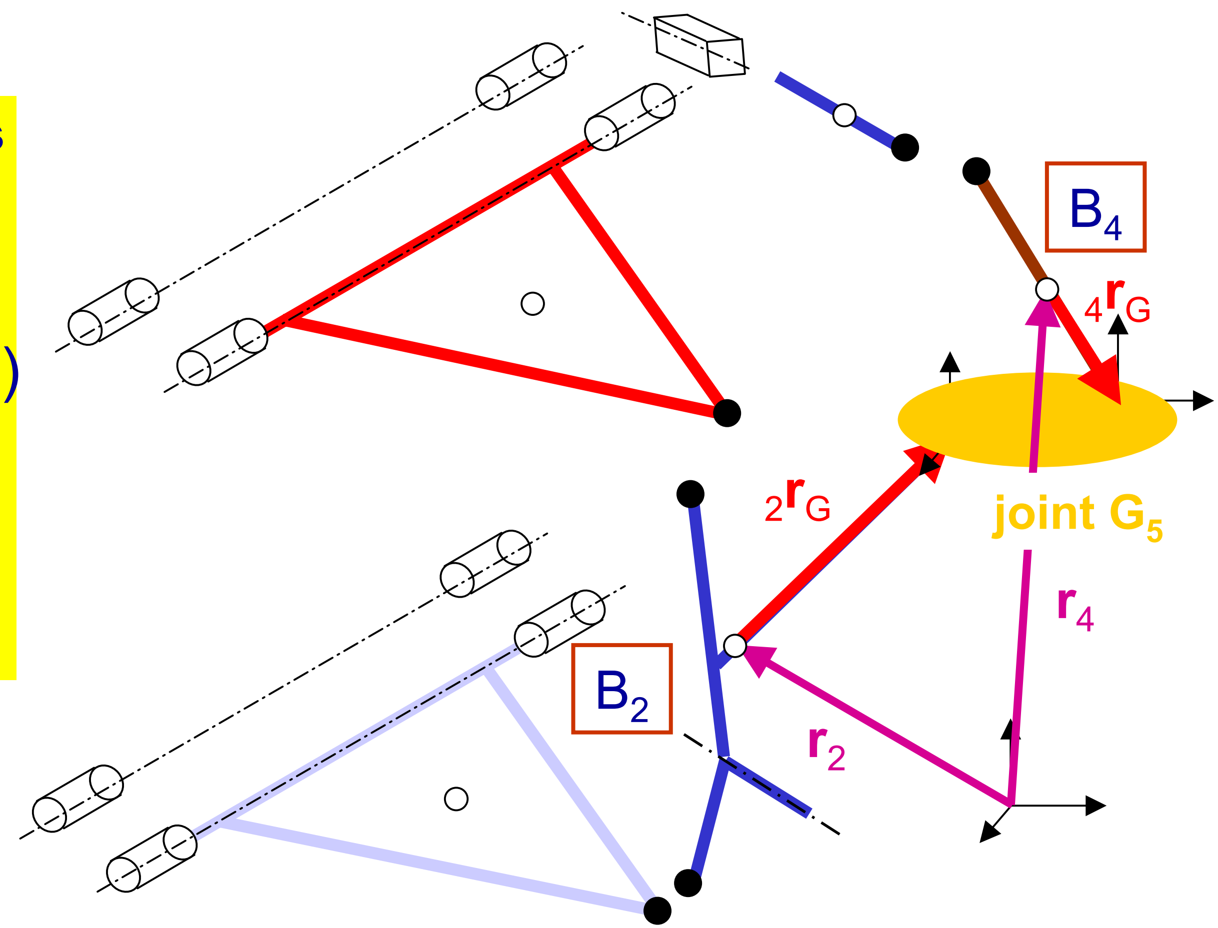
Equation of constraints

e.g. for G_5:

$$\mathbf{r}_2 + {}_2\mathbf{r}_G(\mathbf{w}_i) = \mathbf{r}_4 + {}_4\mathbf{r}_G(\mathbf{w}_j)$$

with

$$\mathbf{w}_i = [x_i \; y_i \; z_i \; \varphi_i \; \psi_i \; \theta_i]^T$$



Double wishbone wheel suspension with cut joints:

a) „sparse“ – method (cutting off all bodies)

$\left.\begin{array}{l}\text{5 bodies}\\ \text{7 joints}\end{array}\right\} \longrightarrow$ all joints disconnected

assembly: each joint has $(6 - f_{G_i})$ constraints

f_{G_i}: degrees of freedom of joint

equations of constraints are identified from the closure conditions for position and orientation:

$$\boxed{\begin{array}{l} \underline{r}_G\,({}_i\underline{r}_G) \overset{!}{=} \underline{r}_G\,({}_j\underline{r}_G) \\ T_G\,({}^iT_G) \overset{!}{=} T_G\,({}^jT_G) \end{array}}$$

2. „vector-loop“ – Methods (disconnecting loops)

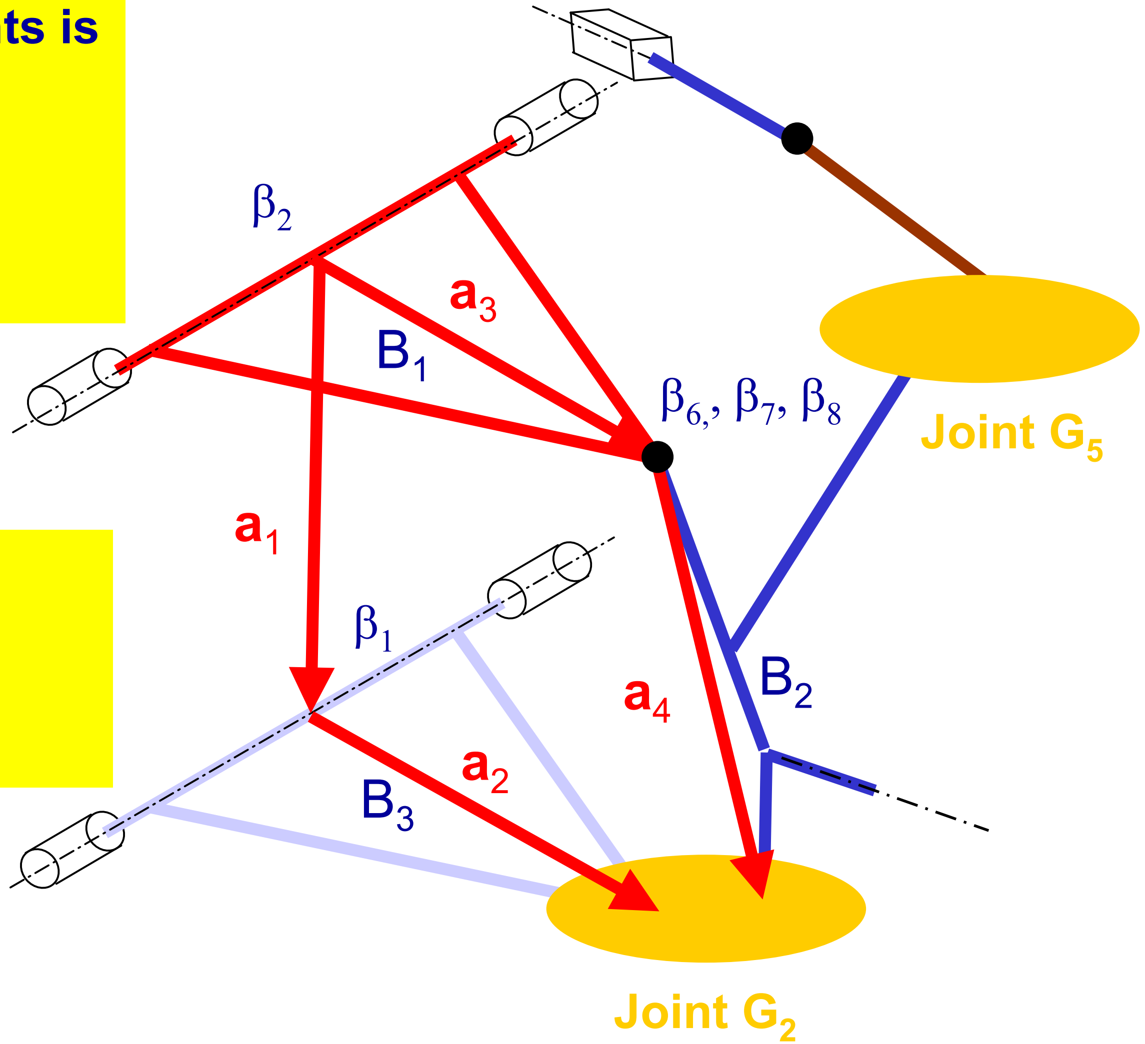
The relative motion in the joints is described through

$$f_G = \sum_{i=1}^{n_g} f_{G_i} \quad Joint\ variables \quad \beta_i,\ s_i$$

Closure Equations

e.g. for G_2:

$$a_1 + a_2(\beta_1) = a_3(\beta_2) + a_4(\beta_6, \beta_7, \beta_8)$$



Double Wishbone Suspension with disjointed loops

„vector-loop“ – Method (disconnecting loops)

5 Bodies
7 Joints

Number of independant loops:
$n_L = n_G - n_B = 7 - 5 = 2$

→ **Joints G_2 and G_5 are disconnected**

Assembly: each Joint has $(6 - f_{G_i})$ Constraints

Degrees of freedom of Joint

Constraint equations from closure conditions for position and orientation:

$$\boxed{\begin{array}{c} \sum\limits_i \underline{a}_i \overset{!}{=} \underline{0} \\ \prod\limits^{i} \mathbf{T}_i \overset{!}{=} \mathrm{I} \end{array}}$$

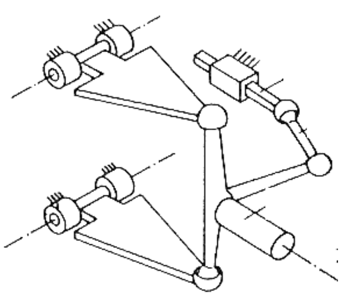
Identitymatrix

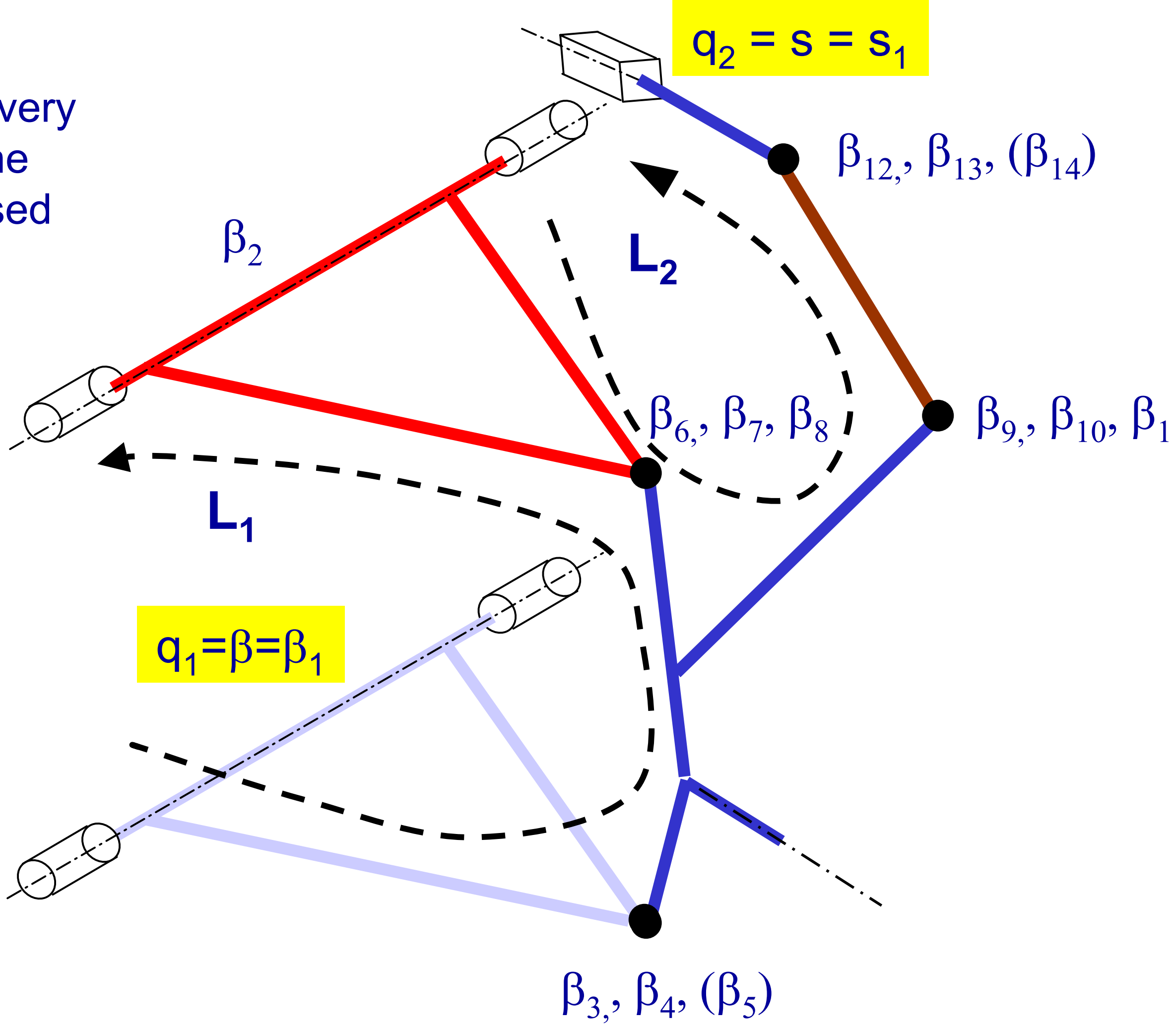
3. „topologische" – Methode (Loops are considered as tranmission elements)

