On simple 3-designs having 2-resolutions

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Abstract

Simple nontrivial s-resolvable t-designs for $t \geq 3$ and $1 < s < t$ are still very sparsely investigated up to now. The problem has been tackled in a recent paper by the author. Here, we continue to explore the problem by focussing on the case $t = 3$ and $s = 2$. In 1963 Shrikhande and Raghavarao published a recursive construction for BIBD. Different authors have studied generalizations of the method for constructing simple 3-designs. In this paper we show that the method can be further extended to studying simple 3-designs having 2-resolutions. As a result, we are able to construct many new infinite families of simple 2-resolvable 3-designs.

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1 Introduction

When the complete k-$(v, k, 1)$ design can be partitioned into s-(v, k, λ_s) designs, for $s < k$, then we have large sets of s-designs. Thus large sets may be viewed as an s-resolution of the complete design. Large sets have been intensively investigated by many researchers for more than three decades. Unlike large sets, the question of partitioning a nontrivial t-(v, k, λ_t) design into s-(v, k, λ_s) designs for $2 \leq s < t$ remains almost unexplored. By contrast much more is known about the case $s = 1$, in particular, when the design can be partitioned into 1-$(v, k, 1)$ designs, which are called *parallel classes*, it is usually said to be *resolvable*. In this case, we will say that the design has a *parallelism*. In a recent paper of the author [13] one can find general recursive methods for constructing s-resolvable t-designs with arbitrary large t. For $t = 3$ very few papers are known in the literature, nonetheless Baker [3] and Teirlinck [11] have handled the most important case of partitioning certain Steiner quadruple systems into Steiner 2-designs. In the present paper we focus on the case $t = 3$, i.e., on 2-resolvable 3-designs. Among papers dealing with constructions of 3-designs with block size large than four, there are two papers, which may be viewed as generalizations of an old construction for BIBD by Shrikhande and Raghavarao [8]. The first one is due to Jimbo et al. [6], where a significant result about conditions for the simplicity of the constructed designs has been obtained, and the second due to Stinson et al. [9]. We show that this approach can be extended to 2-resolvable 3-designs and, in fact, it provides a simple and efficient method for constructing 3-designs with this property. A particular advantage of the method is that large sets of 2-designs can be used in the construction, and thus a great source of ingredients is available.

We assume that the reader is familiar with the concepts of t-designs. For completeness we include the following definition, see also [12, 13].

Definition 1.1 *A t-(v,k,λ)-design $(X,\mathcal{B})$ is said to be s-resolvable, or to have an s-resolution, with $0 < s < t$, if its block set $\mathcal{B}$ can be partitioned into $N \geq 2$ classes $\mathcal{A}_1,\ldots,\mathcal{A}_N$ such that each $(X,\mathcal{A}_i)$ is an s-(v,k,δ) design for $i = 1,\ldots,N$. Each $\mathcal{A}_i$ is called an s-resolution class or simply a resolution class. The set of N classes is called an s-resolution of the design.*

For more information about s-resolvable t-designs, see [13].

The paper is organized as follows. In Section 2, we recall the Shrikhande-Raghavarao construction for BIBD and its generalizations for 3-designs. Section 3 presents the constructions of 2-resolvable 3-designs based on the generalizations. Section 4 summarizes some known results in [6] and [9] and presents some conditions for the simplicity of the constructed designs; to illustrate some small examples for simple 2-resolvable 3-designs are included. Section 5 presents infinite families of simple 2-resolvable 3-designs obtained by applying the construction. Section 6 gives a brief discussion of a construction for disjoint simple 3-designs. Section 7 shows a table of simple 3-designs having 2-resolutions constructed in the paper. The paper closes with a conclusion in Section 8.

2 Brief summary of previous results

We first begin with a description of the construction for balanced incomplete block designs (BIBD) by Shrikhande and Raghavarao published in 1963 [8].

Construction 2.1 *There are two ingredients for the construction:*

1. *Let $(X,\mathcal{B})$ be a resolvable (v,b,r,k,λ)-BIBD and let $\Pi_1,\ldots,\Pi_r$ denote the parallel classes in the resolution of $(X,\mathcal{B})$. There are $w = v/k$ blocks in each parallel class. Let the blocks in Π_i be named B_i^j, $1 \leq j \leq w$. We call $(X,\mathcal{B})$ the **master design**.*

2. *Suppose $(Y,\mathcal{C})$ is a (w,b',r',k',λ')-BIBD, where $Y = \{1,\ldots,w\}$. We call $(Y,\mathcal{C})$ the **indexing design**.*

Now, for each i, $1 \leq i \leq r$, and for each $C \in \mathcal{C}$, define

$$D_{i,C} = \bigcup_{j \in C} B_i^j.$$

That is, for every block C of the indexing design and for every parallel class Π_i of the master design, we construct a block $D_{i,C}$ by taking the union of the blocks in Π_i indexed by C. Define

$$\mathcal{D} = \{D_{i,C} : 1 \le i \le r, C \in \mathcal{C}\}.$$

Then $(X, \mathcal{D})$ is a $(v, b'', r'', k'', \lambda'')$-BIBD with $b'' = rb'$, $r'' = rr'$, $k'' = kk'$, and

$$\lambda'' = \lambda r' + (r - \lambda)\lambda'.$$

$(X, \mathcal{D})$ is called the ***constructed design****.*

It has been proved that the Shrikhande-Raghavarao construction can be extended to 3-designs. In 2011 Jimbo, Kunihara, Laue and Sawa [6] have given a similar construction for simple 3-designs when both master and indexing designs are 3-designs, where the trivial 2-$(v, 2, 1)$ design is viewed as a 3-design with $\lambda_3 = 0$. In 2014 Stinson, Swanson and Tran [9] have studied 3-designs produced by the method where the master design is a resolvable 2-design instead of a 3-design. Both constructions are, in fact, generalizations of the Shrikhande-Raghavarao construction. The following theorem in [9] shows the generalizations.

