On existence theorems for simple t-designs

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Abstract

The paper concerns a study of our previous general construction for simple t-designs, called the basic construction, with the goal to establish existence theorems for t-designs. As a general framework the basic construction involves a great deal of possibilities of combining ingredient designs, and thus computations are necessary for constructing designs by this method. The work shows the results of an investigation finding specified conditions under which the required computations can be avoidable. They thus lead to existence theorems for simple t-designs and many of them have been found. Also a large number of examples are included to illustrate the results.

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1 Introduction

Two recursive methods for constructing simple t-designs with arbitrary t have been presented in [5, 6]. Both methods are of combinatorial nature in which a t-design is built up from other small designs. The first method in [5], also called the basic construction, gives a construction in which the blocks of the constructed t-design are formed as a collection of block unions from a number of appropriate pairs of disjoint ingredient designs. As a result, the construction sets conditions on the indices of the ingredient designs for which a certain set of equalities have to be satisfied. The second method in [6] is a further extension of the basic construction using the concept of s-resolutions for ingredient designs. The explicit applications of the second method as shown in [6] are derived from large sets of s-designs, which are special examples of s-resolutions of the trivial t-designs for $s < t$.

In this work we particularly focus on the first paper [5]. The basic construction is a generic method which may be viewed as a general framework for constructing t-designs by block unions. First, to construct a simple t-(v, k, Λ) design, the method allows all $\lfloor v/2 \rfloor$ possible choices of two disjoint sets X_1 and X_2 of size v_1 and v_2 with

$v_1 + v_2 = v$, on which $(k+1)$ pairs of ingredient designs $\big((X_1, \mathfrak{B}^{(i)}), (X_2, \mathfrak{B}^{(k-i)})\big)$ of block sizes i and $k-i$, for $i = 0, \ldots, k$, are formed. Second, the method also allows the "vanishing" of certain such pairs, which means that they are not involved in the construction. Thus the basic construction produces a great number of possible combinations of ingredient designs. Consequently, the method requires computations to find all the designs for any given set of parameters t, v, k. In spite of this fact, we investigate the equations of the basic construction to determine specified conditions for the existence of solutions, where computations can be avoided. In particular, this leads to establishing existence theorems for simple t-designs. The results of the study are presented by distinguishing between two cases $t < k \leq 2t$ and $2t+1 \leq k$. We also include numerous examples to illustrate the obtained theorems.

For the sake of completeness we recall a few basic definitions. A t-design, denoted by t-(v, k, λ), is a pair $(X, \mathfrak{B})$, where X is a v-set of *points* and $\mathfrak{B}$ is a collection of k-subsets, called *blocks*, of X having the property that every t-set of X is a subset of exactly λ blocks in $\mathfrak{B}$. The parameter λ is called the *index* of the design. A t-design is called *simple* if no two blocks are identical i.e. no block of $\mathcal{B}$ is repeated; otherwise, it is called *non-simple* (i.e. $\mathfrak{B}$ is a multiset). It can be shown by simple counting that a t-(v, k, λ) design is an s-(v, k, λ_s) design for $0 \leq s \leq t$, where $\lambda_s = \lambda\binom{v-s}{t-s}/\binom{k-s}{t-s}$. Since λ_s is an integer, necessary conditions for the parameters of a t-design are $\binom{k-s}{t-s} | \lambda\binom{v-s}{t-s}$, for $0 \leq s \leq t$. For given t, v and k, we denote by $\lambda_{\min}(t, k, v)$, or $\lambda_{\min}$ for short, the smallest positive integer such that these conditions are satisfied for all $0 \leq s \leq t$. By complementing each block in X of a t-(v, k, λ) design, we obtain a t-$(v, v-k, \lambda^*)$ design with $\lambda^* = \lambda\binom{v-k}{t}/\binom{k}{t}$, hence we shall assume that $k \leq v/2$. The largest value for λ for which a simple t-(v, k, λ) design exists is denoted by $\lambda_{\max}$ and we have $\lambda_{\max} = \binom{v-t}{k-t}$. The simple t-$(v, k, \lambda_{\max})$ design is called the *complete* design or the *trivial* design.

We refer the reader to [1, 2, 3] for more information about designs.

1.1 The basic construction

We include a summary of the basic construction as described in [5] in the following theorem. This is necessary for the main investigation in the next section.

Theorem 1.1 (Basic construction) *Let v, k, t be integers with $v > k > t \geq 2$. Let X be a v-set and let $X = X_1 \cup X_2$ be a partition of X with $|X_1| = v_1$ and $|X_2| = v_2$. Let $D_i = (X_1, \mathfrak{B}^{(i)})$ be the complete i-$(v_1, i, 1)$ design for $i = 0, \ldots, t$ and let $D_i = (X_1, \mathfrak{B}^{(i)})$ be a simple t-$(v_1, i, \lambda_t^{(i)})$ design for $i = t+1, \ldots, k$. Similarly, let $\bar{D}_i = (X_2, \bar{\mathfrak{B}}^{(i)})$ be the complete i-$(v_2, i, 1)$ design for $i = 0, \ldots, t$, and let $\bar{D}_i = (X_2, \bar{\mathfrak{B}}^{(i)})$ be a simple t-$(v_2, i, \bar{\lambda}_t^{(i)})$ design for $i = t+1, \ldots, k$. Define*

$$\mathfrak{B} = \mathfrak{B}_{(0,k)} \times [u_0] \cup \mathfrak{B}_{(1,k-1)} \times [u_1] \cup \cdots \cup \mathfrak{B}_{(k-1,1)} \times [u_{k-1}] \cup \mathfrak{B}_{(k,0)} \times [u_k],$$

where

$$\mathfrak{B}_{(i,k-i)} = \{B = B_i \cup \bar{B}_{k-i} \ | \ B_i \in \mathfrak{B}^{(i)}, \bar{B}_{k-i} \in \bar{\mathfrak{B}}^{(k-i)}\}.$$

Assume that

$$L_{0,t} = L_{1,t-1} = L_{2,t-2} = \cdots = L_{t,0} := \Lambda, \tag{1}$$

for a positive integer Λ, *where*

$$L_{r,t-r} \quad = \quad \sum_{i=0}^{k} u_i.\lambda_r^{(i)}.\bar{\lambda}_{t-r}^{(k-i)}, \tag{2}$$

$r = 0, \ldots, t$, *and* $u_i \in \{0, 1\}$, *for* $i = 0, \ldots, k$. *Then* $(X, \mathfrak{B})$ *is a simple* t-(v, k, Λ) *design.*

Some explanations of the symbols in the theorem need to be included.

- Two degenerate cases for designs occur when either $k = t = 0$ or $v = k$.
  - The case $k = t = 0$ gives an "empty" design, denoted by $\emptyset$; note however that the number of blocks of the empty design is 1.
  - The case $v = k$ gives a degenerate k-design having just 1 block consisting of all v points.
- The notation $X \times [u]$, where X is a finite set and $u \in \{0, 1\}$, has the following meaning. $X \times [0]$ is the empty set $\emptyset$, and $X \times [1] = X$. In particular, $\mathfrak{B}_{(i,k-i)} \times [u_i]$ indicates that either it is an empty set $\emptyset$ (when $u_i = 0$) or the set $\mathfrak{B}_{(i,k-i)}$ itself (when $u_i = 1$). The case $u_i = 0$ means that the pair $(D_i, \bar{D}_{k-i})$ is not involved in the construction.
- Any t-subset T of X is denoted by $T_{(r,t-r)}$ where $|T \cap X_1| = r$ and $|T \cap X_2| = t - r$, for $r = 0, \ldots, t$. And $L_{r,t-r}$ is the number of blocks in $\mathfrak{B}$ containing $T_{(r,t-r)}$.

We should mention that Theorem 1 in [4] is a special case of the basic construction, in particular, the easiest case with $v_1 = 1$ and $v_2 = v$ has widely been used to generate new designs from two specific known designs.

2 Existence Theorems

We present in this section various existence theorems which arise from studying specific conditions of the basic construction.

2.1 Designs with $k \leq 2t$

In this section we study the case $k \leq 2t$ and $t \leq 8$, where we restrict to conditions $v_1 = v_2 = v$ and $\lambda_t^{(i)} = \bar{\lambda}_t^{(i)}$ for $i \leq k$. The goal is to find solutions of the basic construction for which u_i can be set to 0, or the ingredient design corresponding to u_i can be chosen as the trivial design.

Theorem 2.1 *(i) Suppose there exist simple* 5-$(v, i, \lambda_5^{(i)})$ *designs,* $i = 7, 8$, *such that*

$$\lambda_5^{(7)} = (v-2)(v-9)/2,$$
$$\lambda_5^{(8)} = 8(v-2).$$

Then there exists a simple 5-$(2v, 8, \Lambda_1)$ *design with*

$$\Lambda_1 = 4\binom{v-2}{3}.$$

(ii) Suppose there exist simple 5-$(v, i, \lambda_5^{(i)})$ *designs,* $i = 6, 7, 8$, *such that*

$$\lambda_5^{(7)} = -v\lambda_5^{(6)} + 2\binom{v-2}{2},$$
$$\lambda_5^{(8)} = 2(v^2 - v + 3)\lambda_5^{(6)}/3 - (v+3)\binom{v-2}{2}.$$

Then there exists a simple 5-$(2v, 8, \Lambda_2)$ *design with*

$$\Lambda_2 = \binom{v-3}{2}\lambda_5^{(6)}/3 + (v-3)\binom{v-2}{2}.$$

Proof. It is traightforward to see that the theorem follows from equalities of the basic construction. However, as an illustration, we show here some details of the proof.