Aim:
to calculate the motion of every body of the system using the input parameters (generalised coordinates)

$\beta_i = \beta_i(q_1,q_2)$, i=1...14

$s_i = s_i(q_2)$, i=1





Double Wishbone Suspension with transmission elements

„topologische“ Method (loops as transmission elements)

5 Bodies
7 Joints

Number of independant loops:

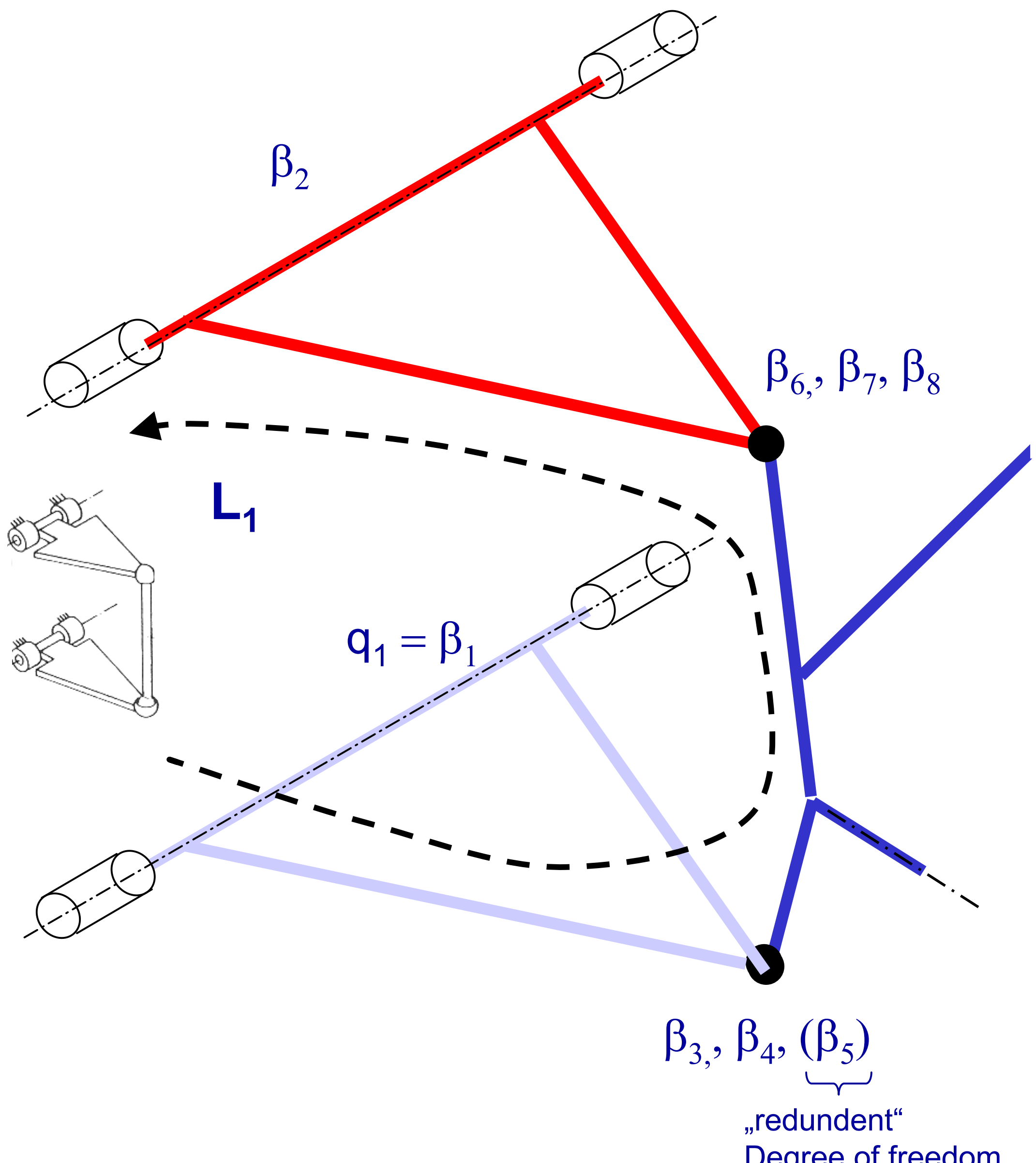
$n_L = n_G - n_B = 7 - 5 = 2$

Assembly: each loop is a kinematic transmission element (kinematic Transformator)

Constraint equations:

- local: in general 6 non-linear contraint equations per loop
- global: linear coupling equations between the loops; Number depends on the order of the coupling between the loops

3.1 Fundamentals of Kinematics



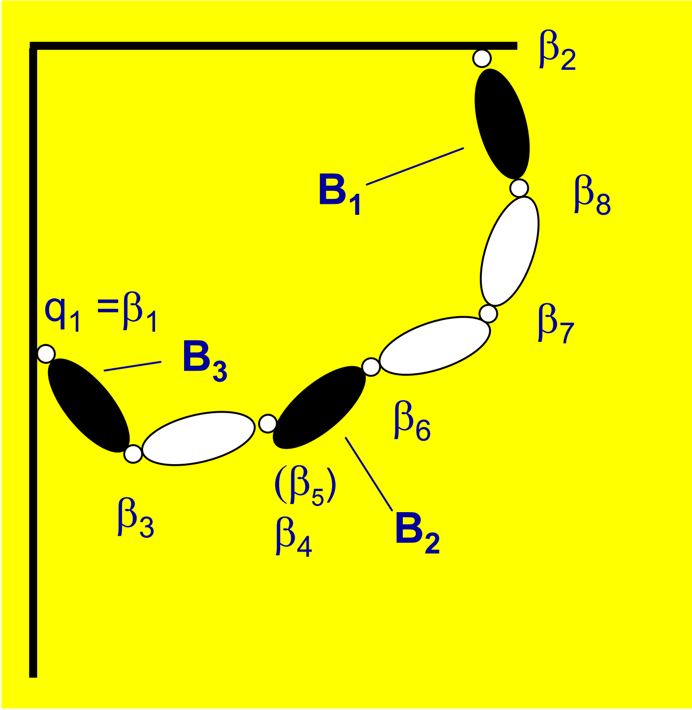
loop 1

$f_G = 8$, $n_L = 1$

$f_{L1} = f_G - 6n_L = 8-6 = 2$

Inputs: $\beta=\beta_1, \beta_6$

Outputs: $\beta_2, \beta_3, \beta_4, (\beta_5), \beta_7, \beta_8$



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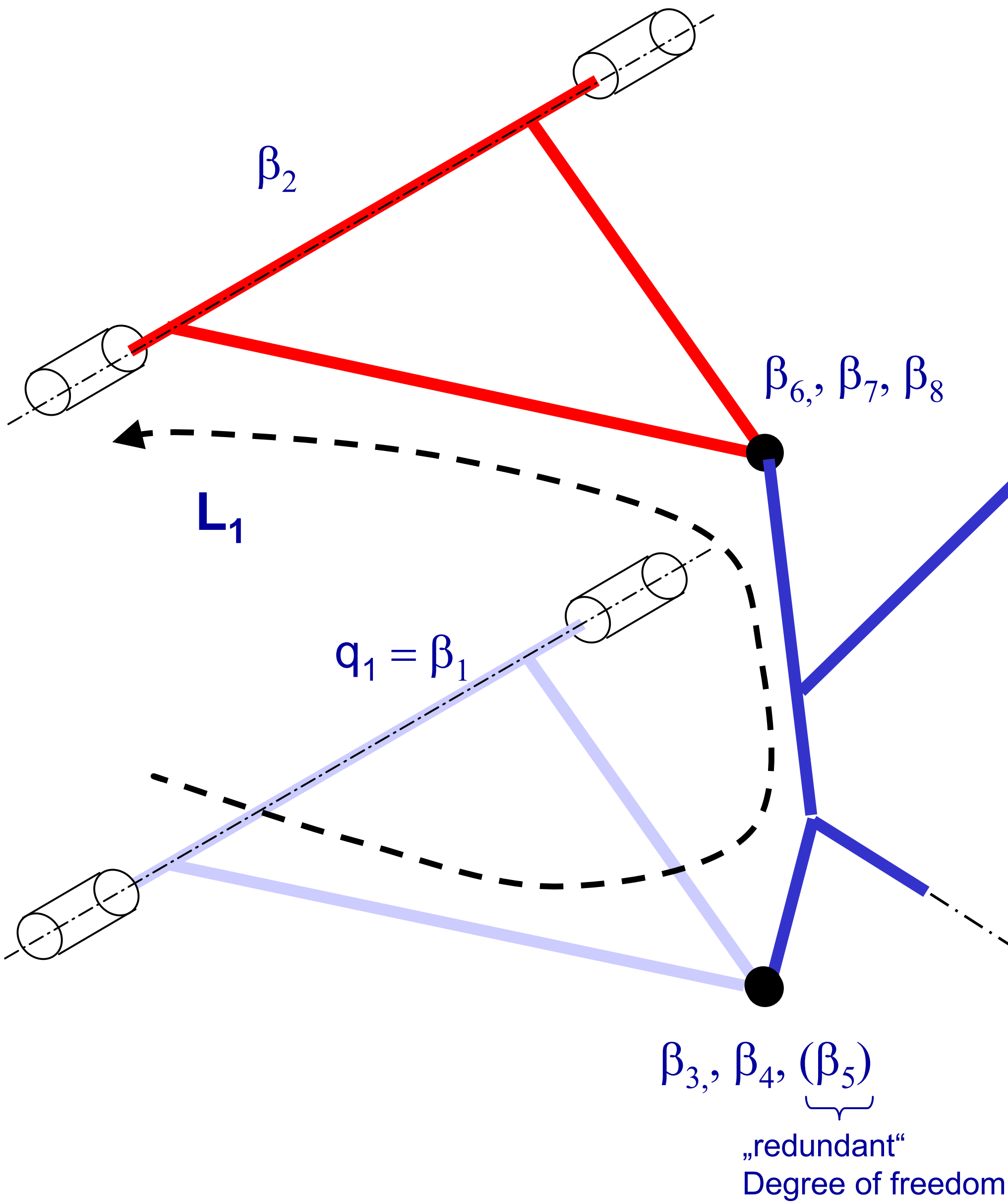

Loop 1

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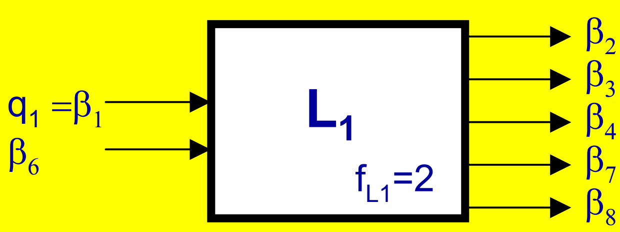
Inputs: $\beta=\beta_1, \beta_6$