Theorem 2.1 *[9] Suppose that $(X, \mathcal{B})$ is a resolvable (v, k, λ)-BIBD and $(Y, \mathcal{C})$ is a 3-(w, k', λ')-design where $w = v/k$. Let $(X, \mathcal{D})$ be defined as in Construction 2.1. Then $(X, \mathcal{D})$ is a 3-design if and only if one of the following conditions is satisfied:*

1. *$(X, \mathcal{B})$ is a 3-design,*
2. *$k = 2$, or*
3. *$k' = v/(2k)$.*

The details of Cases 1. and 2. of Theorem 2.1 is given in the following theorem, which is first presented in [6].

Theorem 2.2 *[6] Suppose the following designs exist: a resolvable 3-(v, k, λ_3) design, and a 3-(w, k', λ'_3) design, where $w = v/k$. Then there exists a 3-(v, kk', λ''_3) design, where*

$$\lambda''_3 = \lambda_3\lambda'_1 + 3(\lambda_2 - \lambda_3)\lambda'_2 + (\lambda_1 - 3\lambda_2 + 2\lambda_3)\lambda'_3.$$

Moreover, if $k = 2$, then

$$\lambda''_3 = 3\lambda'_2 + (v - 4)\lambda'_3.$$

The next corollary gives the details of Case 3. of Theorem 2.1 as shown in [9].

Corollary 2.3 *[9] Suppose there are the following designs: a resolvable (v, b, r, k, λ)-BIBD, and a 3-$(w, w/2, \lambda')$ design with $w = v/k$ even.*

1. *If $w = v/k > 4$, then there exists a 3-$(v, v/2, \lambda'')$ design, where*

$$\lambda'' = \lambda' \left(\frac{3\lambda w}{w - 4} + r \right).$$

2. *If $w = v/k = 4$, then there is a 3-$(v, v/2, 3\lambda)$ design.*

3 Construction of 2-resolvable 3-designs

In this section we show that the constructions in [6] and [9] can be further extended to studying 2-resolvable 3-designs. The first result is given in the next theorem.

Theorem 3.1 *Suppose that the indexing design is 2-resolvable. Then the constructed 3-design is 2-resolvable.*

Proof. Let $(X, \mathcal{B})$ be a resolvable 3-(v, k, λ_3) design. i.e., we deal with Cases 1. and 2. of Theorem 2.1. Note that a trivial BIBD with $k = 2$ is viewed as a 3-design with $\lambda_3 = 0$. For $(X, \mathcal{B})$ we have $\lambda_2 = \lambda_3(v-2)/(k-2)$ and $\lambda_1 = \lambda_3\binom{v-1}{2}/\binom{k-1}{2}$. Let $\Pi_1, \ldots, \Pi_{\lambda_1}$ be the parallel classes from a partition of the blocks of $\mathcal{B}$. Let $w = v/k$ and let 3-(w, k', λ'_3) be the parameters of the indexing design $(Y, \mathcal{C})$. Let $\mathcal{C}_1, \ldots, \mathcal{C}_N$ be a 2-resolution of $(Y, \mathcal{C})$, where each $(Y, \mathcal{C}_i)$ is a 2-(w, k', δ'_2) design. Here $\delta'_2 = \lambda'_2/N$ and $\lambda'_2 = \lambda'_3(w-2)/(k'-2)$. The block set of the constructed design $(X, \mathcal{D})$ is of the form

$$\mathcal{D} = \{D_{i,C} : \ 1 \leq i \leq \lambda_1, \ C \in \mathcal{C}\}$$

with $D_{i,C} = \bigcup_{j \in C} B_i^j$, where B_i^j is a block in Π_i. As $\mathcal{C} = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_N$ is a disjoint union, we may write

$$\mathcal{D} = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_N,$$

where

$$\mathcal{D}_h = \{D_{i,C} : \ 1 \leq i \leq \lambda_1, \ C \in \mathcal{C}_h\},$$

$h = 1, \ldots, N$. It follows that $\mathcal{D}_1, \ldots, \mathcal{D}_N$ form a partition of $\mathcal{D}$. Now each $(X, \mathcal{D}_h)$, $h = 1, \ldots, N$, is 2-design constructed from

- the master design: 2-(v, k, λ_2)= 2-$(v, k, \lambda_3(v-2)/(k-2))$,
- the indexing design : 2-(w, k', δ'_2), $\delta'_2 = \lambda'_2/N = \lambda'_3(w-2)/(k'-2)N$.

Each $(X, \mathcal{D}_h)$ has parameters 2-(v, kk', δ''_2), $\delta''_2 = \lambda_2.\delta'_2\frac{(w-1)}{(k'-1)} + (\lambda_2\frac{(v-1)}{(k-1)} - \lambda_2).\delta'_2$.

The Case 3. of Theorem 2.1 can be treated in a similar way, hence we omit the proof. □

Theorem 3.1 deals with the case where the indexing design is 2-resolvable. We may further proceed with the other case where the master design is not only resolvable but also 2-resolvable. However, the detailed requirements for the master design are given in the following theorem.

Theorem 3.2 *Suppose that the* 3-(v, k, λ_3) *master design* $(X, \mathcal{B})$ *with* $k \geq 3$ *satisfies the following conditions.*

1. $(X, \mathcal{B})$ *is 2-resolvable.*
2. *Each 2-resolution class of* $(X, \mathcal{B})$ *has a parallelism.*

Then the constructed 3-design is 2-resolvable.