(i) For case (i) we choose $v_1 = v_2 = v$, $k = 8$, $u_4 = u_6 = 0$ and $\lambda_5^{(i)} = \bar{\lambda}_5^{(i)}$ for $i = 7, \ldots, 8$. We obtain the following equalities

$$L_{2,3} = L_{3,2} = 3\binom{v-2}{3} + \binom{v-2}{3},$$
$$L_{1,4} = L_{4,1} = (v-4)\lambda_5^{(7)} + \binom{v-1}{2}(v-4),$$
$$L_{0,5} = L_{5,0} = \lambda_5^{(8)} + v\lambda_5^{(7)} + \binom{v}{3}(v-4).$$

The values for $\lambda_5^{(7)}$, $\lambda_5^{(8)}$ and Λ_1 are derived from $L_{2,3} = L_{1,4} = L_{0,5} = \Lambda_1$.

(ii) Similar to case (i), here however, we choose $u_5 = 0$. The equalities become

$$L_{2,3} = L_{3,2} = \binom{v-3}{2}\lambda_5^{(6)}/3 + \binom{v-2}{2}(v-3),$$
$$L_{1,4} = L_{4,1} = (v-4)\lambda_5^{(7)}/3 + (v-1)(v-4)\lambda_5^{(6)}/2 + \binom{v-1}{3},$$
$$L_{0,5} = L_{5,0} = \lambda_5^{(8)} + v\lambda_5^{(7)} + \binom{v}{2}\lambda_5^{(6)}.$$

From $L_{2,3} = L_{1,4} = L_{0,5} = \Lambda_2$ we compute $\lambda_5^{(7)}$, $\lambda_5^{(8)}$ and Λ_2, as desired. $\square$

Remark 2.1 In Case (i) of the theorem, for a given v, if required ingredient designs exist, then the index Λ_1 of the resulting design is unique. In Case (ii), the existence of ingredient designs depends on "free" parameter $\lambda_5^{(6)}$, and thus the resulting designs normally have different values for Λ_2. For example, consider the case $v_1 = v_2 = 24$ for Theorem 2.1. Without any restriction on u_i, there are altogether 7 solutions for 5-$(48, 8, \lambda)$ from the basic construction. One of which is the complete 5-$(48, 8, 1763 \times 7)$ design. One solution for case (i) with $u_4 = u_6 = 0$, namely 5-$(48, 8, 880 \times 7)$ has its supplement 5-$(48, 8, 883 \times 7)$ with $u_5 = 0$. There are two further solutions for case (ii) with $u_5 = 0$, namely 5-$(48, 8, 863 \times 7)$, 5-$(48, 8, 873 \times 7)$ and two of their supplements 5-$(48, 8, 900 \times 7)$, 5-$(48, 8, 890 \times 7)$.

Examples 2.1 Case (i) of Theorem 2.1:

1. $v_1 = v_2 = 18$. There is a simple 5-$(36, 8, 448 \times 5)$ design, since there are simple 5-$(18, i, \lambda_5^{(i)})$ designs for $i = 7, 8$ with $\lambda_5^{(7)} = 12 \times 6$ and $\lambda_5^{(8)} = 64 \times 2$.

2. $v_1 = v_2 = 22$. If there are simple 5-$(22, i, \lambda_5^{(i)})$ designs for $i = 7, 8$ with $\lambda_5^{(7)} = 65 \times 2$ and $\lambda_5^{(8)} = 8 \times 20$, then there is a simple 5-$(44, 8, 4560)$ design.

3. $v_1 = v_2 = 23$. If there are simple 5-$(23, i, \lambda_5^{(i)})$ designs for $i = 7, 8$ with $\lambda_5^{(7)} = 49 \times 3$ and $\lambda_5^{(8)} = 21 \times 8$, then there is a simple 5-$(46, 8, 266 \times 20)$ design.

4. $v_1 = v_2 = 24$. There is a simple 5-$(48, 8, 880 \times 7)$ design, since there are simple 5-$(24, i, \lambda_5^{(i)})$ designs for $i = 7, 8$ with $\lambda_5^{(7)} = 55 \times 3$ and $\lambda_5^{(8)} = 176$.

5. $v_1 = v_2 = 32$. If there are simple 5-$(32, i, \lambda_5^{(i)})$ designs for $i = 7, 8$ with $\lambda_5^{(7)} = 115 \times 3$ and $\lambda_5^{(7)} = 48 \times 5$, then there is a simple 5-$(64, 8, 16240)$ design.

Case (ii) of Theorem 2.1:

1. $v_1 = v_2 = 24$. The following simple 5-$(48, 8, \Lambda_2)$ designs exist with

$$\Lambda_2 = (836 + 10j) \times 7, \;\; j = 0, 1, 2$$

since there exist simple 5-$(24, i, \lambda_5^{(i)})$ designs for $i = 6, 7, 8$ with

$$\lambda_5^{(6)} = (17 + j), \;\; \lambda_5^{(7)} = (18 - 8j) \times 3, \;\; \lambda_5^{(8)} = (53 + 370j), \;\; j = 0, 1, 2.$$

2. $v_1 = v_2 = 32$. If there exist simple 5-$(32, i, \lambda_5^{(i)})$ designs for $i = 6, 7, 8$ with

$$\lambda_5^{(6)} = (8 + j) \times 3, \;\; \lambda_5^{(7)} = (34 - 32j) \times 3, \;\; \lambda_5^{(8)} = (139 + 398j) \times 5, \;\; j = 0, 1,$$

then there exists a simple $(64, 8, \Lambda_2)$ design with

$$\Lambda_2 = (15863 + 406j), \;\; j = 0, 1.$$

Theorem 2.2 *(i) Suppose there exist simple* $6\text{-}(v, i, \lambda_6^{(i)})$ *designs,* $i = 8, 9, 10$*, such that*

$$\begin{aligned}
\lambda_6^{(8)} &= (v-3)(v-10)/2, \\
\lambda_6^{(9)} &= 8(v-3), \\
\lambda_6^{(10)} &= (v-3)(v^3 - 27v^2 + 122v - 480)/24.
\end{aligned}$$

Then there exists a simple $6\text{-}(2v, 10, \Lambda_1)$ *design with*

$$\Lambda_1 = 2(v-3)\binom{v-3}{3}.$$

(ii) Suppose there exist simple $6\text{-}(v, i, \lambda_6^{(i)})$ *designs,* $i = 7, \ldots, 10$*, such that*

$$\begin{aligned}
\lambda_6^{(8)} &= -(v-1)\lambda_6^{(7)} + 2\binom{v-3}{2}, \\
\lambda_6^{(9)} &= 2(v^2 - 3v + 5)\lambda_6^{(7)}/3 - (v+2)\binom{v-3}{2}, \\
\lambda_6^{(10)} &= (-3v^3 + 6v^2 + 9v - 60)\lambda_6^{(7)}/12 + \binom{v-3}{2}(v^2 - v + 12)/2.
\end{aligned}$$

Then there exists a simple $6\text{-}(2v, 10, \Lambda_2)$ *design with*

$$\Lambda_2 = \binom{v-3}{3}\lambda_6^{(7)}/2 + \binom{v-3}{2}^2.$$

Proof. The proof is similar to that of Theorem 2.1.

(i) For the equalities of the basic construction with $v_1 = v_2 = v$, $k = 10$ and $\lambda_6^{(i)} = \bar{\lambda}_6^{(i)}$ for $i = 8, \ldots, 10$ choose $u_5 = u_7 = 0$.

(ii) As in (i), but choose $u_4 = u_6 = 0$.

□

Examples 2.2 Case (i) of Theorem 2.2:

1. $v_1 = v_2 = 24$. If there are simple $6\text{-}(24, i, \lambda_6^{(i)})$ designs for $i = 8, 9, 10$ with $\lambda_6^{(8)} = 49 \times 3$, $\lambda_6^{(9)} = 7 \times 24$ and $\lambda_6^{(10)} = 7 \times 90$, then there is a simple $6\text{-}(48, 10, 266 \times 210)$ design.

2. $v_1 = v_2 = 33$. If there are simple $6\text{-}(33, i, \lambda_6^{(i)})$ designs for $i = 8, 9, 10$ with $\lambda_6^{(8)} = 115 \times 3$, $\lambda_6^{(9)} = 16 \times 15$ and $\lambda_6^{(10)} = 140 \times 90$, then there is a simple $6\text{-}(66, 10, 16269 \times 15)$ design.

3. $v_1 = v_2 = 40$. If there are simple $6\text{-}(40, i, \lambda_6^{(i)})$ designs for $i = 8, 9, 10$ with $\lambda_6^{(8)} = 185 \times 3$, $\lambda_6^{(9)} = 37 \times 8$ and $\lambda_6^{(10)} = 19425 \times 2$, then there is a simple $6\text{-}(80, 10, 95830 \times 6)$ design.

Case (ii) of Theorem 2.2:

1. $v_1 = v_2 = 45$. If there are simple 6-$(45, i, \lambda_6^{(i)})$ designs for $i = 7, \ldots, 10$ with $\lambda_6^{(7)} = 12 \times 3$, $\lambda_6^{(8)} = 46 \times 3$, $\lambda_6^{(9)} = 5013$ and $\lambda_6^{(10)} = 8324 \times 9$, then there is a simple 6-$(90, 10, 15047 \times 63)$ design.