Outputs: $\beta_2, \beta_3, \beta_4, (\beta_5), \beta_7, \beta_8$

Kinematic Transformer



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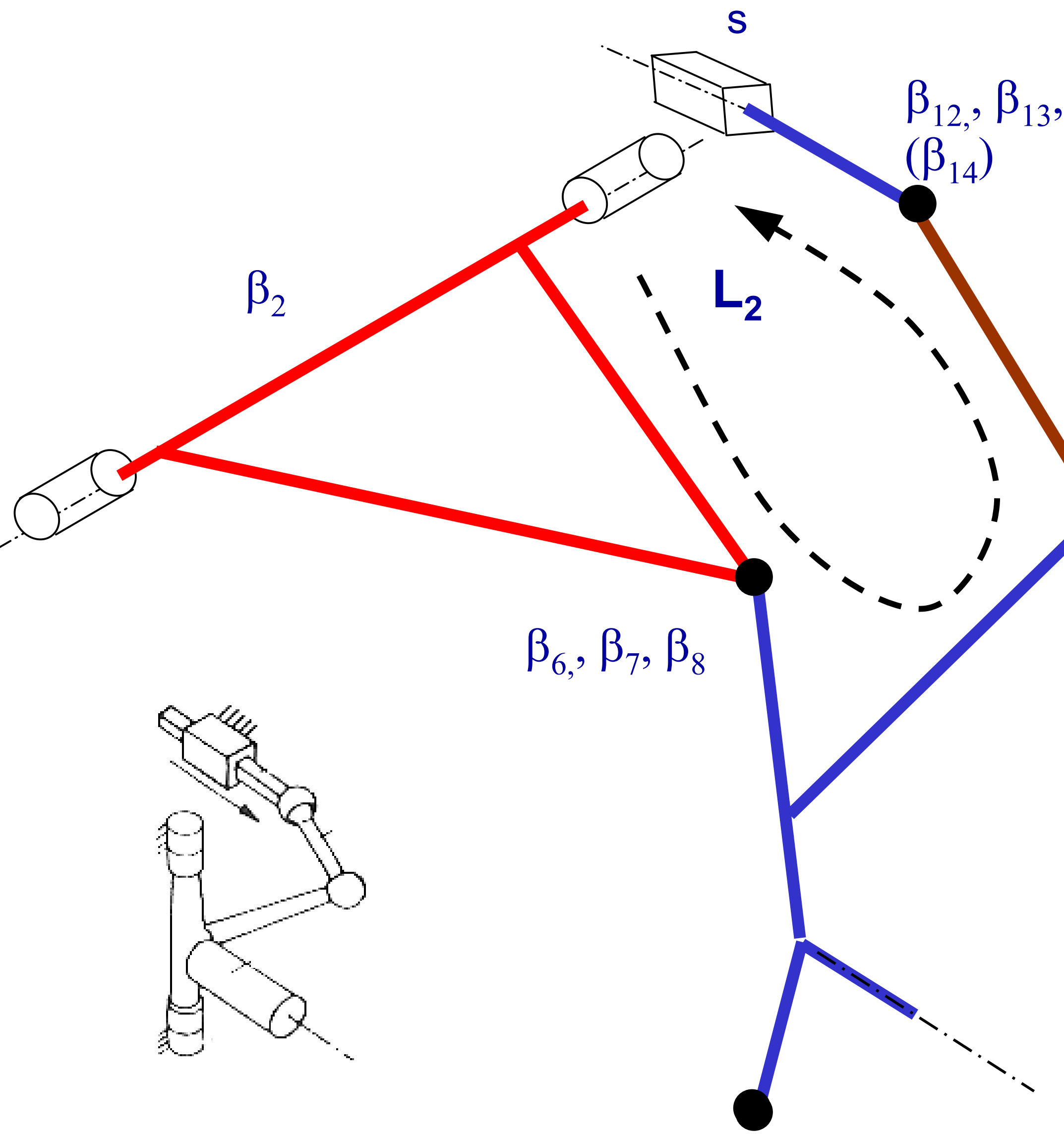


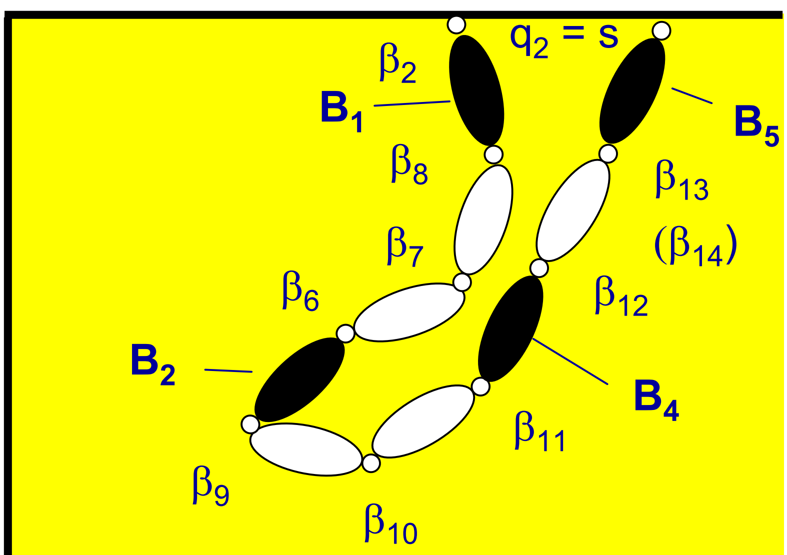
Loop 2

$f_G = 11,\ n_L = 1$

$f_{L2} = f_G - 6n_L = 11-6 = 5$

Inputs: β_2, $s=s_1$, β_7, β_8, (β_{14})

Outputs: β_6, β_9, β_{10}, β_{11}, β_{12}, β_{13}



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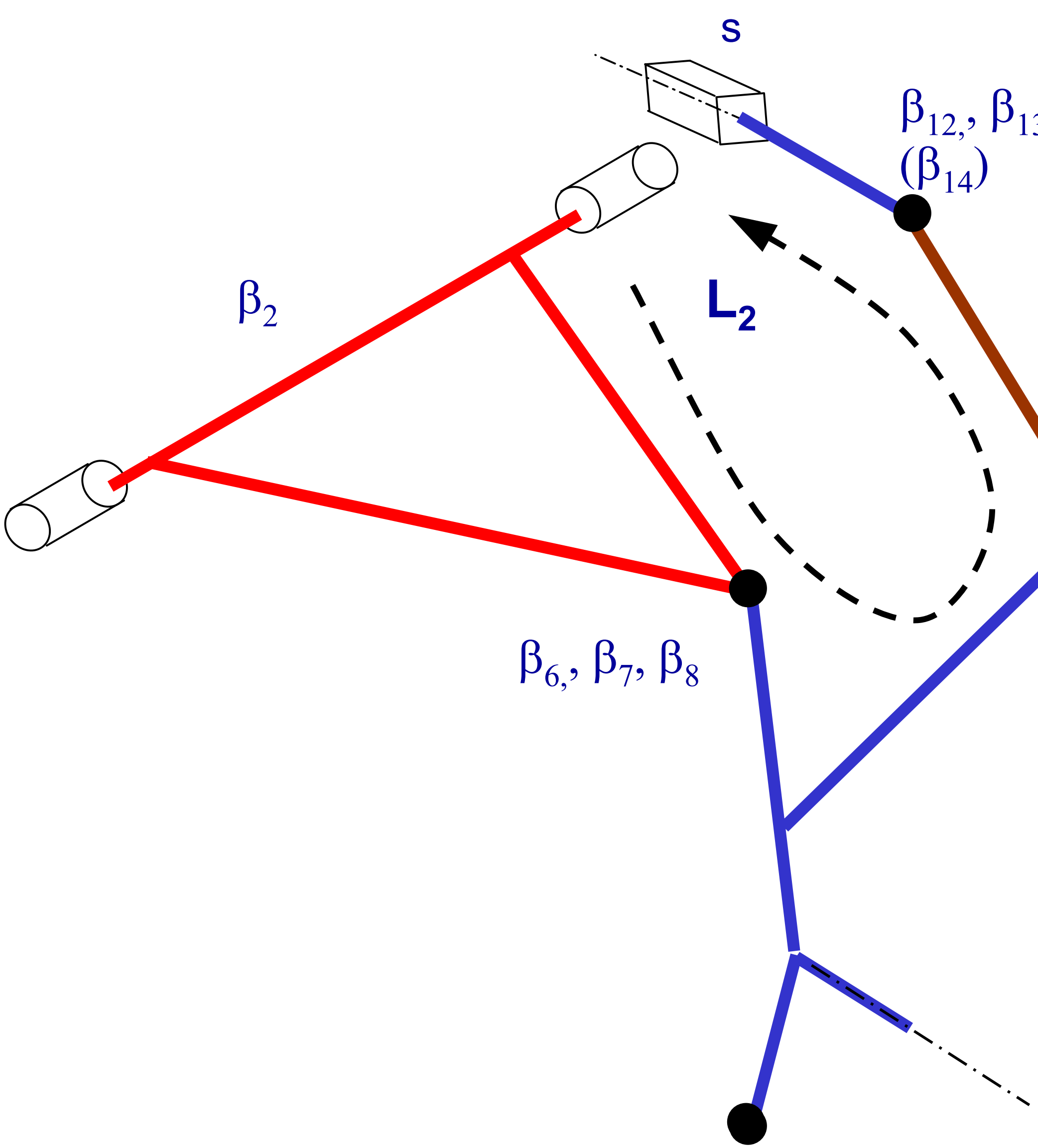


Loop 2

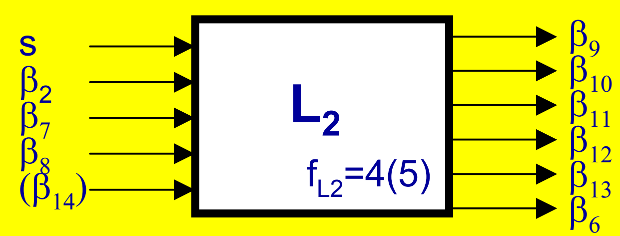
$f_G = 11$, $n_l = 1$

$f_{L2} = f_G - 6n_l = 11 - 6 = 5$

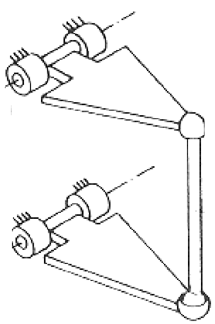
Inputs: β_2, $s = s_1$, β_7, β_8, (β_{14})

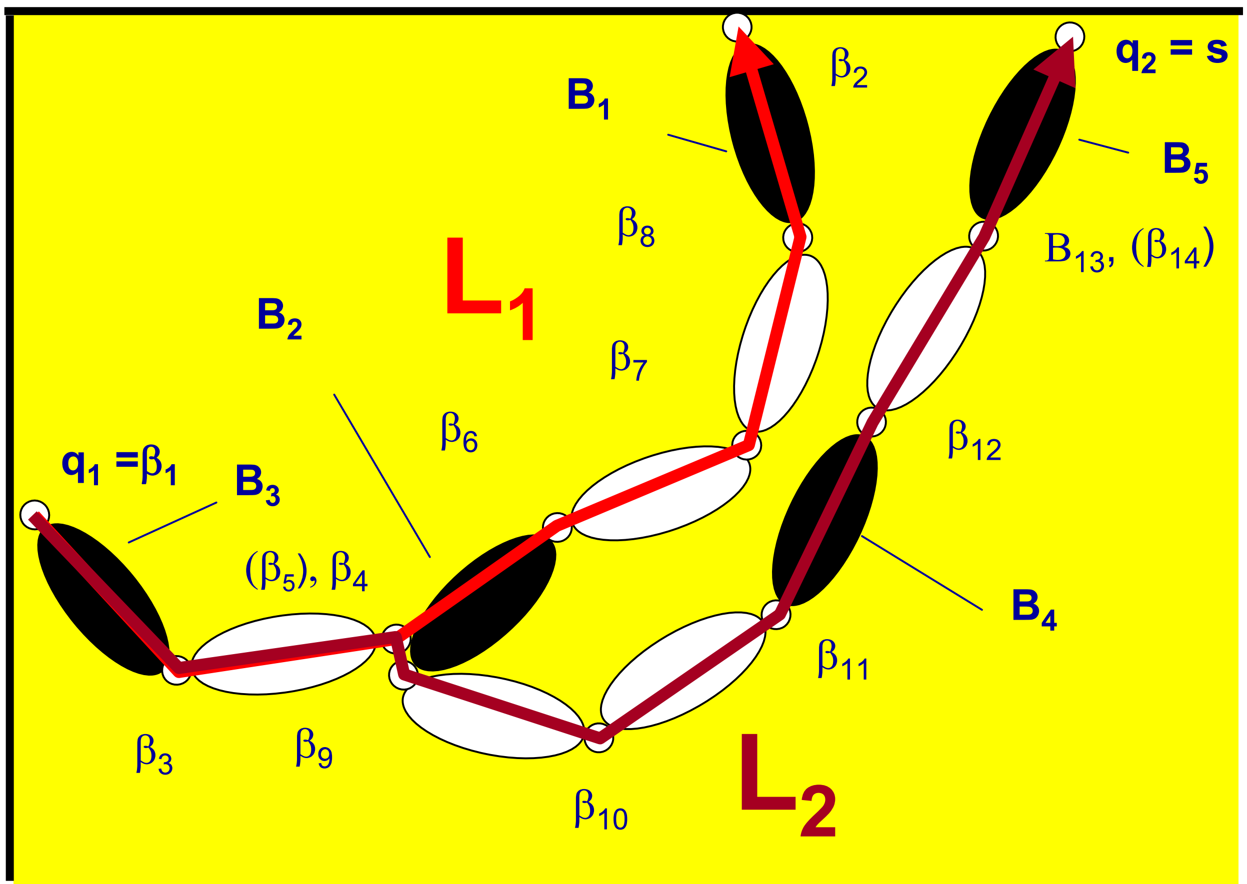
Outputs: β_6, β_9, β_{10}, β_{11}, β_{12}, β_{13}