Proof. Let $\mathcal{B}_1, \ldots, \mathcal{B}_N$ be N 2-resolution classes of $(X, \mathcal{B})$. i.e., for $h = 1, \ldots, N$, $(X, \mathcal{B}_h)$ is a 2-(v, k, δ_2)-design, where $\delta_2 = \lambda_3(v-2)/(k-2)N$. By the assumption, $(X, \mathcal{B}_h)$ is resolvable with parallel classes denoted by

$$\Pi_1^h, \ldots, \Pi_{\delta_1}^h,$$

where $\delta_1 = \delta_2(v-1)/(k-1)$. For the construction we will arrange the parallel classes of $(X, \mathcal{B})$ as the concatenation of N groups of parallel classes, where each group consists of δ_1 parallel classes of the 2-resolution $(X, \mathcal{B}_h)$, $1 \leq h \leq N$. In this way we write the parallel classes of $(X, \mathcal{B})$ as

$$\Pi_1^1, \ldots, \Pi_{\delta_1}^1,\ \Pi_1^2, \ldots, \Pi_{\delta_1}^2, \ldots,\ \Pi_1^N, \ldots, \Pi_{\delta_1}^N.$$

Let $(Y, \mathcal{C})$ be the indexing design with parameters 3-(w, k', λ_3'), where $w = v/k$. Then the constructed design $(X, \mathcal{D})$ can be written as the union of N pairwise disjoint 2-designs $(X, \mathcal{D}_1), \ldots, (X, \mathcal{D}_N)$ with

$$(X, \mathcal{D}_h) = \{D_{i,C}^h :\ 1 \leq i \leq \delta_1,\ C \in \mathcal{C}\},$$

$$D_{i,C}^h = \bigcup_{j \in C} B_i^{h,j},\ 1 \leq h \leq N,\ 1 \leq i \leq \delta_1,\ B_i^{h,j} \in \Pi_i^h.$$

Each $(X, \mathcal{D}_h)$ is a 2-design constructed from

- the master design: 2-(v, k, δ_2),
- the indexing design: 2-(w, k', λ_2'), where $\lambda_2' = \frac{w-2}{k'-2}\lambda_3'$.

In other words, $(X, \mathcal{D})$ is 2-resolvable with N resolution classes. $\square$

When both the master and indexing designs are 2-resolvable, we obtain an interesting result about the number of 2-resolution classes of the constructed design. By combining the proofs of Theorems 3.1 and 3.2 it is straightforward to prove the following result.

Theorem 3.3 *Suppose that the indexing design $(Y, \mathcal{C})$ is a 2-resolvable 3-design with N_2 resolution classes. Further suppose that the master design $(X, \mathcal{B})$ is a 3-design satisfying the conditions of Theorem 3.2 and has N_1 2-resolution classes. Then the constructed design is a 2-resolvable 3-design with $N = N_1.N_2$ 2-resolution classes.*

Proof. (Sketch) Let $\mathcal{B}_1, \ldots, \mathcal{B}_{N_1}$ be a 2-resolution of $(X, \mathcal{B})$ and let $\mathcal{C}_1, \ldots, \mathcal{C}_{N_2}$ be a 2-resolution of $(Y, \mathcal{C})$. Then it is clear that the constructed design $(X, \mathcal{D})$ is a disjoint union of $N = N_1.N_2$ 2-designs $(X, \mathcal{D}_i^h)$ where $(X, \mathcal{D}_i^h)$ is a 2-design constructed from the master design $(X, \mathcal{B}_h)$ and the indexing design $(Y, \mathcal{C}_i)$. $\square$

The following example illustrates Theorem 3.3.

Example 3.1 Let the master and indexing designs be 2-resolvable 3-$(2^{2m}, 4, 1)$ and 3-$(2^{2m-2}, 4, 1)$ Steiner quadruple systems, having $N_1 = (2^{2m-1}-1)$ and $N_2 = (2^{2m-3}-1)$ 2-resolution classes, respectively, as shown by Baker in [3]. Here the master design satisfies the resolvability conditions of Theorem 3.2. Hence, by applying Theorem 3.3 we obtain a 2-resolvable 3-$(2^{2m}, 16, \frac{35}{3}(2^{2m-2}-1)(2^{2m-3}-1))$ design having $N = N_1.N_2 = (2^{2m-1}-1)(2^{2m-3}-1)$ 2-resolution classes, for any $m \geq 3$. Note also that the constructed design has a parallelism.

An interesting problem arises.

Problem Prove or disprove the simplicity of the constructed 3-designs in Example 3.1.

We include a simple lemma, however useful, without proof.

Lemma 3.4 *If the indexing design has a parallelism, then the constructed design has a parallelism.*

The next corollary is an immediate consequence of Theorem 3.1, which presents a connection between large sets of 2-designs and 3-designs having 2-resolutions.

Corollary 3.5 *Suppose that there exists a large set $LS[N](2, k', w)$. Then there exists a 2-resolvable 3-(v, kk', λ_3'') with N resolution classes for any $k \geq 2$, where $v = kw$ and*

1. $\lambda_3'' = \binom{v-3}{k-3}\binom{w-3}{k'-3}\frac{\binom{w-1}{2}}{\binom{k'-1}{2}} + 3\binom{v-3}{k-3}(\frac{v-2}{k-2}-1)\binom{w-3}{k'-3}\frac{w-2}{k'-2} + \binom{v-3}{k-3}\left(\frac{\binom{v-1}{2}}{\binom{k-1}{2}} - 3\frac{v-2}{k-2} + 2\right)\binom{w-3}{k'-3}$, *if* $k \geq 3$,
2. $\lambda_3'' = \binom{w-3}{k'-3}\left(3\frac{w-2}{k'-2} + (2w-4)\right)$, *if* $k = 2$.

Proof. Applying Theorem 3.1 for which

1. master design: $\begin{cases} 3-(v, k, \binom{v-3}{k-3}), & \text{if } k \geq 3, \\ 2-(v, 2, 1), & \text{if } k = 2 \end{cases}$
2. indexing design: $3-(w, k', \binom{w-3}{k'-3})$.