2. $v_1 = v_2 = 59$. If there exist simple 6-$(59, i, \lambda_6^{(i)})$ designs for $i = 7, \ldots, 10$ with

$$\begin{aligned}
&\lambda_6^{(7)} = 48 + j, \;\; \lambda_6^{(8)} = (148 - 29.j) \times 2, \\
&\lambda_6^{(9)} = (5974 + 1103.j) \times 2, \;\; \lambda_6^{(10)} = (53012 - 9913.j) \times 5, \;\; j = 0, \ldots, 5,
\end{aligned}$$

then there exists a simple 6-$(118, 10, \Lambda_2)$ design with

$$\Lambda_2 = (43384 + 198.j) \times 70, \;\; j = 0, \ldots, 5.$$

Theorem 2.3 *(i) Suppose there exist simple* 7-$(v, i, \lambda_7^{(i)})$ *designs,* $i = 9, \ldots, 12$, *such that*

$$\begin{aligned}
\lambda_7^{(10)} &= -2(2v-3)\lambda_7^{(9)}/3 + 2\binom{v-3}{2}(2v-19)/3, \\
\lambda_7^{(11)} &= (11v^2 - 29v + 30)\lambda_7^{(9)}/12 - 5\binom{v-3}{2}(v^2 - 9v - 10)/6, \\
\lambda_7^{(12)} &= -(2v^3 - 5v^2 + 3v + 10)\lambda_7^{(9)}/5 + \binom{v-3}{2}(2v^3 - 21v^2 + 19v - 150)/5.
\end{aligned}$$

Then there exists a simple 7-$(2v, 12, \Lambda_1)$ *design with*

$$\Lambda_1 = \binom{v-4}{3}\lambda_7^{(9)}/10 + 3\binom{v-3}{4}(v-4).$$

(ii) Suppose there exist simple 8-$(v, i, \lambda_8^{(i)})$ *designs,* $i = 10, \ldots, 14$, *such that*

$$\begin{aligned}
\lambda_8^{(11)} &= -(4v-10)\lambda_8^{(10)}/3 + 2\binom{v-4}{2}(2v-21)/3, \\
\lambda_8^{(12)} &= (11v^2 - 51v + 70)\lambda_8^{(10)}/12 - 5\binom{v-4}{2}v(v-11)/6, \\
\lambda_8^{(13)} &= -v(2v^2 - 11v + 19)\lambda_8^{(10)}/5 + \binom{v-4}{2}(2v^3 - 27v^2 + 67v - 192)/5, \\
\lambda_8^{(14)} &= (23v^4 - 128v^3 + 253v^2 - 268v + 840)\lambda_8^{(10)}/180 \\
&\quad - (v-4)(v-5)(3v^4 - 38v^3 + 33v^2 + 242v - 1680)/48.
\end{aligned}$$

Then there exists a simple 8-$(2v, 14, \Lambda_2)$ *design with*

$$\Lambda_2 = 2\binom{v-4}{4}\lambda_8^{(10)}/15 + 2\binom{v-4}{2}\binom{v-4}{4}.$$

Proof.

(i) Choose $u_6 = u_8 = 0$ for equalities of the basic construction with $v_1 = v_2 = v$, $k = 12$ and $\lambda_7^{(i)} = \bar{\lambda}_7^{(i)}$, $i = 9, \ldots, 12$.

(ii) Choose $u_7 = u_9 = 0$ for equalities of the basic construction with $v_1 = v_2 = v$, $k = 14$ and $\lambda_8^{(i)} = \bar{\lambda}_8^{(i)}$, $i = 10, \ldots, 14$.

□

Examples 2.3 Case (i) of Theorem 2.3:

1. $v_1 = v_2 = 60$. If there exist simple 7-$(60, i, \lambda_7^{(i)})$ designs for $i = 9, \ldots, 12$ with

$$\lambda_7^{(9)} = (672 + j) \times 2,\ \ \lambda_7^{(10)} = (1316 - 78.j) \times 2,$$
$$\lambda_7^{(11)} = (37436 + 1263.j) \times 5,\ \ \lambda_7^{(12)} = (392088 - 23668.j) \times 7,\ \ j = 0, \ldots, 16,$$

then there exists a simple 7-$(120, 12, \Lambda_1)$ design with

$$\Lambda_1 = (455112 + 36.j) \times 154,\ \ j = 0, \ldots, 16.$$

2. $v_1 = v_2 = 66$. If there exist simple 7-$(66, i, \lambda_7^{(i)})$ designs for $i = 9, \ldots, 12$ with

$$\lambda_7^{(9)} = (1670 + 11.j),\ \ \lambda_7^{(10)} = (3506 - 946.j),$$
$$\lambda_7^{(11)} = (21410 + 3014.j) \times 14,\ \ \lambda_7^{(12)} = (28898 - 7906.j) \times 154,\ \ j = 0, \ldots, 3,$$

then there exists a simple 7-$(132, 12, \Lambda_1)$ design with

$$\Lambda_1 = (10646330 + 3782.j) \times 11,\ \ j = 0, \ldots, 3.$$

3. $v_1 = v_2 = 69$. If there exist simple 7-$(69, i, \lambda_7^{(i)})$ designs for $i = 9, \ldots, 12$ with

$$\lambda_7^{(9)} = (1841 + 2.j),\ \ \lambda_7^{(10)} = (224 - 9.j) \times 20,$$
$$\lambda_7^{(11)} = (69965 + 1680.j) \times 5,\ \ \lambda_7^{(12)} = (3119008 - 126686.j) \times 2,\ \ j = 0, \ldots, 24,$$

then there exists a simple 7-$(138, 12, \Lambda_1)$ design with

$$\Lambda_1 = (74290944 + 4368.j) \times 2,\ \ j = 0, \ldots, 24.$$

Case (ii) of Theorem 2.3:

1. $v_1 = v_2 = 61$. If there are simple 8-$(61, i, \lambda_8^{(i)})$ designs for $i = 10, \ldots, 14$ with

$$\lambda_8^{(10)} = 686 \times 2,\ \ \lambda_8^{(11)} = 224 \times 2,\ \ \lambda_8^{(12)} = 55118 \times 5,$$
$$\lambda_8^{(13)} = 4672 \times 91,\ \ \lambda_8^{(14)} = 41853 \times 364.$$

then there is a simple 8-$(122, 14, \Lambda_2)$ design with $\Lambda_2 = 8656704 \times 154$.

2. $v_1 = v_2 = 70$. If there are simple 8-$(70, i, \lambda_8^{(i)})$ designs for $i = 10, \ldots, 14$ with

$$\lambda_8^{(10)} = (1869 + 2.j), \;\; \lambda_8^{(11)} = (98 - 9.j) \times 20, \;\; \lambda_8^{(12)} = (93485 + 1680.j) \times 5,$$
$$\lambda_8^{(13)} = (1345404 - 126686.j) \times 2, \;\; \lambda_8^{(14)} = 2221149 + 5661642.j, \;\; j = 0, \ldots, 10,$$

then there is a simple 8-$(140, 14, \Lambda_2)$ design with

$$\Lambda_2 = (5719392 + 336.j) \times 572, \;\; j = 0, \ldots, 10.$$

Theorem 2.4 *Suppose there exist simple* 4-$(v, i, \lambda_4^{(i)})$ *designs,* $i = 5, 7, 8$ *such that*

$$\lambda_4^{(7)} = (v-2)(v-9)\lambda_4^{(5)}/6,$$
$$\lambda_4^{(8)} = 2(v-2)\lambda_4^{(5)}.$$

Then there exists a simple 4-$(2v, 8, \Lambda)$ *design with*

$$\Lambda = 2(v-2)\binom{v-2}{2}\lambda_4^{(5)}/3.$$

Proof. The result follows from equalities of the basic construction with $v_1 = v_2 = v$, $k = 8$, $u_2 = u_4 = u_6 = 0$ and $\lambda_4^{(i)} = \bar{\lambda}_4^{(i)}$, $i = 5, 7, 8$. □

Examples 2.4 1. $v_1 = v_2 = 22$. There exist simple 4-$(44, 8, \Lambda)$ designs with

$$\Lambda = j.1520 \times 10, \;\; j = 1, 2, 3,$$

since simple 4-$(22, i, \lambda_4^{(i)})$ designs for $i = 5, 7, 8$ exist with

$$\lambda_4^{(5)} = j6, \;\; \lambda_4^{(7)} = j4, \;\; \lambda_4^{(8)} = j30, \;\; j = 1, 2, 3.$$

2. $v_1 = v_2 = 23$. There exist simple 4-$(46, 8, \Lambda)$ designs with

$$\Lambda = j.14 \times 210, \;\; j = 1, \ldots, 19,$$

since simple 4-$(23, i, \lambda_4^{(i)})$ designs for $i = 5, 7, 8$ exist with

$$\lambda_4^{(5)} = j, \;\; \lambda_4^{(7)} = j, \;\; \lambda_4^{(8)} = j2, \;\; j = 1, \ldots, 19.$$

In previous theorems we observe that certain u_i are set to 0 for the basic construction. On the other hand, we may view a complete t-$(v, i, \binom{v-t}{i-t})$ design, whose existence is guaranteed for all possible parameters, as a "constant". So, we may assume that the complete design is the corresponding design for certain $u_i = 1$. The next theorem shows how to use complete designs for the case $k = 2t$.