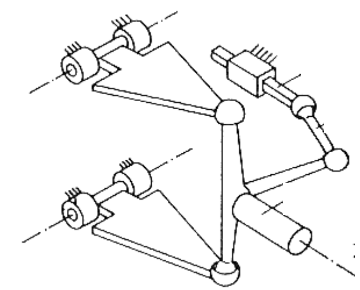
Kinematic Transformer



Topological description of the double wishbone suspension







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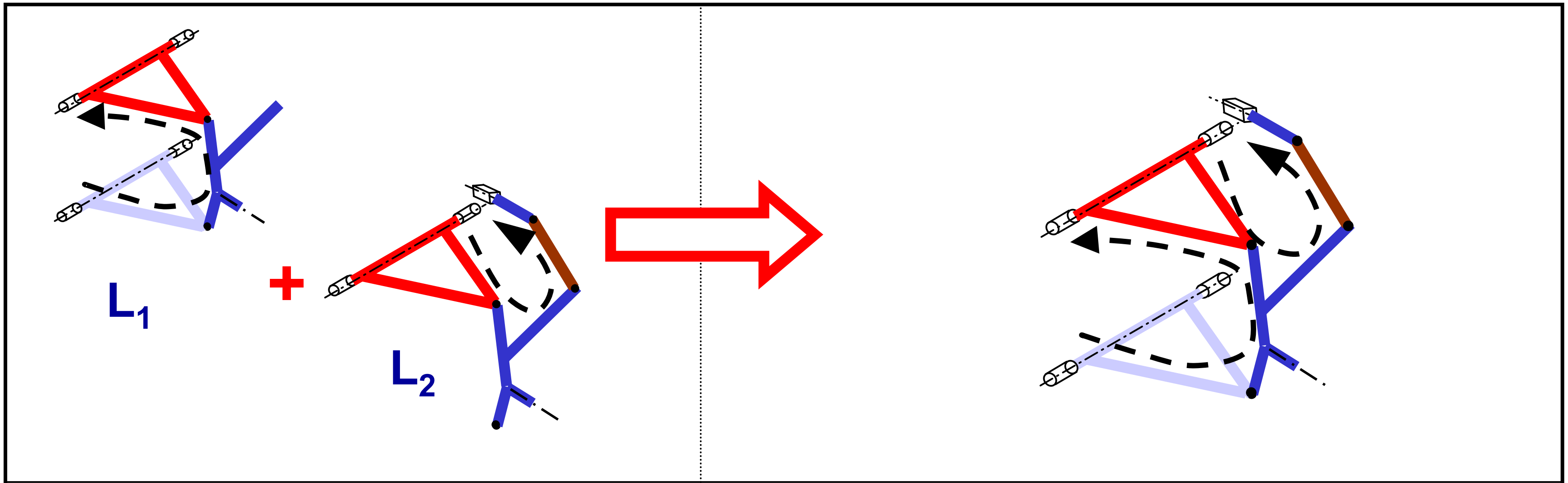
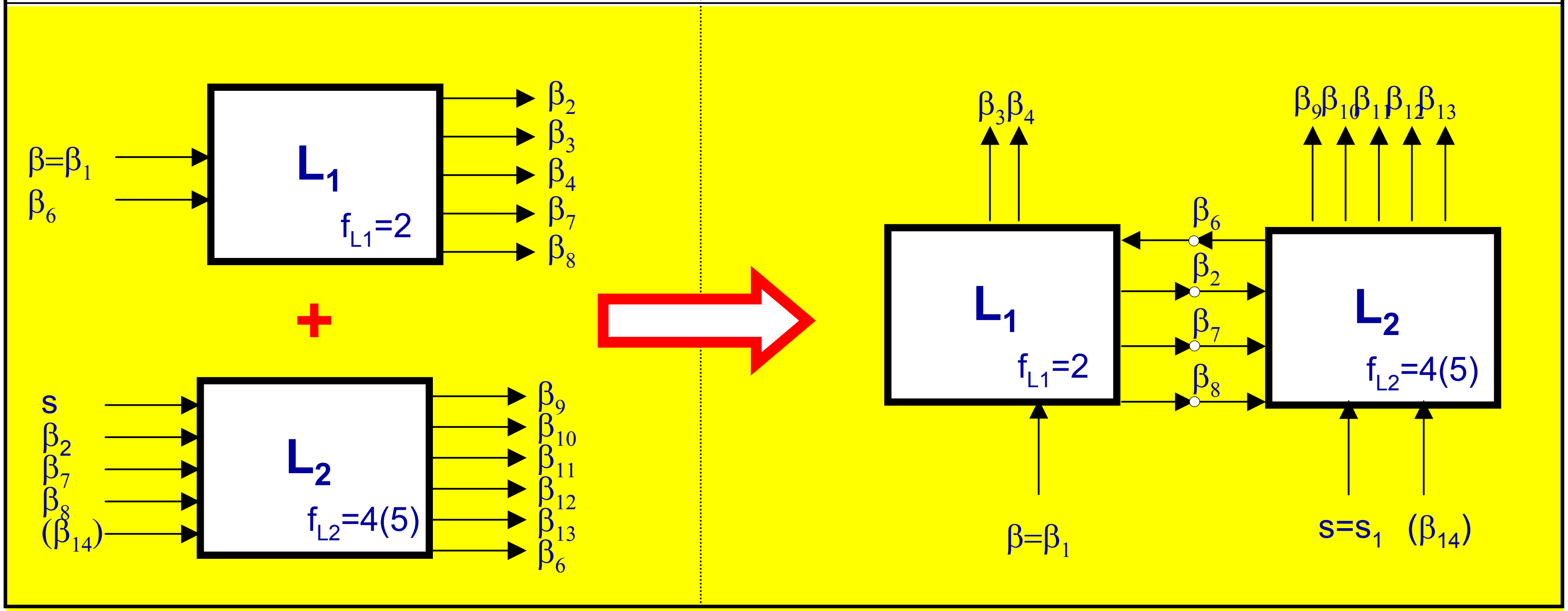
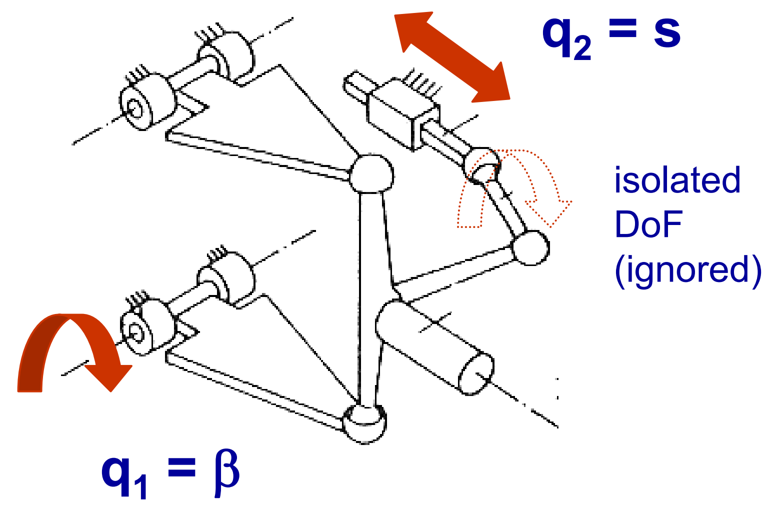
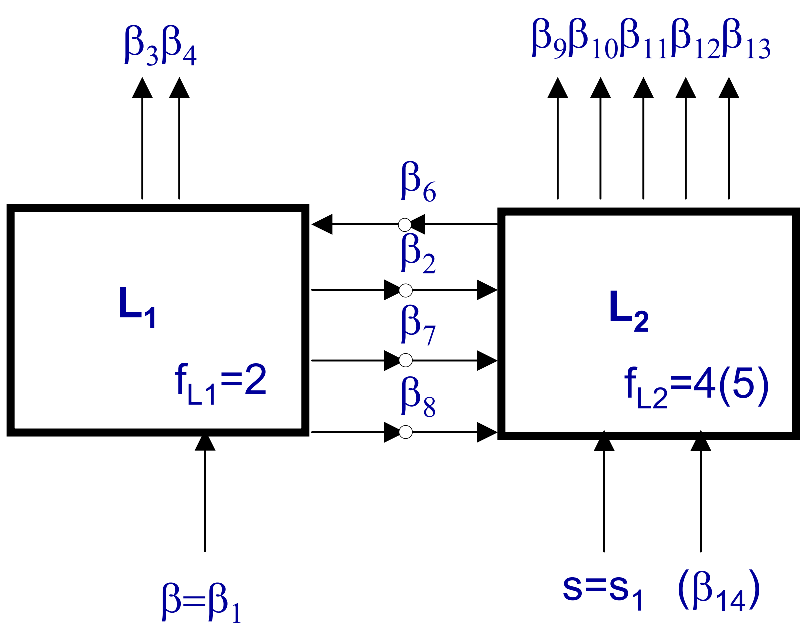




Kinematic Net

(without isolated degree of fredom)





2 kinematic Loops with (2+4) loop degrees of freedom **6**

-4 Coupling equations between the loops **- 4**

System degrees of freedom 2

Vehicle as a normal Multi-body system

Is modelled as a complex multi-body system with kinematic loops, consisting of rigid bodies. To begin with elastic characteristics are modelled as concentrated elasticities.

In the case of holonomic Constraints: Normal/Usual Multi-body systems

The equations of motion in minimal coordinates for a system with f degrees of freedom
In the general form:

$$M\,(q)\ddot{q} + b\,(q,\dot{q}) = Q\,(q,\dot{q},t)$$

q (f x 1) - Vector of general coordinates
M (f x f) - Mass matrix (symmetric)
b (f x 1) - Vector of general centrifugal and gyro Forces
Q (f x 1) - Vector of general applied forces

Specialities in Multi-loop

Multi-body systems and Mechanisms

Because of the kinematic loops, there are comparatively less degrees of freedom in a System with more number of bodies and constraints.

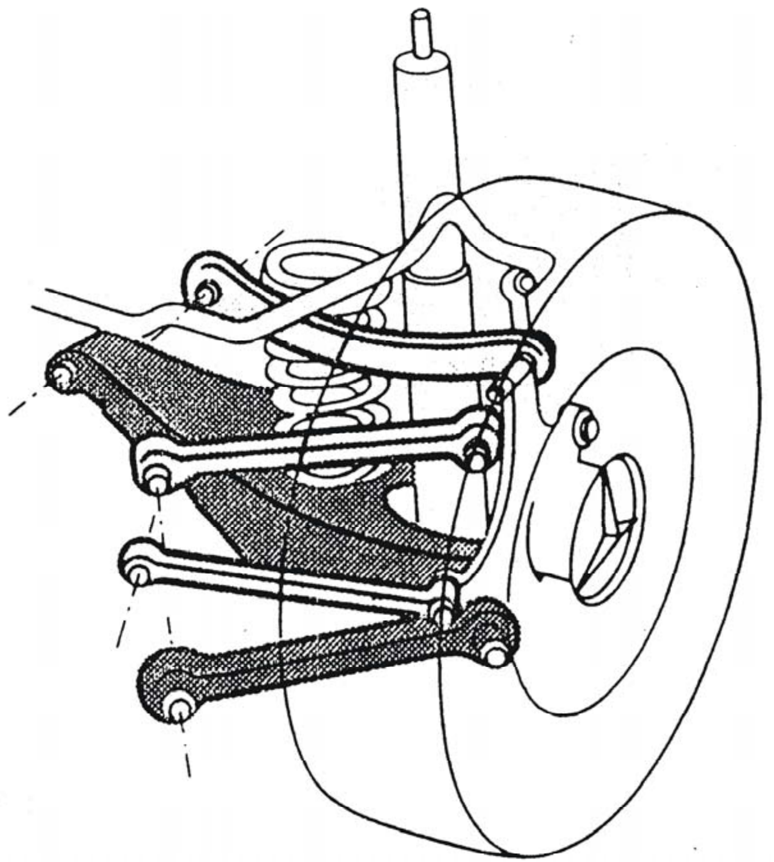
Setting up complex equations of motion:

- **manually (very strenuous!)**
- **numerically and/or using symbols with the help of computers**

Example: The Five-point Wheel Suspension (Spatial)

Equations of motion for the spring compression (f = 1 Degree of freedom

multilink rear suspension Daimler-Benz



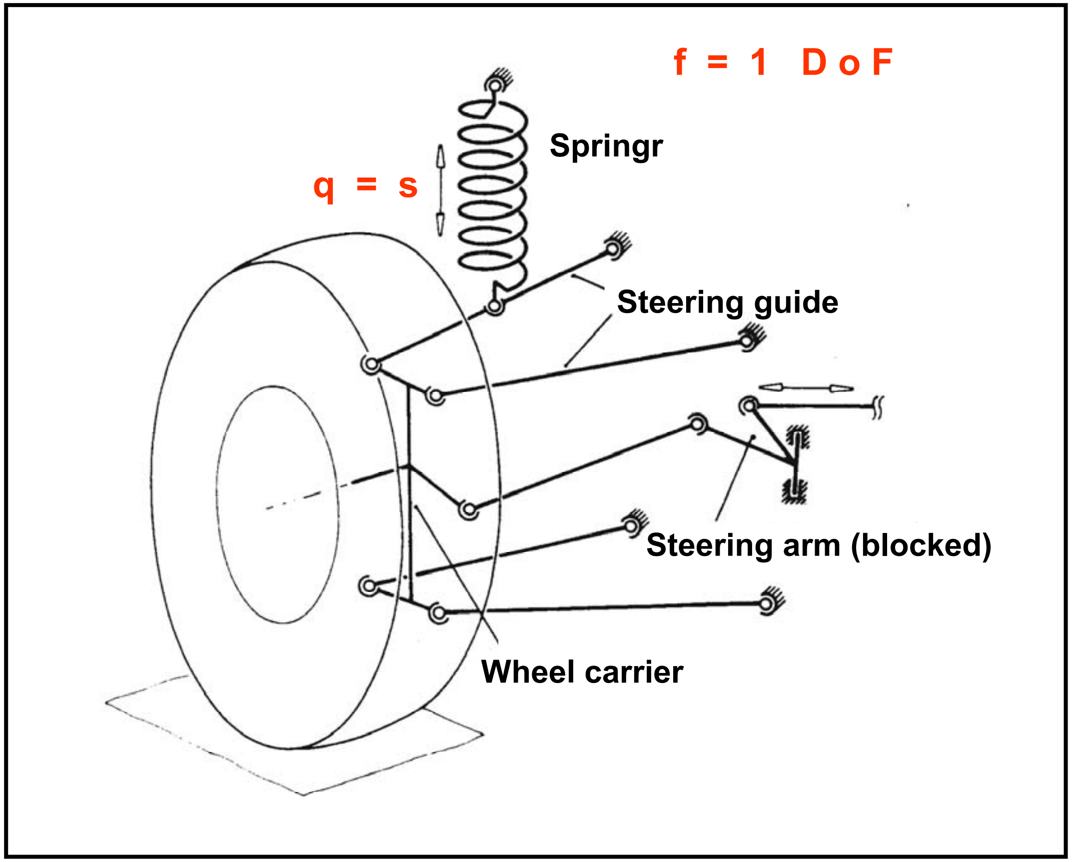
3.2 Equations of motion for Multi-Body systems