By a result of Baranyai [4], the master design is resolvable. The indexing design is 2-resolvable by the assumption of existence of the large set. Hence the constructed design is 2-resolvable, and its index λ_3'' is computed according to Corollary 2.2. $\square$

4 Simplicity of the constructed designs

In [6] the authors have studied the simplicity of the constructed designs in Cases 1. and 2. of Theorem 2.1. In particular, by using the graph-theoretical method the authors prove the following significant results for the case $k = 2$ showing conditions for the simplicity of the constructed designs.

Theorem 4.1 *[6] Suppose that the master design $(X, \mathcal{B})$ is the resolvable trivial 2-$(v, 2, 1)$ design and the indexing design $(Y, \mathcal{C})$ is a 3-(w, k', λ_3') design with $w = v/2$. Then the constructed design $(X, \mathcal{D})$ with parameters 3-$(v, 2k', 3\lambda_2' + (v-4)\lambda_3')$ is simple, if one of the following conditions is satisfied:*

(i) $k' = 2$, *or* $k' = 3$ *and* $v \equiv 2 \pmod 4$,

(ii) $\gcd(2k'-1, v-1) = \gcd(k', v-1) = 1$,

(iii) $\gcd(k', w) = 1$ *and* $v \equiv 2 \pmod 4$.

For Case 1. and 3. (i.e., $k \geq 3$) of Theorem 2.1 a condition for the simplicity of the constructed design can be derived in terms of block intersection numbers of the master design.

Theorem 4.2 *Suppose that the master and indexing designs are simple with block sizes k and k', respectively. Suppose that $|B_1 \cap B_2| \leq u$ for any two different blocks B_1 and B_2 of the master design. If $k \geq uk' + 1$, then the constructed design is simple.*

Proof. It is clear from the assumption that any two blocks of the constructed design obtained from the same parallel class of the master design are distinct. It is sufficient to prove that for any block $D_{j,C} = \bigcup_{h \in C} B_j^h$ constructed from a parallel class j of the master design we have $B_i^{h'} \not\subseteq D_{j,C}$ for any block $B_i^{h'}$ in the parallel class i with $i \neq j$. In fact, we have

$$|B_i^{h'} \cap D_{j,C}| = |B_i^{h'} \cap (\bigcup_{h \in C} B_j^h)| = \sum_{h \in C} |B_i^{h'} \cap B_j^h| \leq uk'.$$

From $k \geq uk' + 1$ it follows that $B_i^{h'} \not\subseteq D_{j,C}$ (i.e., any two blocks constructed from different parallel classes of the master design are distinct). □

It should be noted that Theorem 4.2 has been given in [6], when the index of the master design is 1. It seems that for $k \geq 3$ we are still far away from having general conditions on the master design, which guarantee the simplicity of the constructed design. Some efforts of finding conditions other than intersection numbers have been explored in [9], of which the next theorem is a special case.

Theorem 4.3 *Suppose that the master and indexing designs are simple. If the master design has a unique parallelism, then the constructed design is simple.*

Proof. We keep using the notation in Construction 2.1 with $(X, \mathcal{B})$ as the master design and $(Y, \mathcal{C})$ as the indexing design. Let $\Pi_1, \ldots, \Pi_r$ be the unique parallel classes of $(X, \mathcal{B})$. Recall that a block of the constructed design is of the form $D_{i,C} = \bigcup_{j \in C} B_i^j$, $i = 1, \ldots, r$, $1 \leq j \leq w$. Remark that since the indexing design is simple, any two blocks of the constructed design can only be equal if they are formed from distinct parallel classes of the master design, in other words $D_{i,C} \neq D_{i,C'}$ for any $C, C' \in \mathcal{C}$. So, without loss of generality, we may assume that $D_{1,C} = D_{2,C'}$. Let $C = \{y_1, \ldots, y_{k'}\}$, $C' = \{y_1', \ldots, y_{k'}'\}$.

By reordering the blocks in Π_1 and Π_2, we may set

$$\Pi_1 = \{B_1^{y_1}, B_1^{y_2}, \ldots, B_1^{y_{k'}}, B_1^{y_{k'+1}}, \ldots, B_1^{y_w}\},$$

and

$$\Pi_2 = \{B_2^{y'_1}, B_2^{y'_2}, \ldots, B_2^{y'_{k'}}, B_2^{y'_{k'+1}}, \ldots, B_2^{y'_w}\}.$$

Hence

$$\begin{aligned} D_{1,C} &= B_1^{y_1} \cup B_1^{y_2} \cup \ldots \cup B_1^{y_{k'}}, \\ D_{2,C'} &= B_2^{y'_1} \cup B_2^{y'_2} \cup \ldots \cup B_2^{y'_{k'}}. \end{aligned}$$

Define

$$\begin{aligned} \Pi_1^* &= \{B_2^{y'_1}, B_2^{y'_2}, \ldots, B_2^{y'_{k'}}, B_1^{y_{k'+1}}, \ldots, B_1^{y_w}\}, \\ \Pi_2^* &= \{B_1^{y_1}, B_1^{y_2}, \ldots, B_1^{y_{k'}}, B_2^{y'_{k'+1}}, \ldots, B_2^{y'_w}\}. \end{aligned}$$

Since $D_{1,C} = D_{2,C'}$, it follows that Π_1^* and Π_2^* form two new parallel classes of $(X, \mathcal{B})$. Thus $\{\Pi_1, \Pi_2, \Pi_3, \ldots, \Pi_r\}$ and $\{\Pi_1^*, \Pi_2^*, \Pi_3, \ldots, \Pi_r\}$ are two distinct parallelisms of $(X, \mathcal{B})$, which contradicts the assumption. Hence $D_{1,C} \neq D_{2,C'}$ and the constructed design is simple. □

We illustrate Corollary 3.5 (2.) by presenting some interesting special examples of simple 2-resolvable 3-designs.