Theorem 2.5 *(i) Suppose there exist simple* 5-$(v, i, \lambda_5^{(i)})$ *designs,* $i = 8, 9, 10$, *such that*

$$\lambda_5^{(9)} = -(2v-1)\lambda_5^{(8)}/2 + (v-2)(v-3)(v-5)(2v-17)/12,$$
$$\lambda_5^{(10)} = (11v^2-7v+12)\lambda_5^{(8)}/20 - (v-2)(v-3)(v-5)(v^2-7v-18)/12.$$

Then there exists a simple 5-$(2v, 10, \Lambda_1)$ *design with*

$$\Lambda_1 = \binom{v-3}{2}\lambda_5^{(8)}/10 + \binom{v-2}{2}\binom{v-3}{2}(v-5)/2.$$

(ii) Suppose there exist simple 6-$(v, i, \lambda_6^{(i)})$ *designs,* $i = 9, 10, 11, 12$, *such that*

$$\lambda_6^{(10)} = -(2v-3)\lambda_6^{(9)}/2 + \binom{v-3}{2}(v-6)(2v-19)/6,$$
$$\lambda_6^{(11)} = (11v^2-29v+30)\lambda_6^{(9)}/20 - \binom{v-3}{2}(v-6)(v^2-9v-10)/6,$$
$$\lambda_6^{(12)} = -(2v^3-5v^2+3v+10)\lambda_6^{(9)}/10 + \binom{v-3}{2}(v-6)(2v^3-21v^2+19v-150)/30.$$

Then there exists a simple 6-$(2v, 12, \Lambda_2)$ *design with*

$$\Lambda_2 = \binom{v-3}{3}\lambda_6^{(9)}/10 + \binom{v-3}{2}\binom{v-3}{3}(v-6)/2.$$

(iii) Suppose there exist simple 7-$(v, i, \lambda_7^{(i)})$ *designs,* $i = 11, 12, 13, 14$, *such that*

$$\begin{aligned}
\lambda_7^{(12)} &= -(6v-10)\lambda_7^{(11)}/5 - (v-3)\binom{v-7}{3}(13v-10)/20 \\
&\quad + 7\binom{v-3}{3}(v-7)(v-10)/10, \\
\lambda_7^{(13)} &= (11v^2-29v+24)\lambda_7^{(11)}/15 + (4v-12)\binom{v-7}{3}(2v^2-4v+5)/15 \\
&\quad - 4\binom{v-3}{3}(v-7)(2v^2-20v-3)/15, \\
\lambda_7^{(14)} &= -(31v^3-80v^2+61v+60)\lambda_7^{(11)}/105 \\
&\quad - (v-3)\binom{v-7}{3}(101v^3-199v^2+194v+480)/420 \\
&\quad + \binom{v-3}{3}(v-7)(29v^3-295v^2+74v-960)/120.
\end{aligned}$$

Then there exists a simple 7-$(2v, 14, \Lambda_3)$ *design with*

$$\Lambda_3 = \binom{v-4}{3}\lambda_7^{(11)}/35 + 8\binom{v-3}{4}\binom{v-7}{3}/35 + 8\binom{v-3}{4}\binom{v-4}{2}(v-7)/15.$$

Proof.

(i) Choose $v_1 = v_2 = v$, $k = 10$, $u_5 = u_7 = 0$ and $u_6 = 1$ with the complete 5-$(v, 6, (v-5))$ design for the basic construction.

(ii) Choose $v_1 = v_2 = v$, $k = 12$, $u_6 = u_8 = 0$ and $u_7 = 1$ with the complete 6-$(v, 7, (v-6))$ design.

(iii) Choose $v_1 = v_2 = v$, $k = 14$, $u_7 = u_9 = 0$ and $u_8 = u_{10} = 1$ with the complete 7-$(v, 8, (v-7))$ and 7-$(v, 10, \binom{v-7}{3})$ designs.

□

Examples 2.5 Case (i) of Theorem 2.5:

1. $v_1 = v_2 = 23$. If there are simple 5-$(23, i, \lambda_5^{(i)})$ designs for $i = 8, 9, 10$ with

   $\lambda_5^{(8)} = (98 + j) \times 8,\ \ \lambda_5^{(9)} = (7 - 2.j) \times 90,\ \ \lambda_5^{(10)} = (7 + 9.j) \times 252,\ \ j = 0, \ldots, 3,$

   then there is a simple 5-$(46, 10, \Lambda_1)$ design with

   $$\Lambda_1 = (186998 + 76.j) \times 2,\ \ j = 0, \ldots, 3.$$

2. $v_1 = v_2 = 24$. There are simple 5-$(48, 10, \Lambda_1)$ designs with

   $$\Lambda_1 = (11438 + 6.j) \times 42,\ \ j = 0, 1, 2,$$

   since there are simple 5-$(24, i, \lambda_5^{(i)})$ designs for $i = 8, 9, 10$ with

   $\lambda_5^{(8)} = (931 + 12.j),\ \ \lambda_5^{(9)} = (133 - 47.j) \times 6,\ \ \lambda_5^{(10)} = (133 + 206.j) \times 18,\ \ j = 0, 1, 2.$

3. $v_1 = v_2 = 64$. If there are simple 5-$(64, i, \lambda_5^{(i)})$ designs for $i = 8, 9, 10$ with

   $$\lambda_5^{(8)} = (30267 + 28.j),\ \ \lambda_5^{(9)} = (10148 - 127.j) \times 14,$$
   $$\lambda_5^{(10)} = (1888 + 4462.j) \times 14,\ \ j = 0, \ldots, 79,$$

   then there is a simple 5-$(128, 10, \Lambda_1)$ design with

   $$\Lambda_1 = (2562488 + 122.j) \times 42,\ \ j = 0, \ldots, 79.$$

Case (ii) of Theorem 2.5:

1. $v_1 = v_2 = 59$. If there are simple 6-$(59, i, \lambda_6^{(i)})$ designs for $i = 9, \ldots, 12$ with

   $\lambda_6^{(9)} = (11414 + j) \times 2,\ \ \lambda_6^{(10)} = (6824 - 23.j) \times 5,$
   $\lambda_6^{(11)} = (256122 + 523.j) \times 7,\ \ \lambda_6^{(12)} = (818600 - 2811.j) \times 28,\ \ j = 0, \ldots, 291,$

   then there is a simple 6-$(118, 12, \Lambda_2)$ design with

   $$\Lambda_2 = (1292784 + 6.j) \times 924,\ \ j = 0, \ldots, 291.$$

2. $v_1 = v_2 = 68$. If there are simple 6-$(68, i, \lambda_6^{(i)})$ designs for $i = 9, \ldots, 12$ with

$$\lambda_6^{(9)} = (1840 + j) \times 20, \quad \lambda_6^{(10)} = (13504 - 266.j) \times 5,$$
$$\lambda_6^{(11)} = (2000080 + 24461.j) \times 2, \quad \lambda_6^{(12)} = 60842624 - 1211916.j, \quad j = 0, \ldots, 50,$$

then there is a simple 6-$(136, 12, \Lambda_2)$ design with

$$\Lambda_2 = (14177280 + 416.j) \times 210, \quad j = 0, \ldots, 50.$$

Case (iii) of Theorem 2.5:

1. $v_1 = v_2 = 39$. If there are simple 7-$(39, i, \lambda_7^{(i)})$ designs for $i = 11, \ldots, 14$ with

$$\lambda_7^{(11)} = 5 \times 10, \quad \lambda_7^{(12)} = 3548 \times 56, \quad \lambda_7^{(13)} = 876 \times 84, \quad \lambda_7^{(14)} = 6500 \times 312.$$

then there is a simple 7-$(78, 14, \Lambda_3)$ design with $\Lambda_3 = 930002 \times 715$.

2. $v_1 = v_2 = 48$. If there are simple 7-$(48, i, \lambda_7^{(i)})$ designs for $i = 11, \ldots, 14$ with

$$\lambda_7^{(11)} = (7 + 13.j) \times 10, \quad \lambda_7^{(12)} = (372466 - 3614.j) \times 2,$$
$$\lambda_7^{(13)} = (938 + 1332.j) \times 156, \quad \lambda_7^{(14)} = (24720 - 5154.j) \times 780, \quad j = 0, \ldots, 4,$$

then there is a simple 7-$(96, 14, \Lambda_3)$ design with

$$\Lambda_3 = (78298958 + 1118.j) \times 44, \quad j = 0, \ldots, 4.$$

2.2 Designs with arbitrarily large t and $k \geq 2t + 1$

In the previous section we have dealt with specified conditions for the basic construction for $4 \leq t \leq 8$ and $k \leq 2t$. In this section we investigate specified conditions for the basic construction with arbitrarily large t, and $k \geq 2t+1$. This case is interesting as we obtain statements about internal relationship between ingredient designs and constructed designs.

Theorem 2.6 *Let t, k, and v be positive integers such that $2t + 1 \leq k < v - t$. Assume that there exist simple t-$(v, h, \bar{\lambda}_t^{(h)})$ designs for $h = k - t, \ldots, k$ such that the indices $\bar{\lambda}_t^{(h)}$, for $h = k - t + 1, \ldots, k$, can be computed from $\bar{\lambda}_t^{(k-t)}$ by the recursive formulas*

$$L_{i,t-i} = \binom{v - (t-i)}{i} \sum_{j=0}^{t-i} \frac{\binom{t+1-i}{j}}{\binom{k-t-j}{i}} \bar{\lambda}_t^{(k-i-j)},$$

and equalities $L_{0,t} = L_{1,t-1} = \cdots = L_{t,0}$. Then there is a simple $t-(v+t+1, k, \Lambda)$ design with

$$\Lambda = \frac{\binom{v}{t}}{\binom{k-t}{t}} \bar{\lambda}_t^{(k-t)}.$$

Proof. The theorem follows from the basic construction with $|X_1| = v_1 = t+1$, $|X_2| = v_2 = v$ and $u_i = 1$ for $i = 0, \ldots, t$ and $u_{t+1} = 0$. More precisely, from the general expression for