Example: The Five-point Wheel Suspension (Spatial)

Equations of motion in the form

$$m(s)\ddot{s} + b(s,\dot{s}) = Q(s,\dot{s},t)$$

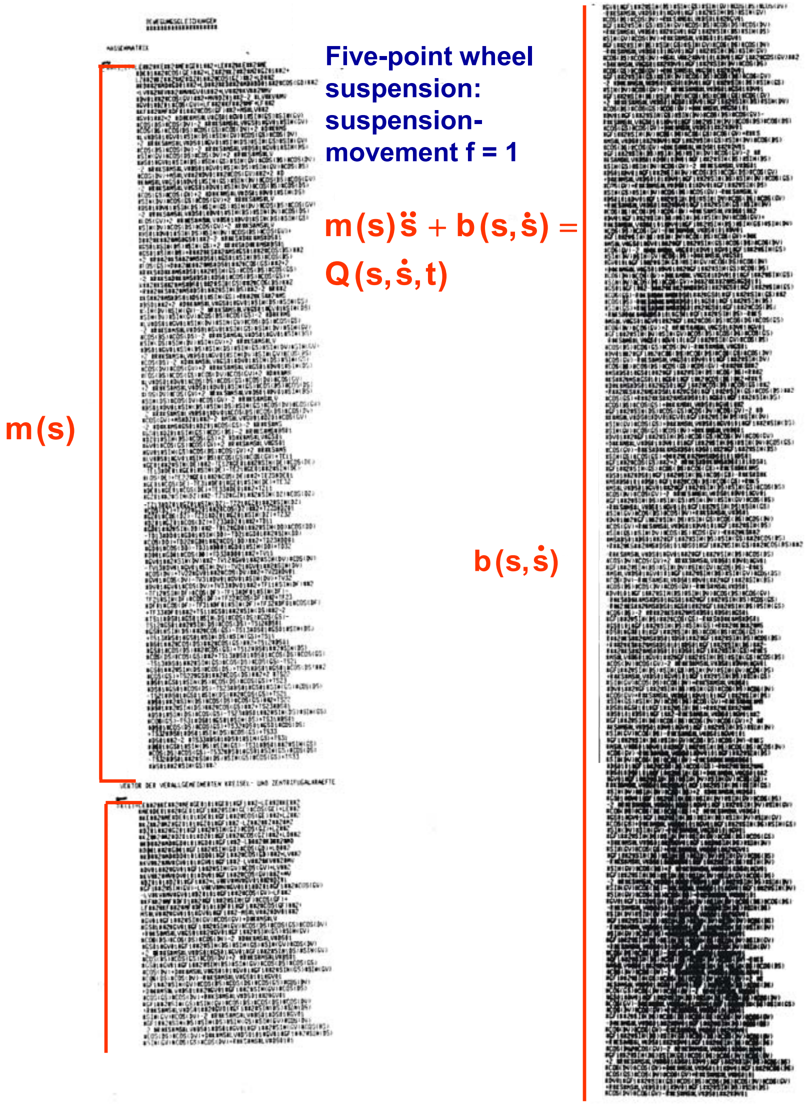
Using symbols with
Programmsystem NEWEUL



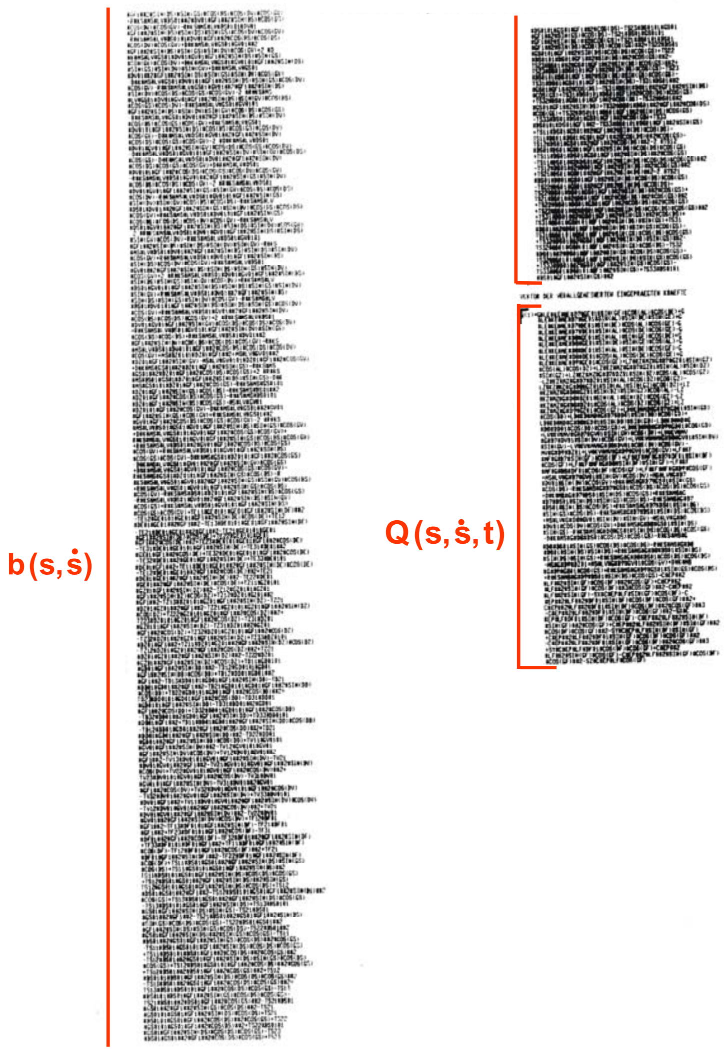
3.2 Equations of motion for Multi-Body systems

Five-point wheel suspension: suspension-movement f = 1

$$m(s)\ddot{s} + b(s,\dot{s}) = Q(s,\dot{s},t)$$



wheel suspension (cont.)



3.2 Equations of motion for Multi-Body systems

Methods to set up the equations of motion for multi-body systems and mechanisms

Holonom Systems ≡ considered as normal Multi-body systems

n Bodies
r Constraints

$\left.\right\}$ **$f = 6n - r$ Degrees of freedom**

- **Principle of linear momentum and Principle of conservation of angular momentum (Newton-Euler-Equation)** →
  Number of equations of motion: $6n > f$ (f)

- **Lagrange Equations of first order** **(Method 1)**
  Number of equations of motion : $6n > f$

- **Lagrange Equations of scond order** **(Method 2)**
  Number of equations of motion : f

- **d`Alembert's Principle** **(Method 3)**
  Number of equations of motion :
  → **$6n$ (Lagrange Multiplicators)**
  → **f (Minimum coordinates)**

Methods to set up the equations of motion

1. Method:
Based on the Fundamental equations of dynamics

$$\sum_{i=1}^{N}(\underline{F}_i - m_i\,\underline{a}_i)\cdot\delta\underline{r}_i = 0 \qquad \textbf{(}\ \underline{\mathbf{F}}_{\mathbf{i}}\ \textbf{- Applied forces)}$$

Will be then applied in the LAGRANGE equations

first order for point masses:

Given is a system with

N Mass points $m_i\,,\ \underline{r}_i\,,$

g geometric Constraints $f_\alpha(t\,;\,r_1,\ldots,r_N) = 0\ \ ;\ \alpha = 1,\ldots,g$

k kinematic Constraints $\phi_\beta \equiv \sum_{i=1}^{N}\ell_{i\beta}\,(t;\underline{r}_1,\ldots,\underline{r}_N)\cdot\underline{v}_i + d_\beta\,(t;\underline{r}_1,\ldots,\underline{r}_N) = 0\ ;$

$\beta = 1,\ldots,k$

⇨ **Hence the System has** $\mathbf{f = 3\,N - g - k}$ **Degrees of freedom**

By applying the LAGRANGE Multiplicators λ_α , μ_β

We get the LAGRANGE Equations of first order:

$$\left.\begin{aligned} & m_i\,\underline{a}_i = \underline{F}_i + \sum_{\alpha=1}^{g} \lambda_\alpha \frac{\partial f_\alpha}{\partial \underline{r}_i} + \sum_{\beta=1}^{k} \mu_\beta\, \underline{\ell}_{i\beta} && ;\ i = 1,\ldots,N \\ & f_\alpha\ (t;\ \underline{r}_1, \ldots, \underline{r}_N) = 0 && ;\ \alpha = 1,\ldots,g \\ & \phi_\beta \equiv \sum_{i=1}^{N} \underline{\ell}_{i\beta} \cdot \underline{v}_i + d_\beta = 0 && ;\ \beta = 1,\ldots,k \end{aligned}\right\}$$

(DAE – Differential Algebraic Equations)

(3N+g+k) Equations for the (3N+g+k) Unknowns

$$\underbrace{x_i, y_i, z_i}_{3N},\ \underbrace{\lambda_\alpha}_{g},\ \underbrace{\mu_\beta}_{k}$$

Advantages:- applicable for <u>holonomic</u> and <u>non-holonomic</u> systems
- Equations can be easily set up
- Reaction forces can be calculated directly

Disadvantages:- more equations than degrees of freedom
- equations are numerically unstable

Equations for rigid body systems are analogous!

2. Method:

By introducing f independant general coordinates (corresponding to the number of degrees of freedom)

$$q_1, q_2, \ldots, q_f$$

One can obtain from the fundamental dynamic equations
The LAGRANGE equations of second order for holonom Systems:

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \quad ; \quad j = 1, \ldots, f}$$

3. Method: Equations of motion obtained from the d`Alembert's Principle with the help of kinematic Differentials:
Systematic procedure to solve the constraint equations making use of the solved loop kinematics

1. Identifying the independant loops

- introduction of natural coordinates β, s
- setting up local, non-linear constraint equations
- local solutions from the outputs

⇨ Keep at hand the „kinematic Transformator“

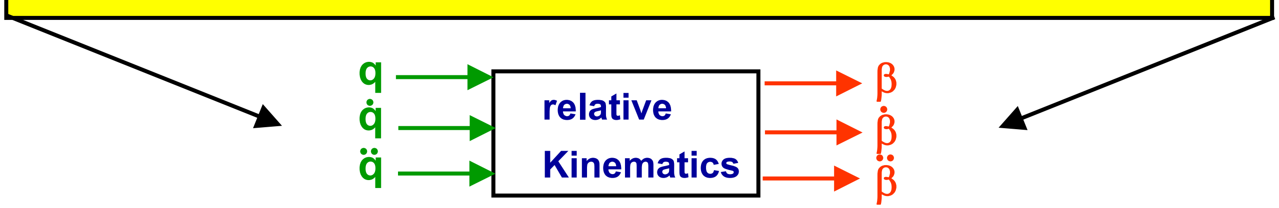
2. Defining the loop network

- setting up the linear coupling equations

⇨ kinematic Network

3. **Choosing the structural inputs**

⇨ Optimising the solution flow

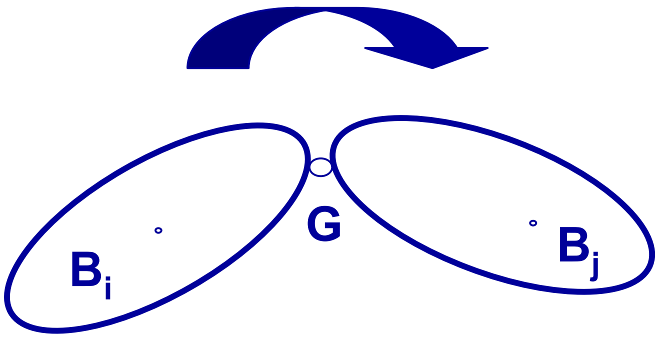


Forward kinematics (recursive)