Examples 4.1

1. The master design is the complete 2-$(32, 2, 1)$ design and the indexing design is the complete 4-$(16, 4, 1)$ design, i.e., a 3-$(16, 4, 13)$ design, which has a large set consisting of $N = 91$ disjoint copies of 2-$(16, 4, 1)$ designs. The constructed design has parameters 3-$(32, 8, 91 \times 7)$ and is 2-resolvable by Theorem 3.1 and it is simple by Theorem 4.1. The 2-resolution of the constructed design has $N = 91$ classes, where each class is a 2-$(32, 8, 35)$ design. Note that the constructed design has a paralellism by Lemma 3.4. More precisely, the 2-design in each resolution class has a parallelism.

2. There exists a simple 2-resolvable 3-design with parameters 3-$(26, 8, 55 \times 7)$ having $N = 55$ resolution classes. Each class is a 2-$(26, 8, 28)$ design. In this case, the complete 2-$(26, 2, 1)$ design is the master design and the complete 4-$(13, 4, 1)$ design is the indexing design, which is known to have a large set comprised of 55 copies of 2-$(13, 4, 1)$ designs. Note that the 2-$(26, 8, 28)$ design in each resolution class has the possible minimum index, i.e., any 2-$(26, 8, \lambda)$ design satisfies $28|\lambda$.

3. There exists a simple 2-resolvable 3-design with parameters 3-$(18, 6, 7 \times 5)$ with $N = 7$ resolution classes. Each class is a 2-$(18, 6, 20)$ design. Here, the complete 2-$(18, 2, 1)$ design is the master design and the complete 3-$(9, 3, 1)$ design is the indexing design, which is known to have a large set comprised of 7 copies of 2-$(9, 3, 1)$ designs. The constructed design as well as the 2-designs of the resolution have parallelisms.

We should remark that the three examples above are related to known large sets of affine and projective planes. Here, the first example comes from a large set of the affine plane of order 4, while the other two from large sets of projective and affine plane of order 3, respectively.

5 Applications

In this section we apply the results from the previous two sections to derive the existence of many infinite classes of simple 2-resolvable 3-designs.

5.1 Master designs with $k = 2$

Here the master design is a resolvable 2-$(v, 2, 1)$ design.

We begin with the second case of Corollary 3.5. First consider the trivial 3-$(w, 3, 1)$ design as the indexing design. It is known that $LS_{\lambda_{\min}}(2, 3, w)$ exists if and only if $v \neq 7$, [7]. From (i) of Theorem 4.1, it follows that the constructed design is simple if $2w \equiv 2 \pmod 4$. Therefore $w \equiv 1, 3 \pmod 6$ or $w \equiv 5 \pmod 6$. Moreover, if $w \equiv 1, 3 \pmod 6$, then $\lambda_{\min} = 1$, whereas if $w \equiv 5 \pmod 6$, then $\lambda_{\min} = 3$.

1. The case $w \equiv 1, 3 \pmod 6$ corresponds to a large set of 2-$(w, 3, 1)$ Steiner triple systems. So, the 3-$(w, 3, 1)$ design is a union of $N = (w - 2)$ disjoint 2-$(w, 3, 1)$ designs. The constructed design has parameters 3-$(2w, 6, 5(w - 2))$ and is 2-resolvable with N resolution classes. Each class is a 2-$(2w, 6, \frac{5}{2}(w - 1))$ design.

2. The case $w \equiv 5 \pmod 6$ corresponds to a large set of 2-$(w, 3, 3)$ designs. The 3-$(w, 3, 1)$ design is a union of $N = (w - 2)/3$ disjoint 2-$(w, 3, 3)$ designs. The constructed design is 2-resolvable with N resolution classes. Each class is a 2-$(2w, 6, \frac{15}{2}(w - 1))$ design.

In summary, we have the following theorem.

Theorem 5.1 *Suppose $w \equiv 1, 3$ or $5 \pmod 6$. Then there exists a simple 2-resolvable 3-$(2w, 6, 5(w - 2))$ design $(X, \mathcal{D})$. More precisely,*

(i) If $w \equiv 1, 3 \pmod 6$, then $(X, \mathcal{D})$ has $N = (w - 2)$ 2-resolution classes. Each class is 2-$(2w, 6, \frac{5}{2}(w - 1))$ design.

(ii) If $w \equiv 5 \pmod 6$, then $(X, \mathcal{D})$ has $N = (w - 2)/3$ 2-resolution classes. Each class is 2-$(2w, 6, \frac{15}{2}(w - 1))$ design.

Next consider the indexing designs with parameters 3-$(2^{2m}, 4, 1)$ and 3-$(2^{2m+1}, 4, 5)$.

1. In [3] Baker proves the 2-resolvability of the 3-$(2^{2m}, 4, 1)$ Steiner quadruple system whose blocks are the planes in an even dimensional affine space over the field of two elements. Each 2-resolution class is a 2-$(2^{2m}, 4, 1)$ design. Thus the number of resolution classes is $N = 2^{2m-1} - 1$. By applying Theorems 2.2 and 3.1 we obtain a 2-resolvable 3-$(2^{2m+1}, 8, 7(2^{2m-1} - 1))$ design with N resolution classes. The constructed design is simple if the condition (ii) : $\gcd(7, 2^{2m+1} - 1) = \gcd(4, 2^{2m+1} - 1) = 1$ of Theorem 4.1 is satisfied. It is straightforward to verify that this is the case if $2m + 1 \not\equiv 0 \pmod 3$.