$$L_{i,t-i} = \sum_{j=0}^{k} u_j.\lambda_i^{(j)}.\bar{\lambda}_{t-i}^{(k-j)},$$

we have

$$\begin{aligned} L_{i,t-i} &= \binom{t+1-i}{i-i}\frac{\binom{v-(t-i)}{i}}{\binom{k-i-(t-i)}{i}}\bar{\lambda}_t^{(k-i)} + \binom{t+1-i}{i+1-i}\frac{\binom{v-(t-i)}{i}}{\binom{k-(i+1)-(t-i)}{i}}\bar{\lambda}_t^{(k-(i+1))} \\ &\quad + \binom{t+1-i}{i+2-i}\frac{\binom{v-(t-i)}{i}}{\binom{k-(i+2)-(t-i)}{i}}\bar{\lambda}_t^{(k-(i+2))} + \cdots + \binom{t+1-i}{i+(t-i)-i}\frac{\binom{v-(t-i)}{i}}{\binom{k-t-(t-i)}{i}}\bar{\lambda}_t^{(k-t)}, \\ &= \binom{v-(t-i)}{i}\Bigg[\frac{\binom{t+1-i}{i-i}}{\binom{k-i-(t-i)}{i}}\bar{\lambda}_t^{(k-i)} + \frac{\binom{t+1-i}{i+1-i}}{\binom{k-(i+1)-(t-i)}{i}}\bar{\lambda}_t^{(k-(i+1))} \\ &\qquad + \frac{\binom{t+1-i}{i+2-i}}{\binom{k-(i+2)-(t-i)}{i}}\bar{\lambda}_t^{(k-(i+2))} + \cdots + \frac{\binom{t+1-i}{i+(t-i)-i}}{\binom{k-t-(t-i)}{i}}\bar{\lambda}_t^{(k-t)}\Bigg], \\ &= \binom{v-(t-i)}{i}\Bigg[\frac{\binom{t+1-i}{0}}{\binom{k-t}{i}}\bar{\lambda}_t^{(k-i)} + \frac{\binom{t+1-i}{1}}{\binom{k-t-1}{i}}\bar{\lambda}_t^{(k-(i+1))} \\ &\qquad + \frac{\binom{t+1-i}{2}}{\binom{k-t-2}{i}}\bar{\lambda}_t^{(k-(i+2))} + \cdots + \frac{\binom{t+1-i}{t-i}}{\binom{k-t-(t-i)}{i}}\bar{\lambda}_t^{(k-t)}\Bigg], \\ &= \binom{v-(t-i)}{i}\sum_{j=0}^{t-i}\frac{\binom{t+1-i}{j}}{\binom{k-t-j}{i}}\bar{\lambda}_t^{(k-i-j)}, \end{aligned}$$

as desired.

The equalities $L_{0,t} = L_{1,t-1} = \cdots = L_{t,0} := \Lambda$ for a positive integer Λ shows in particular that

$$\Lambda = L_{t,0} = \frac{\binom{v}{t}}{\binom{k-t}{t}}\bar{\lambda}_t^{(k-t)}.$$

□

Here is an example to illustrate the recursive computation of $\bar{\lambda}_t^{(h)}$, $h = k-t+1, \ldots, k$ in terms of v, k and $\bar{\lambda}_t^{(k-t)}$ from $L_{i,t-i}$ for $t = 5$. The expressions $L_{i,5-i}$, for

$i = 5, 4, 3, 2, 1, 0$, are then

$$L_{5,0} = \binom{v}{5}\Big(\frac{1}{\binom{k-5}{5}}\bar{\lambda}_5^{(k-5)}\Big),$$
$$L_{4,1} = \binom{v-1}{4}\Big(\frac{1}{\binom{k-5}{4}}\bar{\lambda}_5^{(k-4)} + \frac{2}{\binom{k-6}{4}}\bar{\lambda}_5^{(k-5)}\Big),$$
$$L_{3,2} = \binom{v-2}{3}\Big(\frac{1}{\binom{k-5}{3}}\bar{\lambda}_5^{(k-3)} + \frac{3}{\binom{k-6}{3}}\bar{\lambda}_5^{(k-4)} + \frac{3}{\binom{k-7}{3}}\bar{\lambda}_5^{(k-5)}\Big),$$
$$L_{2,3} = \binom{v-3}{2}\Big(\frac{1}{\binom{k-5}{2}}\bar{\lambda}_5^{(k-2)} + \frac{4}{\binom{k-6}{2}}\bar{\lambda}_5^{(k-3)} + \frac{6}{\binom{k-7}{2}}\bar{\lambda}_5^{(k-4)} + \frac{4}{\binom{k-8}{2}}\bar{\lambda}_5^{(k-5)}\Big),$$
$$L_{1,4} = (v-4)\Big(\frac{1}{k-5}\bar{\lambda}_5^{(k-1)} + \frac{5}{k-6}\bar{\lambda}_5^{(k-2)} + \frac{10}{k-7}\bar{\lambda}_5^{(k-3)} + \frac{10}{k-8}\bar{\lambda}_5^{(k-4)} + \frac{5}{k-9}\bar{\lambda}_5^{(k-5)}\Big),$$
$$L_{0,5} = \bar{\lambda}_5^{(k)} + 6\bar{\lambda}_5^{(k-1)} + 15\bar{\lambda}_5^{(k-2)} + 20\bar{\lambda}_5^{(k-3)} + 15\bar{\lambda}_5^{(k-4)} + 6\bar{\lambda}_5^{(k-5)}.$$

It is clear that the equalities $L_{5,0} = L_{4,1} = L_{3,2} = L_{2,3} = L_{1,4} = L_{0,5}$ yield a recursive computation of $\bar{\lambda}_5^{(k-4)}$, $\bar{\lambda}_5^{(k-3)}$, $\bar{\lambda}_5^{(k-2)}$, $\bar{\lambda}_5^{(k-1)}$, $\bar{\lambda}_5^{(k)}$ in terms of v, k and $\bar{\lambda}_5^{(k-5)}$.

Remark 2.2 Observe that in Theorem 2.6 the number of ingredient designs whose indices need to be computed from $\bar{\lambda}_t^{(k-t)}$ is always t independent of the given value k in the interval $[t+1, v-t-1]$.

As an illustration of Theorem 2.6 we show two corollaries for $t = 5$, $k = 12$ and $t = 6$, $k = 15$, with explicit expressions for computing the indices of the ingredient designs.

Corollary 2.7 *Suppose there are simple* 5-$(v, h, \bar{\lambda}_5^{(h)})$ *designs,* $h = 7, \ldots, 12$, *such* *that*

$$
\begin{aligned}
\bar\lambda_5^{(8)} &= \frac{1}{3\binom{v-1}{4}}\bar\lambda_5^{(7)}\left(5\binom{v}{5}-14\binom{v-1}{4}\right),\\
\bar\lambda_5^{(9)} &= \frac{1}{12\binom{v-2}{3}\binom{v-1}{4}}\bar\lambda_5^{(7)}\left(-105\binom{v-2}{3}\binom{v}{5}+168\binom{v-2}{3}\binom{v-1}{4}+20\binom{v}{5}\binom{v-1}{4}\right),\\
\bar\lambda_5^{(10)} &= \frac{1}{15\binom{v-3}{2}\binom{v-2}{3}\binom{v-1}{4}}\bar\lambda_5^{(7)}\left(420\binom{v-3}{2}\binom{v-2}{3}\binom{v}{5}-504\binom{v-3}{2}\binom{v-2}{3}\binom{v-1}{4}\right.\\
&\quad\left.-140\binom{v-3}{2}\binom{v}{5}\binom{v-1}{4}+15\binom{v-2}{3}\binom{v}{5}\binom{v-1}{4}\right),\\
\bar\lambda_5^{(11)} &= \frac{1}{18\binom{v-3}{2}\binom{v-2}{3}\binom{v-1}{4}(v-4)}\bar\lambda_5^{(7)}\left(6\binom{v}{5}\binom{v-3}{2}\binom{v-2}{3}\binom{v-1}{4}\right.\\
&\quad+1260v\binom{v-3}{2}\binom{v-2}{3}\binom{v-1}{4}-1260v\binom{v-3}{2}\binom{v-2}{3}\binom{v}{5}\\
&\quad+560v\binom{v-3}{2}\binom{v}{5}\binom{v-1}{4}-105v\binom{v}{5}\binom{v-2}{3}\binom{v-1}{4}\\
&\quad-5040\binom{v-3}{2}\binom{v-2}{3}\binom{v-1}{4}+5040\binom{v-3}{2}\binom{v-2}{3}\binom{v}{5}\\
&\quad\left.-2240\binom{v-3}{2}\binom{v}{5}\binom{v-1}{4}+420\binom{v-2}{3}\binom{v}{5}\binom{v-1}{4}\right),\\
\bar\lambda_5^{(12)} &= \frac{1}{21\binom{v-3}{2}\binom{v-2}{3}\binom{v-1}{4}(v-4)}\bar\lambda_5^{(7)}\left(-46\binom{v}{5}\binom{v-3}{2}\binom{v-2}{3}\binom{v-1}{4}\right.\\
&\quad-2772v\binom{v-3}{2}\binom{v-2}{3}\binom{v-1}{4}+3150v\binom{v-3}{2}\binom{v-2}{3}\binom{v}{5}\\
&\quad-1680v\binom{v-3}{2}\binom{v}{5}\binom{v-1}{4}+420v\binom{v}{5}\binom{v-2}{3}\binom{v-1}{4}\\
&\quad+\binom{v}{5}\binom{v-3}{2}\binom{v-2}{3}\binom{v-1}{4}v+11088\binom{v-3}{2}\binom{v-2}{3}\binom{v-1}{4}\\
&\quad+6720\binom{v-3}{2}\binom{v}{5}\binom{v-1}{4}-1680\binom{v-2}{3}\binom{v}{5}\binom{v-1}{4}\\
&\quad\left.-12600\binom{v-3}{2}\binom{v-2}{3}\binom{v}{5}\right).
\end{aligned}
$$

Then there is a simple 5-$(v+6,12,\Lambda)$ *design with* $\Lambda = \frac{1}{21}\binom{v}{5}\bar\lambda_5^{(7)}$.