- **Absolute coordinates, Body B_i**

$$w_i = [x_i, y_i, z_i, \varphi_i, \psi_i, \theta_i]$$

Joint Coordinate β^*



Position:

$$w_j = w_i(w_1, w_2, \ldots, w_i, \beta, t)$$

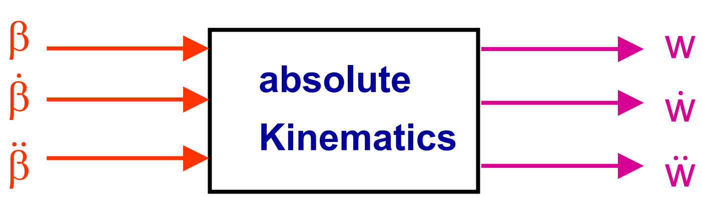
Velocity:

$$\dot{w}_j = \dot{w}_j(w_1, \ldots, w_i; \dot{w}_1, \ldots, \dot{w}_i; \beta, \dot{\beta}; t)$$

•It represents here translation (s) and rotation (β)

Acceleration:

$$\ddot{w}_j = \ddot{w}_j(w_1, \ldots, w_i; \dot{w}_1, \ldots, \dot{w}_i; \ddot{w}_1, \ldots, \ddot{w}_i; \beta, \dot{\beta}, \ddot{\beta}; t)$$



Kinematics of two bodies joined together with a joint

Parameters of Motion of B_i are known

Parameters of motion of B_J are required

Rotational transition from B_i to B_j

$$\boldsymbol{\omega}_j = \boldsymbol{\omega}_i + \boxed{{}_i\boldsymbol{\omega}_j}$$

$$\boldsymbol{\alpha}_j = \boldsymbol{\alpha}_i + \boldsymbol{\omega}_i \times \boxed{{}_i\boldsymbol{\omega}_j} + {}_i\boldsymbol{\alpha}_j$$

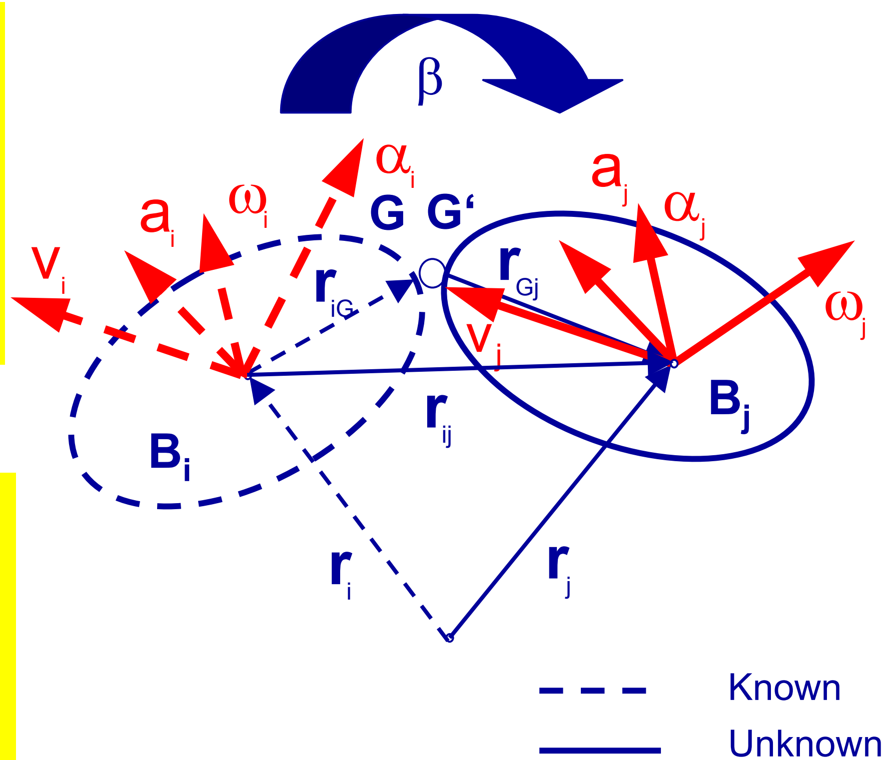
Translatory transition from B_i to B_j

$$\mathbf{r}_j = \mathbf{r}_i + \mathbf{r}_{ij}$$

$$\mathbf{v}_j = \mathbf{v}_i + \boldsymbol{\omega}_i \times \mathbf{r}_{ij} + {}_i\mathbf{v}_j$$

$$\mathbf{a}_j = \mathbf{a}_i + \boldsymbol{\alpha}_i \times \mathbf{r}_{ij} + 2\boldsymbol{\omega}_i \times \boxed{{}_i\mathbf{v}_j} + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \mathbf{r}_{ij}) + \boxed{{}_i\mathbf{a}_j}$$

Elementary standard joint



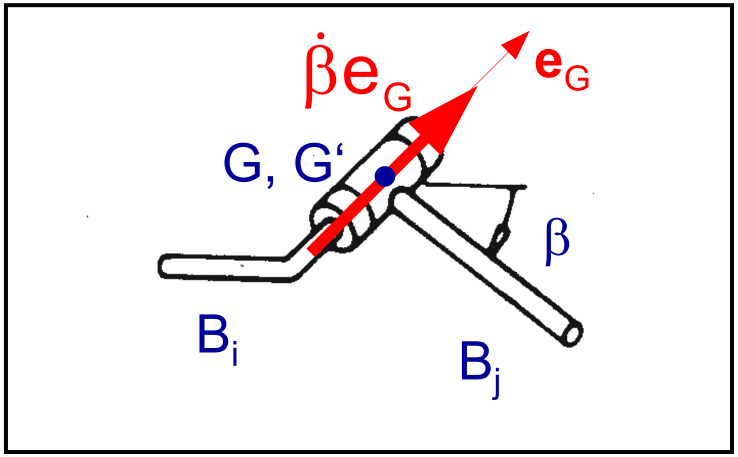
Only one dimensional joints are allowed. In the case of multi dimensional joints, they have to be represented with single dimensional joints with virtual bodies inbetween them.

Used representaions

r_i, r_j	Position vector to the reference/considered point,
v_i, v_j	Absolute velocity of the reference/considered point,
a_i, a_j	Absolute acceleration of the reference/considered point,
ω_i, ω_j	Absolute angular velocity of the body
α_i, α_j	Absolute angular acceleration of the body,
r_{ij}	connecting vector between the reference/considered points,
${}_iv_j$, ${}_ia_j$	velocity / Acceleration of B_j relative to B_i,
${}_i\omega_j$, ${}_i\alpha_j$	Angular velocity and angular acceleration of B_j relative B_i.

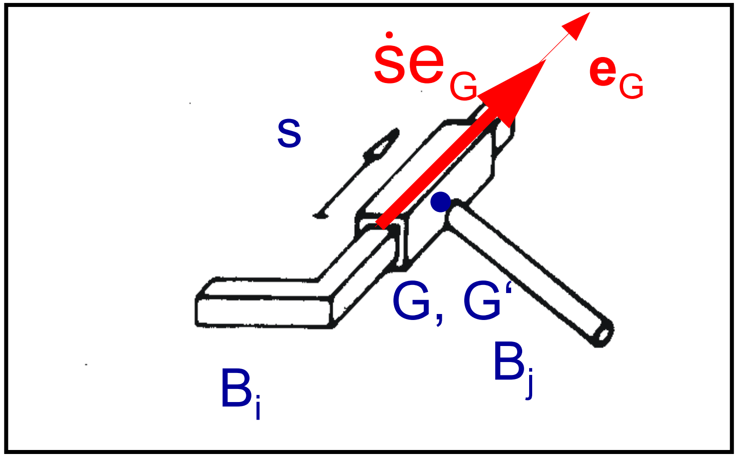
3.2 Equations of motion for Multi-Body systems

Rotational joint



$$r_{ij} = r_{iG} + r_{G'j}$$
$${}_iv_j = {}_G\omega_{G'} \times r_{G'j}$$
$${}_ia_j = {}_G\alpha_{G'} \times r_{G'j} + {}_G\omega_{G'} \times ({}_G\omega_{G'} \times r_{G'j})$$
$${}_i\omega_j = {}_G\omega_{G'}$$
$${}_i\alpha_j = {}_G\alpha_{G'}$$
$${}_G\omega_{G'} = \dot{\beta}e_G$$
$${}_G\alpha_{G'} = \ddot{\beta}e_G$$

Translatory joint



$$r_{ij} = r_{iG} + r_{G'j}$$
$${}_iv_j = {}_Gv_{G'}$$
$${}_ia_j = {}_Ga_{G'}$$
$${}_i\omega_j = 0$$
$${}_i\alpha_j = 0$$
$${}_Gv_{G'} = \dot{s}e_G$$
$${}_Ga_{G'} = \ddot{s}e_G$$

Translation

$$v_j = v_i + \omega_i \times r_{ij} + {}_i v_j = v_i + \omega_i \times (r_{iG} + r_{G'j}) + {}_G v_{G'}$$

$$= v_i + \omega_i \times (r_{iG} + r_{G'j}) + \dot{s} e_G$$

$$a_j = a_i + \alpha_i \times r_{ij} + 2\omega_i \times {}_i v_j + \omega_i \times (\omega_i \times r_{ij}) + {}_i a_j =$$

$$= a_i + \alpha_i \times (r_{iG} + r_{G'j}) + 2\omega_i \times {}_G v_{G'} + \omega_i \times (\omega_i \times (r_{iG} + r_{G'j})) + {}_G a_{G'} =$$

$$= a_i + \alpha_i \times (r_{iG} + r_{G'j}) + 2\omega_i \times \boxed{\dot{s} e_G} + \omega_i \times (\omega_i \times (r_{iG} + r_{G'j})) + \boxed{\ddot{s} e_G}$$

Rotation

$$\omega_j = \omega_i + {}_j\omega_i = \omega_i + {}_G\omega_{G'} = \omega_i + \dot{\beta} e_G$$

$$\omega_j = \alpha_i + \omega_i \times {}_i\omega_j + {}_i\alpha_j = \alpha_i + \omega_i \times {}_G\omega_{G'} + {}_G\alpha_{G'} =$$

$$= \alpha_i + \omega_i \times \boxed{\dot{\beta} e_G} + \boxed{\ddot{\beta} e_G}$$

Used representation

r_{ij}	Vektor from reference point B_i to the reference point B_j
r_{iG}	Vektor from reference point B_i to the joint point G,
$r_{G'j}$	Vektor vom joint point G‘ to the reference point B_j,
${}_G v_{G'} = \dot{s} e_G$	Velocity of G‘ relative to G (translatory Joint),
${}_G a_{G'} = \ddot{s} e_G$	Acceleration of G‘ relative to G (translatory joint),
${}_G \omega_{G'} = \dot{\beta} e_G$	Angular velocity of G‘ relative to G (rotational joint),
${}_G \alpha_{G'} = \ddot{\beta} e_G$	Angular acceleration of G‘relative to G (rotational joint).

$$ {}^{i}\underline{x} = {}^{i}\underline{T}_{j}\,{}^{j}\underline{x} \tag{3.23} $$

$$ {}^{j}\underline{x} = {}^{j}\underline{T}_{i}\,{}^{i}\underline{x} = {}^{i}\underline{T}_{j}^{T}\,{}^{i}\underline{x} \tag{3.24} $$

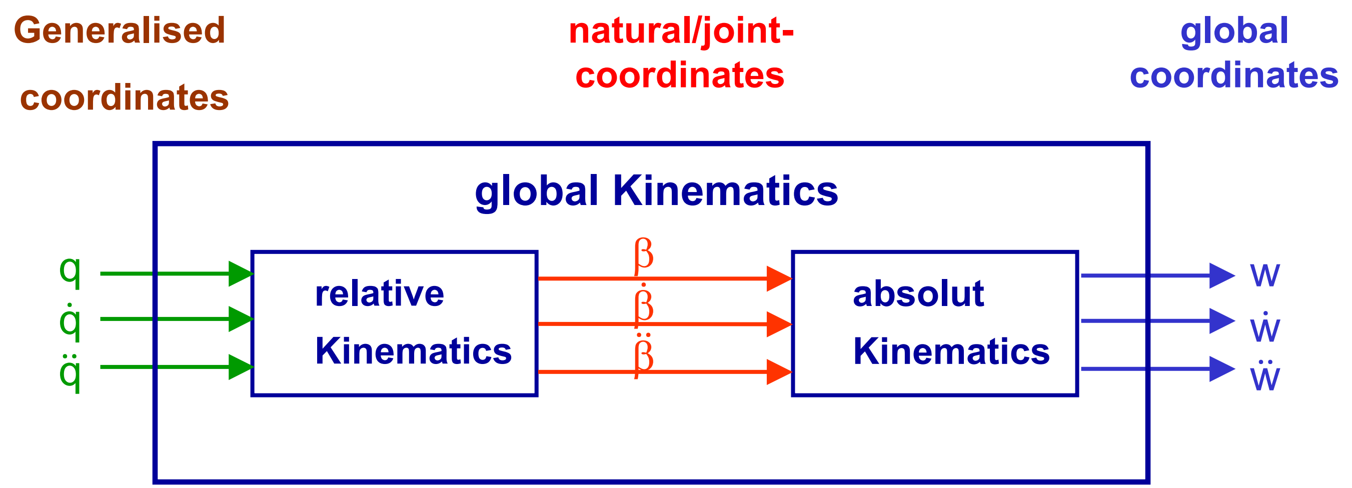
$$ {}^{i}\underline{T}_{j} = \begin{bmatrix} {}^{i}\underline{e}_{xj} & {}^{i}\underline{e}_{yj} & {}^{i}\underline{e}_{zj} \end{bmatrix} = \begin{bmatrix} {}^{j}\underline{e}_{xi}^{T} \\ \hline {}^{j}\underline{e}_{yi}^{T} \\ \hline {}^{j}\underline{e}_{zi}^{T} \end{bmatrix} \tag{3.25} $$

$$ {}^{i}\underline{T}_{j} = {}^{i}\underline{T}_{G}\,{}^{G}\underline{T}_{G`}\,{}^{G`}\underline{T}_{j} \tag{3.26} $$

$$ {}^{G}\underline{T}_{G`}({}^{G}\underline{u},\varphi) = \cos\varphi\,\underline{I} + \sin\varphi \begin{bmatrix} 0 & -{}^{G}u_{z} & {}^{G}u_{y} \\ {}^{G}u_{z} & 0 & -{}^{G}u_{x} \\ -{}^{G}u_{y} & {}^{G}u_{x} & 0 \end{bmatrix} + (1-\cos\varphi)\,{}^{G}\underline{u}\,{}^{G}\underline{u}^{T} \tag{3.27} $$

$$ {}^{G}\underline{T}_{G`} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.28} $$