2. In [2] Alltop constructs simple 4-designs with paramters 4-$(2^{2m+1} + 1, 5, 5)$ for $2m + 1 \geq 5$. Each design is 3-resolvable with $N = (2^{2m+1} - 2)/6$ resolution classes, see [13]. Hence its derived design $(Y, \mathcal{C})$ with parameters 3-$(2^{2m+1}, 4, 5)$ is 2-resolvable with N resolution classes. By taking $(Y, \mathcal{C})$ as the indexing design, we obtain a 2-resolvable 3-design with parameters 3-$(2^{2m+2}, 8, 35(2^{2m} - 1))$ having $N = (2^{2m+1} - 2)/6$ resolution classes. Again, the constructed design is simple if $\gcd(7, 2^{2m+2} - 1) = \gcd(4, 2^{2m+2} - 1) = 1$. It follows that $2m + 2 \equiv 2, 4 \pmod 6$.

Thus we have the following.

Theorem 5.2 *Let n be a positive integer. Suppose that $n \not\equiv 0 \pmod 3$, if n is odd, and $n \equiv 2, 4 \pmod 6$, if n is even. Then there exists the following simple 2-resolvable 3-design $(X, \mathcal{D})$.*

(i) If n is odd and $n \geq 5$, then $(X, \mathcal{D})$ is a 3-$(2^n, 8, 7(2^{n-2} - 1))$ design with $N = (2^{n-2} - 1)$ resolution classes. Each class is a 2-$(2^n, 8, \frac{7}{3}(2^{n-1} - 1))$ design,

(ii) If n is even and $n \geq 6$, then $(X, \mathcal{D})$ is a 3-$(2^n, 8, 35(2^{n-2} - 1))$ design with $N = (2^{n-1} - 2)/6$ resolution classes. Each class is a 2-$(2^n, 8, 35(2^{n-1} - 1))$ design.

The following families are derived from 2-resolvable indexing designs with parameters 3-$(2(7^n + 1), 4, 1)$ and 3-$(2(31^n + 1), 4, 1)$ due to Teirlinck [11].

1. The 2-resolvable 3-$(2.(7^n + 1), 4, 1)$ design for $n \geq 1$ has $N = 7^n$ resolution classes and each class is a 2-$(2(7^n + 1), 4, 1)$ design. Corollary 2.2 and Theorem 3.1 thus give a 2-resolvable 3-design with parameters 3-$(4(7^n + 1), 8, 7^{n+1})$ with N resolution classes. Each class is a 2-$(4(7^n + 1), 8, \frac{7}{3}(2.7^n + 1))$ design. The constructed design is simple because the condition (ii) : $\gcd(7, 4.7^n + 3) = \gcd(4, 4.7^n + 3) = 1$ of Theorem 4.1 is satisfied.

2. The case with 2-resolvable 3-$(2.(31^n + 1), 4, 1)$ design for $n \geq 1$ can be handled similarly.

Thus we obtain the following.

Theorem 5.3 *(i) There exists a simple 2-resolvable* 3-$(4(7^n+1), 8, 7^{n+1})$ *design having* $N = 7^n$ *resolution classes, for any integer* $n \geq 1$*. Each class is a* 2-$(4(7^n+1), 8, \frac{7}{3}(2.7^n+1))$ *design.*

(ii) There exists a simple 2-resolvable 3-$(4(31^n+1), 8, 7.31^n)$ *design having* $N = 31^n$ *resolution classes, for any integer* $n \geq 1$*. Each class is a* 2-$(4(31^n+1), 8, \frac{7}{3}(2.31^n+1))$ *design.*

Consider a further family. In [5] Bierbrauer constructs a simple 4-$(2^{2m-1}+1, 8, 35)$ design for $2m-1 \geq 5$ and $2m-1 \not\equiv 0 \pmod 3$. The design is 3-resolvable with $N = (2^{2m-1}-2)/6$ resolution classes, see [13]. Hence its derived design $(Y, \mathcal{C})$ with parameters 3-$(2^{2m-1}, 7, 35)$ is 2-resolvable with N resolution classes. Taking $(Y, \mathcal{C})$ as the indexing design will yield a 2-resolvable 3-$(2^{2m}, 14, 91(2^{2m-1}-2))$ design with N resolution classes. Each class is a 2-$(2^{2m}, 14, 91(2^{2m-1}-1))$ design. The constructed design is simple if the condition (ii): $\gcd(13, 2^{2m}-1) = \gcd(7, 2^{2m}-1) = 1$ of Theorem 4.1 is satisfied. Now, it is straightforward to check that if $\gcd(7, 2^{2m}-1) = 1$, then $2m \equiv 2, 4 \pmod 6$ and if $\gcd(13, 2^{2m}-1) = 1$, then $2m \equiv 2, 4, 6, 8, 10 \pmod{12}$. From these two congruences it follows that if $2m \equiv 2, 4, 8, 10 \pmod{12}$, then the condition (ii) is satisfied.

We have proved the following.

Theorem 5.4 *Let* $2m$ *be a positive integer such that* $2m \equiv 2, 4, 8, 10 \pmod{12}$*. Then there exists a simple 2-resolvable 3-design with parameters* 3-$(2^{2m}, 14, 91(2^{2m-1}-2))$ *having* $N = (2^{2m-1}-2)/6$ *resolution classes. Each class is a* 2-$(2^{2m}, 14, 91(2^{2m-1}-1))$ *design.*

5.2 Master designs with $k \geq 3$

Here the master designs will have block size $k \geq 3$. We will differentiate the case with 3-designs from the case with 2-designs for the master designs.

The following noteworthy result can be found in [6].