Examples 2.6 (Corollary 2.7):

1. $v_1 = 6,\ v_2 = 25$. If there are simple 5-$(25, i, \bar\lambda_5^{(i)})$ designs for $i = 7, \ldots, 12$ with

$$
\begin{aligned}
&\bar\lambda_5^{(7)} = 6.j\times 10,\ \ \bar\lambda_5^{(8)} = 11.j\times 20,\ \ \bar\lambda_5^{(9)} = 81.j\times 15,\\
&\bar\lambda_5^{(10)} = 141.j\times 24,\ \ \bar\lambda_5^{(11)} = 155.j\times 60,\ \ \bar\lambda_5^{(12)} = 144.j\times 120,\ \ j=1,2,3,
\end{aligned}
$$

then there is a simple 5-$(31,12,\Lambda)$ design with

$$\Lambda = 345.j\times 440,\ \ j = 1,2,3.$$

A close look at these 3 solutions shows that a 5-$(31,12,690\times 440)$ design exists for $j=2$ which was not known before. When $j=1$ and 3, all the ingredient 5-$(25,i,\bar\lambda_5^{(i)})$ designs exist, except for 5-$(25,11,\bar\lambda_5^{(11)})$ designs with $\bar\lambda_5^{(11)} = 155\times 60$ and 465×60, whose existence is still undecided.

2. $v_1 = 6$, $v_2 = 28$. If there exist simple 5-$(28, i, \bar{\lambda}_5^{(i)})$ designs for $i = 7, \ldots, 12$ with

$$\bar{\lambda}_5^{(7)} = 15.j, \;\; , \;\; \bar{\lambda}_5^{(8)} = 10.j \times 7, \;\; \bar{\lambda}_5^{(9)} = 12.j \times 35,$$
$$\bar{\lambda}_5^{(10)} = 210.j \times 7, \;\; \bar{\lambda}_5^{(11)} = 20.j \times 231, \;\; \bar{\lambda}_5^{(12)} = 330.j \times 33, \;\; j = 1, \ldots, 16,$$

then there exists a simple 5-$(34, 12, \Lambda)$ design with

$$\Lambda = 5850.j \times 12, \;\; j = 1, \ldots, 16.$$

Corollary 2.8 *Suppose there are simple* 6-$(v, h, \bar{\lambda}_6^{(h)})$ *designs,* $h = 9, \ldots, 15$, *such that*

$$\bar{\lambda}_6^{(10)} = \frac{3}{2\binom{v-1}{5}}\bar{\lambda}_6^{(9)}\left(-\binom{v}{6} + 3\binom{v-1}{5}\right),$$

$$\bar{\lambda}_6^{(11)} = \frac{3}{10\binom{v-2}{4}\binom{v-1}{5}}\bar{\lambda}_6^{(9)}\left(5\binom{v}{6}\binom{v-1}{5} - 27\binom{v-2}{4}\binom{v}{6} + 45\binom{v-2}{4}\binom{v-1}{5}\right),$$

$$\begin{aligned}\bar{\lambda}_6^{(12)} = {} & \frac{1}{\binom{v-3}{3}\binom{v-2}{4}\binom{v-1}{5}}\bar{\lambda}_6^{(9)}\left(\binom{v}{6}\binom{v-2}{4}\binom{v-1}{5} - 9\binom{v-3}{3}\binom{v}{6}\binom{v-1}{5}\right. \\ & \left. + 27\binom{v-3}{3}\binom{v-2}{4}\binom{v}{6} - 33\binom{v-3}{3}\binom{v-2}{4}\binom{v-1}{5}\right),\end{aligned}$$

$$\begin{aligned}\bar{\lambda}_6^{(13)} = {} & \frac{3}{7\binom{v-4}{2}\binom{v-3}{3}\binom{v-2}{4}\binom{v-1}{5}}\bar{\lambda}_6^{(9)}\left(\binom{v-3}{3}\binom{v-2}{4}\binom{v}{6}\binom{v-1}{5}\right. \\ & - 15\binom{v-4}{2}\binom{v}{6}\binom{v-2}{4}\binom{v-1}{5} + 75\binom{v-4}{2}\binom{v-3}{3}\binom{v}{6}\binom{v-1}{5} \\ & \left. - 165\binom{v-4}{2}\binom{v-3}{3}\binom{v-2}{4}\binom{v}{6} + 165\binom{v-4}{2}\binom{v-3}{3}\binom{v-2}{4}\binom{v-1}{5}\right),\end{aligned}$$

$$\begin{aligned}\bar{\lambda}_6^{(14)} = {} & \frac{3}{28\binom{v-4}{2}\binom{v-3}{3}\binom{v-2}{4}\binom{v-1}{5}(v-5)}\bar{\lambda}_6^{(9)}\left(1485v\binom{v-4}{2}\binom{v-3}{3}\binom{v-2}{4}\binom{v}{6}\right. \\ & - 1287v\binom{v-4}{2}\binom{v-3}{3}\binom{v-2}{4}\binom{v-1}{5} - 27v\binom{v}{6}\binom{v-3}{3}\binom{v-2}{4}\binom{v-1}{5} \\ & + 225v\binom{v-4}{2}\binom{v}{6}\binom{v-2}{4}\binom{v-1}{5} - 825v\binom{v-4}{2}\binom{v-3}{3}\binom{v}{6}\binom{v-1}{5} \\ & - 7425\binom{v-4}{2}\binom{v-3}{3}\binom{v-2}{4}\binom{v}{6} + 6435\binom{v-4}{2}\binom{v-3}{3}\binom{v-2}{4}\binom{v-1}{5} \\ & + 135\binom{v}{6}\binom{v-3}{3}\binom{v-2}{4}\binom{v-1}{5} - 1125\binom{v-4}{2}\binom{v}{6}\binom{v-2}{4}\binom{v-1}{5} \\ & \left. + 4125\binom{v-4}{2}\binom{v-3}{3}\binom{v}{6}\binom{v-1}{5} + \binom{v-4}{2}\binom{v-3}{3}\binom{v-2}{4}\binom{v}{6}\binom{v-1}{5}\right),\end{aligned}$$

$$\begin{aligned}\bar{\lambda}_6^{(15)} = {} & \frac{1}{84\binom{v-4}{2}\binom{v-3}{3}\binom{v-2}{4}\binom{v-1}{5}(v-5)}\bar{\lambda}_6^{(9)}\left(-68\binom{v}{6}\binom{v-4}{2}\binom{v-3}{3}\binom{v-2}{4}\binom{v-1}{5}\right. \\ & - 27027v\binom{v-4}{2}\binom{v-3}{3}\binom{v-2}{4}\binom{v}{6} + 21021v\binom{v-4}{2}\binom{v-3}{3}\binom{v-2}{4}\binom{v-1}{5} \\ & + 945v\binom{v}{6}\binom{v-3}{3}\binom{v-2}{4}\binom{v-1}{5} - 5775v\binom{v-4}{2}\binom{v}{6}\binom{v-2}{4}\binom{v-1}{5} \\ & + 17325v\binom{v-4}{2}\binom{v-3}{3}\binom{v}{6}\binom{v-1}{5} + \binom{v}{6}\binom{v-4}{2}\binom{v-3}{3}\binom{v-2}{4}\binom{v-1}{5}v \\ & - 4725\binom{v}{6}\binom{v-3}{3}\binom{v-2}{4}\binom{v-1}{5} - 105105\binom{v-4}{2}\binom{v-3}{3}\binom{v-2}{4}\binom{v-1}{5} \\ & + 135135\binom{v}{6}\binom{v-4}{2}\binom{v-3}{3}\binom{v-2}{4} - 86625\binom{v}{6}\binom{v-4}{2}\binom{v-3}{3}\binom{v-1}{5} \\ & \left. + 28875\binom{v}{6}\binom{v-4}{2}\binom{v-2}{4}\binom{v-1}{5}\right).\end{aligned}$$

Then there is a simple 6-$(v+7,15,\Lambda)$ *design with* $\Lambda = \frac{1}{84}\binom{v}{6}\bar{\lambda}_6^{(9)}$.

Examples 2.7 (Corollary 2.8):

$v_1 = 7,\ v_2 = 30$. If there are simple 6-$(30, i, \bar{\lambda}_6^{(i)})$ designs for $i = 9, \ldots, 15$ with

$$\bar{\lambda}_6^{(9)} = 14.j \times 8,\ \ \bar{\lambda}_6^{(10)} = 8.j \times 42,\ \ \bar{\lambda}_6^{(11)} = j \times 1848,\ \ \bar{\lambda}_6^{(12)} = 16.j \times 308,$$
$$\bar{\lambda}_6^{(13)} = 18.j \times 792,\ \ \bar{\lambda}_6^{(14)} = 280.j \times 99,\ \ \bar{\lambda}_6^{(15)} = 1211.j \times 44,\ \ j = 1, \ldots, 18,$$

then there is a simple 6-$(37, 15, \Lambda)$ design with

$$\Lambda = 12180.j \times 65, \ \ j = 1, \ldots, 18.$$

Remark 2.3 The expressions for $\bar{\lambda}_t^{(i)}$ in Corollaries 2.7 and 2.8 were computed by using Maple 14, Maple April 5, 2010.