Summary of Kinematics



Equations of motion derived from the d`ALEMBERT Principle

Reaction forces on B_i

Reaction moments on B_i

$$\sum_{i=1}^{n_B} \left\{ \left(m_i \, \ddot{\underline{s}}_i - \underline{F}_i \right) \delta \underline{s}_i + \left(\underline{\underline{\theta}}_{si} \, \dot{\underline{\omega}}_i + \underline{\omega}_i \times \underline{\underline{\theta}}_{si} \, \underline{\omega}_i - \underline{T}_i \right) \delta \underline{\phi}_i \right\} = 0$$

Virtual work done by the Reaction forces and moments on B_i

m_i , $\underline{\underline{\theta}}_{si}$	:	Mass und inertial tensor (relative to the center of mass) of the body i
$\ddot{\underline{s}}_i$	:	acceleration of the centre of mass of the body i
$\underline{\omega}_i$, $\dot{\underline{\omega}}_i$	:	angular velocity and acceleration
$\underline{F}_i$, $\underline{T}_i$	:	resulting applied forces and moments
$\delta \underline{s}_i$, $\delta \underline{\phi}_i$	:	virtual translation or rotational orische, bzw. rotatorische displacements

Principle of linear momentum

$$m_i\ddot{s}_i = F_i + F_{Ri} \Rightarrow$$

$$F_{Ri} = m_i\ddot{s}_i - F_i$$

$$F_{Ri} = m_i\ddot{s}_i - F_i = m_i J_{Si}\ddot{q} + m_i a_{Si} - F_i$$

Principle of conservation of angular momentum

$$\theta_{Si}\dot{\omega}_i + \omega_i \times \theta_{Si}\omega_i = T_i + T_{Ri} \Rightarrow$$

$$T_{Ri} = \theta_{Si}\dot{\omega}_i + \omega_i \times \theta_{Si}\omega_i - T_i$$

$$T_{Ri} = \theta_{Si}J_{\phi i}\ddot{q} + \theta_{Si}a_{\phi i} + \omega_i \times \theta_{Si}\omega_i - T_i$$

D'Alembert's Principle

virtual work of the Reaction forces vanishes

$$\delta A = \delta \sum_{i=1}^{i=n_B} (s_i^T F_R + \phi_i^T T_R) = \sum_{i=1}^{i=n_B} \delta s_i^T F_R + \delta\phi_i^T T_R =$$

$$= \sum_{i=1}^{i=n_B} \left(\left(\frac{\partial s_i}{\partial q}\delta q \right)^T F_R + \left(\frac{\partial \phi_i}{\partial q}\delta q \right)^T T_R \right) = (\delta q)^T \sum_{i=1}^{i=n_B} \left(\left(\frac{\partial s_i}{\partial q} \right)^T F_R + \left(\frac{\partial \phi_i}{\partial q} \right)^T T_R \right) =$$

$$= (\delta q)^T \sum_{i=1}^{i=n_B} \left(J_{Si}{}^T F_R + J_{\phi i}{}^T T_R \right) =$$

$$= (\delta q)^T \sum_{i=1}^{i=n_B} \left(J_{Si}{}^T (m_i J_{Si}\ddot{q} + m_i a_{Si} - F_i) + J_{\phi i}{}^T (\theta_{Si}J_{\phi i}\ddot{q} + \theta_{Si}a_{\phi i} + \omega_i \times \theta_{Si}\omega_i - T_i) \right) = 0$$

3.2 Equations of motion for Multi-Body systems

One requires the transfoemation of

Equations of motion

$$\boxed{\delta q^T [\, M \ddot{q} + b \,] = \delta q^T Q} \xrightarrow{\delta q_i \ \text{independant}} \boxed{M \ddot{q} + b = Q}$$

q : [f x 1] - Vector of generalised coordinates

(f = number of degrees of freedom)

M : [f x f] - System – Mass matrix (symmetric, regular)

b : [f x 1] - Vector of generalised gyro and centrifugal forces

Q : [f x 1] - Vector of generalised applied forces

→ Introducing the kinematic relations

$$\left.\begin{array}{lll} \delta\, {}^{\circ}\underline{s}_i = {}^{\circ}J_{s_i} \delta q \quad ; & {}^{\circ}\underline{\ddot{s}}_i = {}^{\circ}J_{s_i} \ddot{q} + {}^{\circ}\underline{b}_{s_i} \quad ; & {}^{\circ}J_{s_i} = \dfrac{\partial\, {}^{\circ}\underline{s}_i}{\partial\, q} \\ \delta\, {}^{\circ}\underline{\phi}_i = {}^{\circ}J_{\phi_i} \delta q \quad ; & {}^{\circ}\underline{\dot{\omega}}_i = {}^{\circ}J_{\phi_i} \ddot{q} + {}^{\circ}\underline{b}_{\phi_i} \quad ; & {}^{\circ}J_{\phi_i} = \dfrac{\partial\, {}^{\circ}\underline{\phi}_i}{\partial\, q} \end{array}\right\} \text{Jacobi-Matrices}$$

Elements of the equation of motion

- **Mass matrix**

$$\mathbf{M} = \sum_{i=1}^{n_B} \left\{ {}^{\circ}J_{s_i}^T \, {}^{\circ}J_{s_i} m_i + {}^{\circ}J_{\phi_i}^T \, {}^{\circ}\underline{\underline{\theta}}_{s_i} \, {}^{\circ}J_{\phi_i} \right\}$$

- **Generalised gyro forces**

$$\mathbf{b} = \sum_{i=1}^{n_B} \left\{ {}^{\circ}J_{s_i}^T \, {}^{\circ}\underline{a}_{s_i} \, m_i + {}^{\circ}J_{\phi_i}^T \left[{}^{\circ}\underline{\underline{\theta}}_{s_i} \, {}^{\circ}\underline{a}_{\phi_i} + {}^{\circ}\underline{\omega}_i \times {}^{\circ}\underline{\underline{\theta}}_{s_i} \, {}^{\circ}\underline{\omega}_i \right] \right\}$$

- **Generalised applied forces**

$$\mathbf{Q} = \sum_{i=1}^{n_B} \left\{ {}^{\circ}J_{s_i}^T \, {}^{\circ}\underline{F}_i + {}^{\circ}J_{\phi_i}^T \, {}^{\circ}\underline{T}_i \right\}$$

Partial differentiation of the absolute values with respect to the generalised coordinates

Motivation:

$$v_i = \dot{s}_i(q) = \underbrace{\frac{\partial s_i}{\partial q}}_{J_{Si}} \dot{q} = \underbrace{\frac{\partial s_i}{\partial q_1}}_{Column1} \dot{q}_1 + \cdots + \underbrace{\frac{\partial s_i}{\partial q_f}}_{Column\ f} \dot{q}_f$$

V_i is already known through the Kinematics dependant on q_i

- $${}^{\circ}J_{s_i} = \frac{\partial\, {}^{\circ}\underline{s}_i}{\partial\ q} \quad ; \quad {}^{\circ}J_{\phi_i} = \frac{\partial\, {}^{\circ}\underline{\phi}_i}{\partial\ q}$$ **(Jacobi-Matrices)**

- $${}^{\circ}\underline{a}_{s_i} = \sum_{j=1}^{f}\sum_{k=1}^{f} \frac{\partial^2\, {}^{\circ}\underline{s}_i}{\partial q_j\ \partial q_k}\ \dot{q}_j\ \dot{q}_k$$

- $${}^{\circ}\underline{a}_{\phi_i} = \sum_{j=1}^{f}\sum_{k=1}^{f} \frac{\partial^2\, {}^{\circ}\underline{\phi}_i}{\partial q_j\, \partial q_k}\ \dot{q}_j\ \dot{q}_k$$

Kinematic Differentials

1. **Differentiation to determine:** $\boxed{J_{w_i} = \dfrac{\partial\, w_i}{\partial\, q}}$

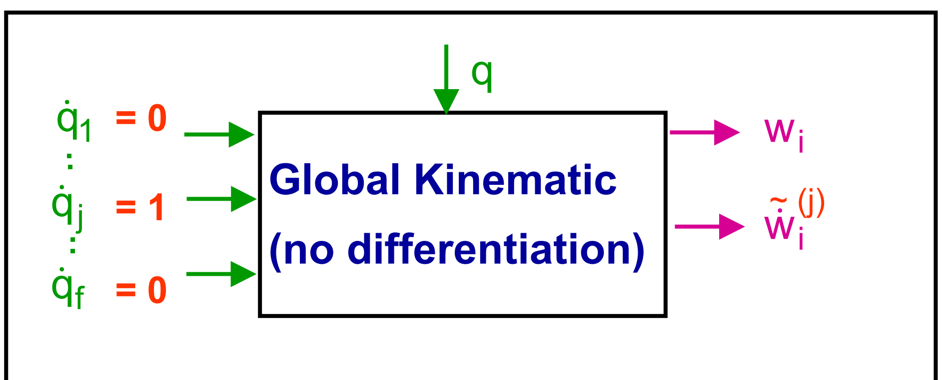
$$w_i = \begin{bmatrix} x_i \\ y_i \\ z_i \\ \psi_i \\ \theta_i \\ \varphi_i \end{bmatrix} = \begin{bmatrix} s_i \\ \hline \phi_i \end{bmatrix}$$

Formal:

$$\boxed{\dot{\tilde{w}}_i^{(j)} = \cancel{\frac{\partial\, w_i}{\partial\, q_1}}\, \underbrace{\dot{q}_1}_{\to 0} + \ldots + \frac{\partial\, w_i}{\partial\, q_j}\, \underbrace{\dot{q}_j}_{\to 1} + \ldots + \cancel{\frac{\partial\, w_i}{\partial\, q_f}}\, \underbrace{\dot{q}_f}_{\to 0}}$$

$$J_{w_i} = \begin{bmatrix} J_{s_i} \\ \hline J_{\phi_i} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x_i}{\partial q_1} & \cdots & \dfrac{\partial x_i}{\partial q_f} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \varphi_i}{\partial q_1} & \cdots & \dfrac{\partial \varphi_i}{\partial q_f} \end{bmatrix}$$

kinematic:



kinematic Differentials

$$\boxed{\frac{\partial\, w_i}{\partial\, q_j} = \tilde{\dot{w}}_i^{(j)}}$$

$$\boxed{\tilde{\dot{w}}_i^{(j)} = \dot{w}_i \Big|_{\substack{\dot{q}_j = 1 \\ \text{else} \quad \dot{q}_k = 0}}}$$

$$\boxed{\text{j – th Column } \{J_{w_i}\} = \tilde{\dot{w}}_i^{(j)}}$$

Seperated into translation and rotation:

$$\boxed{\begin{aligned} \frac{\partial\, \underline{s}_i}{\partial\, q_j} &= \tilde{\dot{\underline{s}}}_i^{(j)} \\ \frac{\partial\, \phi_i}{\partial\, q_j} &= \tilde{\underline{\omega}}_i^{(j)} \end{aligned}}$$

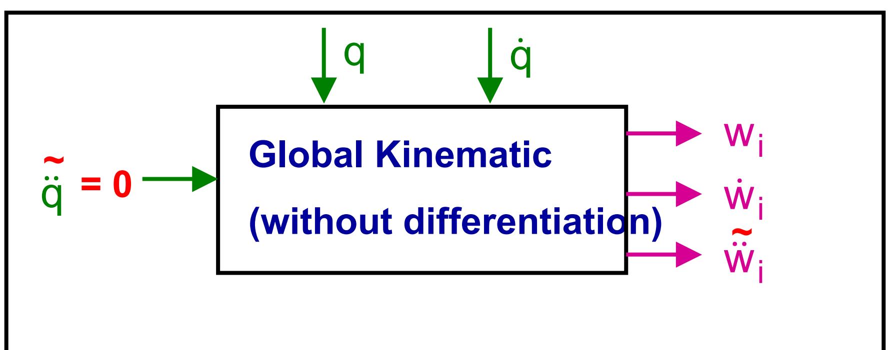
2. Differentiation to determine:

$$a_{wi} = \sum_j \sum_k \frac{\partial^2 w_i}{\partial q_j \, \partial q_k} \dot{q}_j \dot{q}_k$$