Proposition 5.5 *Let* q *be a prime power and* n *be a positive integer. Then the* 3-$(q^n+1, q+1, 1)$ *design with* $PGL(2, q^n)$ *as an automorphism group has a parallelism if and only if* $n \equiv 1 \pmod 2$*.*

Now take the master design as a resolvable 3-$(q^n+1, q+1, 1)$ design, $n \equiv 1 \pmod 2$ and the indexing design is the trivial 3-$(w, 3, 1)$ design, where $w = \frac{q^n+1}{q+1}$. Then, the constructed design has parameters 3-$(q^n+1, 3(q+1), \Lambda)$, where

$$\Lambda = \binom{\frac{q^n+1}{q+1} - 1}{2} + 3\left(\frac{q^n-1}{q-1} - 1\right)\left(\frac{q^n+1}{q+1} - 2\right) + \left(q^{n-1}\frac{q^n-1}{q-1} - 3\frac{q^n-1}{q-1} + 2\right).$$

Recall that there is a large set $LS_{\lambda_{\min}}(2, 3, w)$. Now as w is odd, it follows that $w \equiv 1, 3,$ or $5 \pmod 6$. Thus, if $w \equiv 1, 3 \pmod 6$, then $\lambda_{\min} = 1$ and the trivial 3-$(w, 3, 1)$ design is the union of $N = (w-2)$ disjoint 2-$(w, 3, 1)$ designs. If $w \equiv 5$

(mod 6), then $\lambda_{\min} = 3$ and it is the union of $N = (w-2)/3$ disjoint 2-$(w, 3, 3)$ designs.

From Theorem 4.2, the constructed design is simple for $q > 5$, and is 2-resolvable with N resolution classes, where $N = (w-2)$, if $w \equiv 1, 3 \pmod 6$ and $N = (w-2)/3$, if $w \equiv 5 \pmod 6$. The result is recorded in the following theorem.

Theorem 5.6 *Let q be a prime power with $q > 5$ and let n be any positive odd integer. Then there is a simple 2-resolvable 3-design with parameters* 3-$(q^n + 1, 3(q + 1), \Lambda)$, *where*

$$\Lambda = \binom{\frac{q^n+1}{q+1} - 1}{2} + 3\Big(\frac{q^n - 1}{q - 1} - 1\Big)\Big(\frac{q^n + 1}{q + 1} - 2\Big) + \big(q^{n-1} - 3\big)\frac{q^n - 1}{q - 1} + 2.$$

Moreover,

1. *if* $\frac{q^n+1}{q+1} \equiv 1, 3 \pmod 6$, *the design has* $N = (\frac{q^n+1}{q+1} - 2)$ *2-resolution classes,*
2. *if* $\frac{q^n+1}{q+1} \equiv 5 \pmod 6$, *the design has* $N = (\frac{q^n+1}{q+1} - 2)/3$ *2-resolution classes.*

The following examples are about the case 3 of Theorem 2.1, i.e., the master design is 2-(v, k, λ)- BIBD and not necessary a 3-design with $k \geq 3$.

Let the affine resolvable BIBD with parameters 2-$(8^m, 8^{m-1}, \frac{8^{m-1}-1}{8-1})$ be the master design and the indexing design be the complete 3-$(8, 4, 5)$ design. It is known that there exists a $LS[5](2, 4, 8)$, i.e., the indexing design is 2-resolvable with $N = 5$ resolution classes. By Corollary 2.3 and Theorem 4.3 the constructed design is a simple 2-resolvable 3-$(8^m, 4.8^{m-1}, 5(2.8^{m-1} - 1))$ design with $N = 5$ resolution classes.

Further, a $LS[13](2, 7, 15)$ is known to exist, see [7]. By Corollary 4.3 of [13] it follows that there is a $LS[13](2, 8, 16)$. Taking the affine resolvable BIBD with parameters 2-$(16^m, 16^{m-1}, \frac{16^{m-1}-1}{16-1})$, as the master design and the complete 3-$(16, 8, 13.99)$ design as the indexing design, will give a 2-resolvable 3-$(16^m, 8.16^{m-1}, 13.33(4.16^{m-1} - 1))$ design with $N = 13$ resolution classes. The constructed design is simple by Theorem 4.3. In summary, we have obtained the following.

Theorem 5.7 *Let $m \geq 2$ be an integer.*

1. *There exists a simple 2-resolvable* 3-$(8^m, 4.8^{m-1}, 5(2.8^{m-1} - 1))$ *design with* $N = 5$ *resolution classes.*
2. *There exists a simple 2-resolvable* 3-$(16^m, 8.16^{m-1}, 13.33(4.16^{m-1} - 1))$ *design with* $N = 13$ *resolution classes.*

Remark 5.1 The case of master designs with $k \geq 3$ is worthy of a comment. Actually, for this case Theorems 4.2, 4.3 merely present two specific conditions for the simplicity of the constructed design. We will give an example for $k = 3$ to illustrate the more involved situation in general. Take the master design as a simple resolvable

2-$(24,3,2)$ design having a cyclic automorphism group of order 23, see [1]. Its six base blocks form a parallel class:

$$\{\infty,16,20\},\{0,7,21\},\{1,3,11\},\{4,5,18\},\{6,12,17\},\{2,10,13\},\{8,9,14\},\{15,19,22\}.$$

It is easy to check that $|A\cap B| = 0$, or 1, or 2, for any two distinct blocks A, B. Moreover, a computer search shows that there are 9 parallelisms for this design. So, neither of the conditions of Theorems 4.2 and 4.3 is satisfied. However, when we take the complete 3-$(8,4,5)$ design as the indexing design, then we get a simple 2-resolvable 3-$(24,12,175)$ design with $N = 5$ resolution classes. The simplicity of the latter can be checked with a computer, see [9]. The example displays a paradigm that is not yet well investigated. It would seem that finding general necessary conditions for the simplicity of the constructed design when $k \geq 3$ is a very challenging problem.

6 Disjoint simple 3-designs

In this section, we briefly discuss a consequence of the method for constructing 2-resolvable 3-designs in the paper. The construction gives a connection to the problem of finding mutually disjoint 3-designs. It is routine to check that when the requirement for the indexing design as a union of disjoint 2-designs is replaced by the union of disjoint 3-designs, then obviously the constructed design is the union of disjoint 3-designs. In particular, the results for large sets of 3-designs will give a source for constructing mutually disjoint simple 3-designs. The following example illustrates the idea. Take the 2-$(24,2,1)$ design as the master design and the complete 3-$(12,6,84)$ design as the indexing design. It is known that there is a $LS[42](3,6,12)$, i.e., the 3-$(12,6,84)$ design is a union of 42 mutually disjoint 3-$(12,6,2)$ designs. The constructed design is a simple 3-$(24,12,42.55)$ design and is a union of 42 pairwise disjoint 3-$(24,12,55)$ designs. To put it another way, there are 42 pairwise disjoint simple 3-$(24,12,55)$ designs.