Theorem 2.9 *Let* t, k, v_1, *and* v_2 *be positive integers with* $v = v_1 + v_2$ *such that* $2t + 1 \le k < v - t$. *Assume that there are simple* t-$(v_1, j, \lambda_t^{(j)})$ *and* t-$(v_2, j, \bar{\lambda}_t^{(j)})$ *designs for* $j = 0, \ldots, k$ *having the property that either* $\lambda_t^{(j)} = \lambda_{\max}^{(j)}$ *or* $\bar{\lambda}_t^{(k-j)} = \bar{\lambda}_{\max}^{(k-j)}$ *but not both (i.e. in each pair* $(\mathfrak{B}^{(j)}, \bar{\mathfrak{B}}^{(k-j)})$ *of the basic construction, exactly one design is not the complete design). Let*

$$J := \{j \ \mid \ \lambda_t^{(j)} \ne \lambda_{\max}^{(j)} \ \text{ for } \ j = 0, \ldots, k\},$$

$$\bar{J} := \{j \ \mid \ \bar{\lambda}_t^{(j)} \ne \bar{\lambda}_{\max}^{(j)}, \ \text{ for } \ j = 0, \ldots, k\}$$

denote the subsets of non-trivial designs in all such pairs. Let

$$m_j := \lambda_{\max}^{(j)} / \lambda_{\min}^{(j)}, \quad \bar{m}_j := \bar{\lambda}_{\max}^{(j)} / \bar{\lambda}_{\min}^{(j)},$$

for $j \in J \cup \bar{J}$.

Let

$$n = \gcd(m_{i_1}, \ldots, m_{i_{|J|}}, \bar{m}_{j_1}, \ldots, \bar{m}_{j_{|\bar{J}|}}),$$

where $J = \{i_1, \ldots, i_{|J|}\}$ *and* $\bar{J} = \{j_1, \ldots, j_{|\bar{J}|}\}$

For $i \in J$ *and* $j \in \bar{J}$, *define*

$$\varrho^{(i)} := m_i/n, \ \ \bar{\varrho}^{(j)} := \bar{m}_j/n.$$

Assume that

$$\lambda_t^{(i)} = \lambda_{\min}^{(i)}.\varrho^{(i)}.h \quad \text{and} \quad \bar{\lambda}_t^{(j)} = \bar{\lambda}_{\min}^{(j)}.\bar{\varrho}^{(j)}.h,$$

$i \in J$, $j \in \bar{J}$ *and* $1 \le h \le n$. *Then there is a simple* t-(v, k, Λ) *design, with* $\Lambda = L(0, t) = L(1, t-1) = \cdots = L(t, 0) = \binom{v_1 + v_2 - t}{k - t} * h/n$.

Proof. Recall that the expressions $L(0,t), L(1,t-1), \ldots, L(t-1,1), L(t,0)$ from the basic construction are obtained by taking the union of blocks from $(k+1)$ pairs of the ingredient designs

$$(\mathfrak{B}^{(0)}, \bar{\mathfrak{B}}^{(k)}), \ldots, (\mathfrak{B}^{(t)}, \bar{\mathfrak{B}}^{(k-t)}), (\mathfrak{B}^{(t+1)}, \bar{\mathfrak{B}}^{(k-(t+1))}), \ldots, (\mathfrak{B}^{(k-(t+1))}, \bar{\mathfrak{B}}^{(t+1)}),$$
$$(\mathfrak{B}^{(k-t)}, \bar{\mathfrak{B}}^{(t)}), \ldots, (\mathfrak{B}^{(k)}, \bar{\mathfrak{B}}^{(0)}),$$

where

$$\begin{aligned} L(s, t-s) &= \sum_{j=0}^{k} u_j \lambda_s^j \bar{\lambda}_{t-s}^{k-j}, \\ &= \sum_{j=0}^{k} u_j c_{j,k-j} \lambda_t^j \bar{\lambda}_t^{k-j}. \end{aligned}$$

with $c_{j,k-j}$ as constant.

Here is the crucial point. Set $u_j = 1$ for $j = 0, \ldots, k$. From each pair $(\mathfrak{B}^{(j)}, \bar{\mathfrak{B}}^{(k-j)})$ choose exactly one, either $\mathfrak{B}^{(j)}$ or $\bar{\mathfrak{B}}^{(k-j)}$, as the trivial design. In other words, in all the terms $c_{j,k-j}\lambda_t^{(j)}\bar{\lambda}_t^{(k-j)}$ we have either $\lambda_t^{(j)} = \lambda_{\max}^{(j)}$ or $\bar{\lambda}_t^{(k-j)} = \bar{\lambda}_{\max}^{(k-j)}$.

In doing so we treat the index of the trivial design in each pair $(\mathfrak{B}^{(j)}, \bar{\mathfrak{B}}^{(k-j)})$ as a constant, and the index of the other design as a variable.

Now, observe that if all ingredient designs are trivial designs, we then obtain the trivial t-$(v_1+v_2, k, \binom{v_1+v_2-t}{k-t})$ design from the basic construction. In this case, we have either

$$\lambda_t^{(j)} = \lambda_{\min}^{(j)}.\varrho^{(j)}.n \quad \text{or} \quad \bar{\lambda}_t^{(j)} = \bar{\lambda}_{\min}^{(j)}.\bar{\varrho}^{(j)}.n,$$

for $j = 0, \ldots, k$ and

$$L(0,t) = L(1,t-1) = \cdots = L(t,0) = \binom{v_1+v_2-t}{k-t}.$$

This implies that each of the expressions $L(0,t) = L(1,t-1) = \cdots = L(t,0)$ is a multiple of n. In particular, if the non-trivial ingredient designs have indices

$$\lambda_t^{(j)} = \lambda_{\min}^{(j)}.\varrho^{(j)} \quad \text{or} \quad \bar{\lambda}_t^{(k-j)} = \bar{\lambda}_{\min}^{(k-j)}.\bar{\varrho}^{(k-j)},$$

for $j = 0, \ldots, k$, then we obtain a non-trivial t-(v_1+v_2, k, Λ) design with

$$L(0,t) = L(1,t-1) = \cdots = L(t,0) = \Lambda = \binom{v_1+v_2-t}{k-t}/n.$$

Therefore, if

$$\lambda_t^{(j)} = \lambda_{\min}^{(j)}.\varrho^{(j)}.h \quad \text{or} \quad \bar{\lambda}_t^{(k-j)} = \bar{\lambda}_{\min}^{(k-j)}.\bar{\varrho}^{(k-j)}.h,$$

for $1 \leq h < n$ and $j = 0, \ldots, k$, the construction will yield a non-trivial t-$(v_1 + v_2, k, \Lambda)$= t-$(v_1+v_2, k, \binom{v_1+v_2-t}{k-t} * h/n)$ design. $\square$

Examples 2.8 1. $t = 5$, $v_1 = 18$, $v_2 = 19$ and $k = 12$. For the pair $(\mathfrak{B}^{(6)}, \bar{\mathfrak{B}}^{(6)})$ we choose $(X_2, \bar{\mathfrak{B}}^{(6)})$ as the trivial 5-$(19, 6, 14)$ design. Here we have

$$(\lambda^{(6)}_{\min}, \lambda^{(7)}_{\min}, \lambda^{(8)}_{\min}, \lambda^{(9)}_{\min}, \lambda^{(10)}_{\min}, \lambda^{(11)}_{\min}, \lambda^{(12)}_{\min}) = (1, 6, 2, 5, 9, 132, 132),$$
$$(\bar{\lambda}^{(6)}_{\min}, \bar{\lambda}^{(7)}_{\min}, \bar{\lambda}^{(8)}_{\min}, \bar{\lambda}^{(9)}_{\min}, \bar{\lambda}^{(10)}_{\min}, \bar{\lambda}^{(11)}_{\min}, \bar{\lambda}^{(12)}_{\min}) = (2, 7, 28, 7, 14, 231, 264),$$

so, $n = \gcd(x_0, \ldots, x_{12}) = 13$ and

$$(\varrho^{(6)}, \varrho^{(7)}, \varrho^{(8)}, \varrho^{(9)}, \varrho^{(10)}, \varrho^{(11)}, \varrho^{(12)}) = (1, 1, 11, 11, 11, 1, 1),$$
$$(\bar{\varrho}^{(7)}, \bar{\varrho}^{(8)}, \bar{\varrho}^{(9)}, \bar{\varrho}^{(10)}, \bar{\varrho}^{(11)}, \bar{\varrho}^{(12)}) = (1, 1, 11, 11, 1, 1).$$

Hence, if

$$(\lambda^{(6)}, \lambda^{(7)}, \lambda^{(8)}, \lambda^{(9)}, \lambda^{(10)}, \lambda^{(11)}, \lambda^{(12)}) = (h, h6, 11h2, 11h5, 11h9, h132, h132),$$
$$(\bar{\lambda}^{(7)}, \bar{\lambda}^{(8)}, \bar{\lambda}^{(9)}, \bar{\lambda}^{(10)}, \bar{\lambda}^{(11)}, \bar{\lambda}^{(12)}) = (h7, h28, 11h7, 11h14, h231, h264),$$

for $h = 1, \ldots, 12$, then there exists a non-trivial 5-$(37, 12, \Lambda)$ design with $\Lambda = \binom{v_1+v_2-t}{k-t} h/n) = \binom{32}{7} h/13$. It turns out that for $h = 4, 5$ all the ingredient designs exist, hence there exist a 5-$(37, 12, 43152{\times}24)$ and a 5-$(37, 12, 53940{\times}24)$ design. Both designs were unknown.