Formal:

$$\tilde{\ddot{w}}_i = \sum \cancel{\frac{\partial w_i}{\partial q_j}} \underbrace{\ddot{q}_j}_{0} + a_{wi}$$

Kinematic:



2. Kinematic Differentials

$$\boxed{a_{w_i} = \tilde{\ddot{w}}_i}$$

$$\boxed{\tilde{\ddot{w}}_i = \ddot{w}_i \Big|_{\ddot{q}=0}}$$

Seperated into TRANSLATION and ROTATION

$$\boxed{\begin{aligned} \underline{a}_{s_i} &= \underline{\tilde{\ddot{s}}}_i \\ \underline{a}_{\phi_i} &= \underline{\tilde{\dot{\omega}}}_i \end{aligned}}$$

Equations of motion
with kinematic Differentials

Newton-Euler-equation

$$\sum_{i=1}^{n_B}\left\{(m_i \ddot{\underline{s}}_i - \underline{F}_i)\cdot\delta\underline{s}_i + (\underline{\underline{\theta}}_{si}\,\dot{\underline{\omega}}_i + \underline{\omega}_i \times \underline{\underline{\theta}}_{si}\,\underline{\omega}_i - \underline{T}_i)\cdot\delta\underline{\phi}_i\right\} \overset{!}{=} 0$$

Differential relationships

$$\delta\underline{s}_i = \sum_{j=1}^{f} \tilde{\dot{\underline{s}}}_i^{(j)}\delta q_j \quad ; \quad \ddot{\underline{s}}_i = \sum_{j=1}^{f} \tilde{\dot{\underline{s}}}_i^{(j)}\ddot{q}_j + \tilde{\ddot{\underline{s}}}_i$$

$$\delta\underline{\phi}_i = \sum_{j=1}^{f} \tilde{\underline{\omega}}_i^{(j)}\delta q_j \quad ; \quad \dot{\underline{\omega}}_i = \sum_{j=1}^{f} \tilde{\underline{\omega}}_i^{(j)}\ddot{q}_j + \tilde{\dot{\underline{\omega}}}_i$$

$\Rightarrow$ $\boxed{M\ddot{q} + b = Q}$ **Differential equations of motion of minimal order**

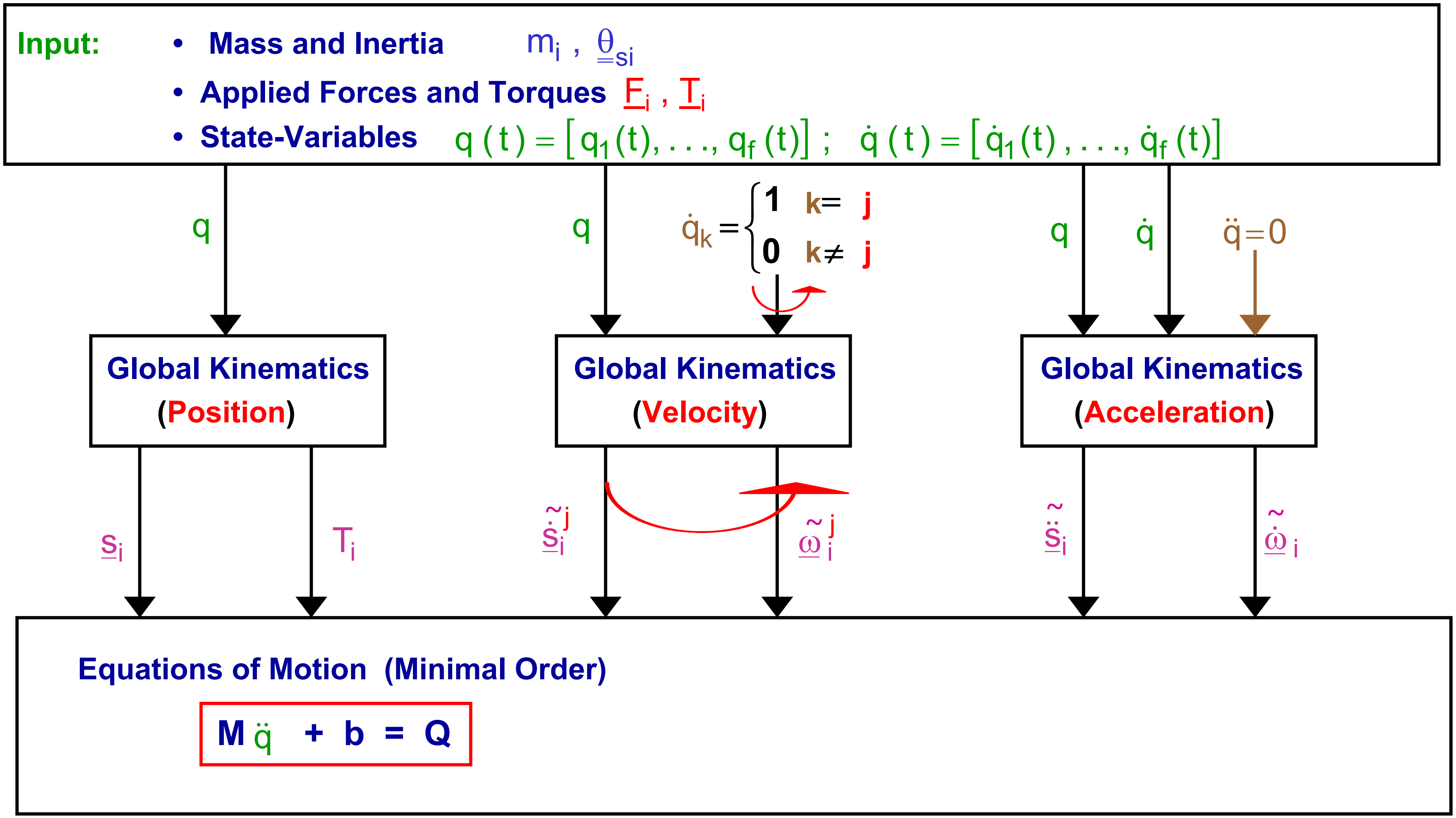
coefficients

$$M_{j,k} = \sum_{i=1}^{n_B} \left\{ m_i \, \tilde{\dot{\underline{s}}}_i^{(j)} \cdot \tilde{\dot{\underline{s}}}_i^{(k)} + \tilde{\underline{\omega}}_i^{(j)} \cdot \left(\underline{\underline{\theta}}_{si} \, \tilde{\underline{\omega}}_i^{(k)} \right) \right\}$$

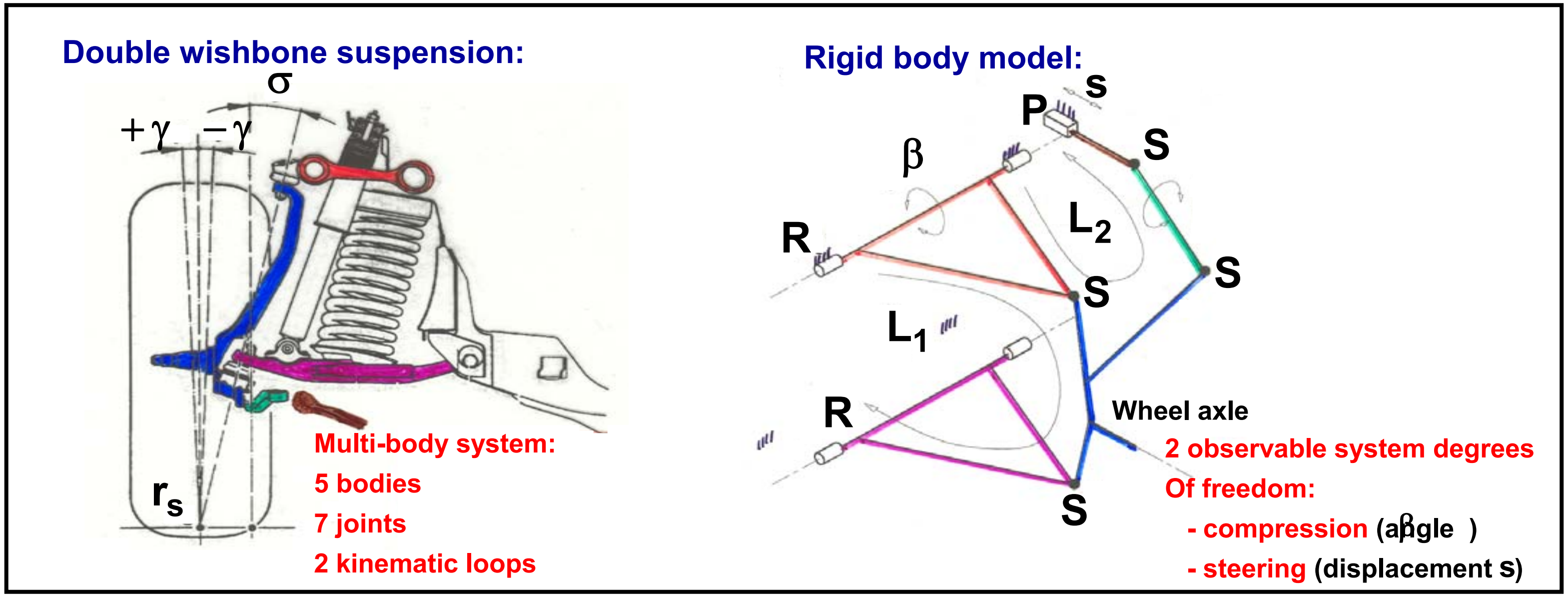
$$b_j = \sum_{i=1}^{n_B} \left\{ m_i \, \tilde{\dot{\underline{s}}}_i^{(j)} \cdot \tilde{\ddot{\underline{s}}}_i + \tilde{\underline{\omega}}_i^{(j)} \cdot \left[\underline{\underline{\theta}}_{si} \, \tilde{\underline{\omega}}_i + \underline{\omega}_i \times \underline{\underline{\theta}}_{si} \, \underline{\omega}_i \right] \right\}$$

$$Q_j = \sum_{i=1}^{n_B} \left\{ \tilde{\dot{\underline{s}}}_i^{(j)} \cdot \underline{F}_i + \underline{\omega}_i^{(j)} \cdot \underline{T}_i \right\}$$

Equations of Motion of Complex Multibody System using „Kinematical Differentials“



3.2 Equations of motion for Multi-Body systems



Lagrange 1. order	Lagrange 2. order	Kinematic Differentials

Lagrange 1. order

Dynamic und kinematic
Equations are set up and worked
with in parallel.

Description results in
Body coordinates and/or
Relative coordinates.

Lagrange 2. order

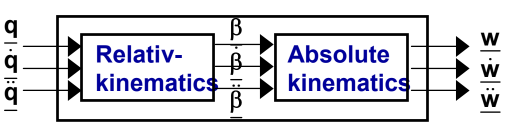
Only dynamic equations.
Kinematic equations will be set up
Earlier und incorporated into the
Dynamic equations.

Description results in
Minimal coordinates
(Relative coordinates
in the most cases):

Applicable only for holonomic Systems.

Kinematic Differentials

Using d`Alembert‘s Principle
Transition over to dynamic
Equations in Minimal coordinates;
i. e. Setting up the kinematic
equations beforehand
in Minimal coordinates und its
Incorporation into the dynamic
Equations.



Global Kinematics

Fundamental problems of the Dynamics

1. „Direct“ Problems

given: forces $\hat{=}$ generalised forces $Q = [Q_1, \ldots, Q_f]$

required: motion $\hat{=}$ generalised accelerations $\ddot{q} = [\ddot{q}_1, \ldots, \ddot{q}_f]$
and generalised coordinates respectively $q = [q_1, \ldots, q_f]$
as a solution of the Differential equation

$$\begin{bmatrix} M_{11} & \cdots & M_{1f} \\ \vdots & M_{22} \ddots & \vdots \\ M_{f1} & \cdots & M_{ff} \end{bmatrix} \underbrace{\begin{bmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_f \end{bmatrix}}_{\text{required}} + \begin{bmatrix} b_1 \\ \vdots \\ b_f \end{bmatrix} = \underbrace{\begin{bmatrix} Q_1 \\ \vdots \\ Q_f \end{bmatrix}}_{\text{given}}$$

„Non-linear Problem“

2. „Inverse“ Problem

given: motion $\hat{=}$ **generalised coordinates** $q = [q_1, \ldots, q_f]$

required: loads $\hat{=}$ **generalised forces** $Q = [Q_1, \ldots, Q_f]$

$$\begin{bmatrix} M_{11} & \cdots & M_{1f} \\ \vdots & M_{22} \ddots & \vdots \\ M_{f1} & \cdots & M_{ff} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_f \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_f \end{bmatrix} = \begin{bmatrix} Q_1 \\ \vdots \\ Q_f \end{bmatrix}$$

given required

„Linear Problem“

3. Reactive forces (Zwangskräfte)

given: loads and motion

required: reactive forces

→ **Principle of linear momentum and principle of conservation of angular momentum (Newton – Euler)**