Based on a result of Teirlinck [10] about the existence of $LS_{\lambda_{\min}}(3,4,w)$ for $w \equiv 0 \pmod 3$ we can prove the following theorem.

Theorem 6.1 *Let w be an integer such that $w \equiv 0 \pmod 3$ and $\gcd(7, 2w-1) = 1$. Then there exists $N = (w-3)/\lambda_{min}$ mutually disjoint simple 3-$(2w, 8, \frac{7}{2}\lambda_{min}(w-2))$ designs, where λ_{min} is the smallest λ of a 3-$(w,4,\lambda)$ quadruple system.*

Proof. Take 2-$(2w,2,1)$ design as the master design and the complete 3-$(w,4,w-3)$ design as the indexing design, where $w \equiv 0 \pmod 3$ and $\gcd(7,2w-1) = 1$. Then the constructed design has parameters 3-$(2w, 8, \frac{7}{2}(w-2)(w-3))$, which is simple by Theorem 4.1. As there is a $LS_{\lambda_{\min}}(3,4,w)$, the constructed design is a union of $N = (w-3)/\lambda_{\min}$ pairwise disjoint 3-$(2w, 8, \frac{7}{2}\lambda_{\min}(w-2))$ designs. $\square$

7 Table of simple 3-designs having 2-resolutions

The table below summarizes families of simple 2-resolvable 3-designs constructed in the paper.

Table 1: Families of simple 2-resolvabe 3-designs in Section 5.

No.	Parameters	Conditions	2-resolutions	Theorems
1	3-$(2w, 6, 5(w-2))$	$w \equiv 1, 3 \bmod 6$	$N = (w-2)$ 2-$(2w, 6, \frac{5}{2}(w-1))$	Th.5.1
2	3-$(2w, 6, 5(w-2))$	$w \equiv 5 \bmod 6$	$N = (w-2)/3$ 2-$(2w, 6, \frac{15}{2}(w-1))$	Th.5.1
3	3-$(2^n, 8, 7(2^{n-2}-1))$	$n \not\equiv 0 \bmod 3$ $n \geq 5$ odd	$N = (2^{n-2}-1)$ 2-$(2^n, 8, \frac{7}{3}(2^{n-1}-1))$	Th.5.2
4	3-$(2^n, 8, 35(2^{n-2}-1))$	$n \equiv 2, 4 \bmod 6$, $n \geq 6$	$N = (2^{n-1}-2)/6$ 2-$(2^n, 8, 35(2^{n-1}-1))$	Th.5.2
5	3-$(4(7^n+1), 8, 7^{n+1})$	$n \geq 1$	$N = 7^n$ 2-$(4(7^n+1), 8, \frac{7}{3}(2.7^n+1))$	Th.5.3
6	3-$(4(31^n+1), 8, 7.31^n)$	$n \geq 1$	$N = 31^n$ 2-$(4(31^n+1), 8, \frac{7}{3}(2.31^n+1))$	Th.5.3
7	3-$(2^{2m}, 14, 91(2^{2m-1}-2))$	$2m \equiv 2, 4, 8, 10 \bmod 12$	$N = (2^{2m-1}-2)/6$ 2-$(2^{2m}, 14, 91(2^{2m-1}-1))$	Th.5.4
8	3-$(q^n+1, 3(q+1), \Lambda)$ $\Lambda = \binom{\frac{q^n+1}{q+1}-1}{2}$ $+3(\frac{q^n-1}{q-1}-1)(\frac{q^n+1}{q+1}-2)$ $+(q^{n-1}-3)\frac{q^n-1}{q-1}+2$	$q > 5$ prime power $n \equiv 1 \bmod 2$ $\frac{q^n+1}{q+1} \equiv 1, 3 \bmod 6$	$N = \frac{q^n+1}{q+1} - 2$ 2-$(q^n+1, 3(q+1), \frac{\Lambda(q^n-1)}{N(3q+1)})$	Th. 5.6
9	3-$(q^n+1, 3(q+1), \Lambda)$ (Λ as in 8)	$\frac{q^n+1}{q+1} \equiv 5 \bmod 6$	$N = (\frac{q^n+1}{q+1} - 2)/3$ 2-$(q^n+1, 3(q+1), \frac{\Lambda(q^n-1)}{N(3q+1)})$	Th. 5.6
10	3-$(8^m, 4.8^{m-1}, \Lambda)$ $\Lambda = 5(2.8^{m-1}-1)$	$m \geq 2$	$N = 5$ 2-$(8^m, 4.8^{m-1}, 4.8^{m-1}-1)$	Th. 5.7
11	3-$(16^m, 8.16^{m-1}, \Lambda)$ $\Lambda = 13.33(4.16^{m-1}-1)$	$m \geq 2$	$N = 13$ 2-$(16^m, 8.16^{m-1}, 33(8.16^{m-1}-1))$	Th. 5.7

8 Conclusion

The paper concerns 2-resolvable 3-designs, for which very little was known up to now. Based on the two main papers [6] and [9] which generalize an old recursive construction of Shrikhande and Raghavarao for BIBD to 3-designs, we further extend the method for studying 2-resolvable 3-designs. It turns out that the approach is very efficient as it provides a simple way to construct these 3-designs. As an application we obtain many infinite families of simple 2-resolvable 3-designs. In general, however, the question of simplicity of the constructed designs remains a challenging problem, which is worth further studying.

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