2. $t = 5$, $v_1 = v_2 = 21$, $k = 12, 13, 14, 15$. It is straightforward to check that $n = \gcd(x_0, \ldots, x_{15}) = 2$ and

$$(\varrho^{(6)}, \varrho^{(7)}, \varrho^{(8)}, \varrho^{(9)}, \varrho^{(10)}, \varrho^{(11)}, \varrho^{(12)}, \varrho^{(13)}, \varrho^{(14)}, \varrho^{(15)}) = (2, 2, 1, 13, 13, 13, 13, 1, 2, 2),$$

with $\varrho^{(i)} = \bar{\varrho}^{(i)}$, $i = 7, \ldots, 15$. Since the 5-$(21, i, \lambda^{(i)})$ ingredient designs with the following indices

$$(\lambda^{(6)}, \lambda^{(7)}, \lambda^{(8)}, \lambda^{(9)}, \lambda^{(10)}, \lambda^{(11)}, \lambda^{(12)}, \lambda^{(13)}, \lambda^{(14)}, \lambda^{(15)}) =$$
$$(2 \times 4, 2 \times 30, 1 \times 180, 13 \times 70, 13 \times 168, 13 \times 308, 13 \times 440, 1 \times 6435, 2 \times 2860, 2 \times 2002)$$

exist, there exist non-trivial simple designs with the following parameters

$$5 - (42, 12, 38998 \times 132),$$
$$5 - (42, 13, 38998 \times 495),$$
$$5 - (42, 14, 1130942 \times 55),$$
$$5 - (42, 15, 1130942 \times 154).$$

Note that these designs are all halvings of the complete 5-$(42, k, \binom{37}{k-5})$ designs for $k = 12, \ldots, 15$.

3. $t = 5$, $v_1 = v_2 = 25$, $k = 13, \ldots, 18$. Observe that $n = \gcd(x_0, \ldots, x_{18}) = 19$ and

$$(\varrho^{(7)}, \varrho^{(8)}, \varrho^{(9)}, \varrho^{(10)}, \varrho^{(11)}, \varrho^{(12)}, \varrho^{(13)}, \varrho^{(14)}, \varrho^{(15)}, \varrho^{(16)}, \varrho^{(17)}, \varrho^{(18)}) =$$
$$(1, 3, 17, 34, 34, 34, 34, 34, 34, 17, 3, 1),$$

with $\varrho^{(i)} = \bar{\varrho}^{(i)}$, $i = 7, \ldots, 18$. Since the 5-$(25, i, \lambda^{(i)})$ ingredient designs with the following indices

$$(\lambda^{(7)}, \lambda^{(8)}, \lambda^{(9)}, \lambda^{(10)}, \lambda^{(11)}, \lambda^{(12)}, \lambda^{(13)}, \lambda^{(14)}, \lambda^{(15)}, \lambda^{(16)}, \lambda^{(17)}, \lambda^{(18)}) =$$
$$(8 \times 10, 24 \times 20, 136 \times 15, 272 \times 24, 272 \times 60, 272 \times 120,$$
$$272 \times 195, 272 \times 260, 272 \times 286, 136 \times 520, 24 \times 2210, 8 \times 4080)$$

exist, there exist non-trivial simple designs with the following parameters

$$\begin{aligned}
&5-(50, 13, 14104 \times 6435),\\
&5-(50, 14, 521848 \times 715),\\
&5-(50, 15, 1565544 \times 858),\\
&5-(50, 16, 1565544 \times 2730),\\
&5-(50, 17, 1565544 \times 7735),\\
&5-(50, 18, 5740328 \times 5355).
\end{aligned}$$

Remark 2.4 1. It should be noted that if appropriate t-$(v, i, \lambda^{(i)})$ designs would exist for $i \in \{t+1, \ldots, v-(t+1)\}$, then Theorem 2.9 could be able to construct t-$(2v, k, \Lambda)$ designs for $2t + 1 \leq k \leq v - (t + 1)$.

2. Observe that in Example 2.8 (3) above if we would consider the case $k = 12$, then Theorem 2.9 would only yield the trivial design. This is because the trivial 5-$(25, 6, 20)$ design is the only simple design for 5-$(25, 6, \lambda_5^{(6)})$. So, $n = \gcd(x_0, \ldots, x_{12}) = 1$, hence the theorem will give the trivial 5-$(50, 12, \binom{45}{7})$ design as the single solution.

The next proposition is useful with regard to the application of Theorem 2.9, as it will show us that $n = \gcd(m_{i_1}, \ldots, m_{i_{|J|}}, \bar{m}_{j_1}, \ldots, \bar{m}_{j_{|\bar{J}|}}) \neq 1$ under a specific condition.

Proposition 2.10 *Let t, k, v be integers with $0 < t < k < v - t$ such that $(v - t)$ is a prime. Then there is a permissible parameter t-(v, k, σ) with $\lambda_{\min} \leq \sigma < \lambda_{\max}$; in other words $\lambda_{\min} \neq \lambda_{\max}$. In particular, $\lambda_{\min}$ divides $\binom{v-(t+1)}{k-(t+1)}/(k - t)$ and $(v - t)$ divides $\lambda_{\max}/\lambda_{\min}$.*

Proof. We have

$$\lambda_{\max} = \binom{v-t}{k-t} = (v - t)\binom{v-(t+1)}{k-(t+1)}/(k - t).$$

Set

$$\sigma = \binom{v-(t+1)}{k-(t+1)}/(k - t).$$

Because $(v-t)$ is a prime and $k-i < v-t$ for all $i = 0, \ldots, k$, it follows that σ is an integer. For the trivial design t-$(v, k, \lambda_{\max})$=t-$(v, k, \binom{v-t}{k-t})$, we have

$$\begin{aligned}\lambda_s &= \lambda_{\max} \binom{v-s}{t-s} / \binom{k-s}{t-s} \\ &= (v-t)\sigma \binom{v-s}{t-s} / \binom{k-s}{t-s},\end{aligned}$$

for $0 \le s \le t$. Again, because $(v-t)$ is a prime larger than $k-s$ and $t-s$, it follows that $\sigma\binom{v-s}{t-s}/\binom{k-s}{t-s}$ is an integer. In other words, t-(v, k, σ) is a permissible parameter. Since $\lambda_{\min}|\sigma$ and $\sigma < \lambda_{\max}$, we have $\lambda_{\min} \neq \lambda_{\max}$. □

The next examples show an application of Theorem 2.9 together with Proposition 2.10.

Examples 2.9 1. $t = 5$, $v_1 = v_2 = 22$ and $k = 11, 12, 13, 14, 15, 16$. By Proposition 2.10 it follows that $n = \gcd(x_0, \ldots, x_k) = 17$. Since the 5-$(22, i, \lambda^{(i)})$ ingredient designs with the following indices

$(\lambda^{(6)}, \lambda^{(7)}, \lambda^{(8)}, \lambda^{(9)}, \lambda^{(10)}, \lambda^{(11)}, \lambda^{(12)}, \lambda^{(13)}, \lambda^{(14)}, \lambda^{(15)}, \lambda^{(16)}) =$
$(\{5, 6\} \times 1, \{20, 24\} \times 2, \{10, 12\} \times 20, \{10, 12\} \times 70, \{130, 156\} \times 14, \{130, 156\} \times 28,$
$\{130, 156\} \times 44, \{10, 12\} \times 715, \{10, 12\} \times 715, \{20, 24\} \times 286, \{5, 6\} \times 728)$

exist, there exist non-trivial simple designs with the following parameters

$$\begin{gathered}5-(44, 11, 45695 \times 21),\ \ 5-(44, 11, 54834 \times 21),\\ 5-(44, 12, 45695 \times 99),\ \ 5-(44, 12, 54834 \times 99),\\ 5-(44, 13, 14060 \times 1287),\ \ 5-(44, 13, 16872 \times 1287),\\ 5-(44, 14, 435860 \times 143),\ \ 5-(44, 14, 523032 \times 143),\\ 5-(44, 15, 435860 \times 429),\ \ 5-(44, 15, 523032 \times 429),\\ 5-(44, 16, 6319970 \times 78),\ \ 5-(44, 16, 7583964 \times 78).\end{gathered}$$

2. $t = 6$, $v_1 = v_2 = 23$ and $k = 13, 14, 15, 16$. By Proposition 2.10 it follows that $n = \gcd(x_0, \ldots, x_k) = 17$. Since the 6-$(23, i, \lambda^{(i)})$ ingredient designs with the following indices

$(\lambda^{(7)}, \lambda^{(8)}, \lambda^{(9)}, \lambda^{(10)}, \lambda^{(11)}, \lambda^{(12)}, \lambda^{(13)}, \lambda^{(14)}, \lambda^{(15)}, \lambda^{(16)}) =$
$(\{5, 6, 8\} \times 1,\ \{10, 12, 16\} \times 4,\ \{10, 12, 16\} \times 20,\ \{10, 12, 16\} \times 70,$
$\{130, 156, 208\} \times 14,\ \{130, 156, 208\} \times 28,\ \{10, 12, 16\} \times 572,\ \{10, 12, 16\} \times 715,$
$\{10, 12, 16\} \times 715,\ \{5, 6, 8\} \times 1144)$

exist, there exist non-trivial simple designs with the following parameters

$6-(46, 13, 3515 \times 1560),\ 6-(46, 13, 4218 \times 1560),\ 6-(46, 13, 5624 \times 1560),$
$6-(46, 14, 3515 \times 6435),\ 6-(46, 14, 4218 \times 6435),\ 6-(46, 14, 5624 \times 6435),$
$6-(46, 15, 28120 \times 2860),\ 6-(46, 15, 33744 \times 2860),\ 6-(46, 15, 44992 \times 2860),$
$6-(46, 16, 217930 \times 1144),\ 6-(46, 16, 261516 \times 1144),\ 6-(46, 16, 348688 \times 1144).$

3 Conclusion

The main purpose of this work was an investigation of specified conditions for the basic construction, under which the existence of solutions can be proved without computations. The results have led to various existence theorems for simple t-designs, and would contribute to a better understanding of the solutions emerged from this general construction. We think there are further cases which are worth studying.

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