Simple t-designs: A recursive construction for arbitrary t

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Abstract

The aim of this paper is to present a recursive construction of simple $t-$designs for arbitrary t. The construction is of purely combinatorial nature and it requires finding solutions for the indices of the ingredient designs that satisfy a certain set of equalities. We give a small number of examples to illustrate the construction, whereby we have found a large number of new $t-$designs, which were previously unknown. This indicates that the method is useful and powerful.

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1 Introduction

One of the most challenging problems in design theory is the problem of constructing simple $t-$designs for large t. There are several major approaches to the problem. These are constructing $t-$designs from large sets of $t-$designs, for instance [1], [11], [14], [15], [20], [21], [25]; constructing $t-$designs by using prescribed automorphism groups, for example [3], [4], [5], [6], [7], [9], [13], [16]; or contructing $t-$designs via recursive construction methods, see for instance [10], [12], [17], [18], [19], [22], [23], [24].

In this paper we present a new recursive method for constructing simple $t-$designs for arbitrary t. The method is of combinatorial nature, which is a composition technique where a $t-$design is built up from other smaller ingredient designs. Which ingredient designs will be necessary are determined by the solutions to a set of equalities involving their indices. The method proves to be very useful and powerful. Our experimental results obtained from its application have shown that, even for a small number of chosen parameters for the ingredient designs, plentiful new simple designs can be constructed, which were previously unknown.

We recall some basic definitions. A $t-$design, denoted by $t-(v,k,\lambda)$, is a pair $(X,\mathcal{B})$, where X is a $v-$set of *points* and $\mathcal{B}$ is a collection of $k-$subsets, called *blocks*, of X having the property that every $t-$set of X is a subset of exactly λ blocks in $\mathcal{B}$. The parameter λ is called the *index* of the design. A $t-$design is called *simple* if no two blocks are identical i.e. no block of $\mathcal{B}$ is repeated; otherwise, it is called *non-simple* (i.e. $\mathcal{B}$ is a multiset). It can be shown by simple counting that a $t-(v,k,\lambda)$ design is an $s-(v,k,\lambda_s)$ design for $0\leq s\leq t$, where $\lambda_s=\lambda\binom{v-s}{t-s}/\binom{k-s}{t-s}$. Since λ_s is an integer, necessary conditions for the parameters of a $t-$design are $\binom{k-s}{t-s}|\lambda\binom{v-s}{t-s}$, for $0\leq s\leq t$. For given t,v and k, we denote by $\lambda_{\text{min}}(t,k,v)$, or λ_{min} for short, the smallest positive integer such that these conditions are satisfied for all $0\leq s\leq t$. By complementing each block in X of a $t-(v,k,\lambda)$ design, we obtain a $t-(v,v-k,\lambda^*)$ design, where $\lambda^*=\lambda\binom{v-k}{t}/\binom{k}{t}$, hence we shall assume that $k\leq v/2$. The largest value for λ for which a simple $t-(v,k,\lambda)$ design exists is denoted by λ_{max} and we have $\lambda_{\text{max}}=\binom{v-t}{k-t}$. The simple $t-(v,k,\lambda_{\text{max}})$ design is called the *complete* design or the *trivial* design. A $t-(v,k,1)$ design is called a *t-Steiner system.*

We refer the reader to [2], [8] for more information about designs.

1.1 The Construction

We first introduce ingredients and notation used in the construction.

Let $t,\ v,\ k$ be non-negative integers such that $v\geq k\geq t\geq 0$. Let X be a v-set and let $X=X_1\cup X_2$ be a partition of X (i.e $X_1\cap X_2=\emptyset$) with $|X_1|=v_1$ and $|X_2|=v_2$.

Throughout the paper the parameter set $t-(v_2,j,\bar{\lambda}_t^{(j)})$ for a design indicates that the point set of the design is X_2. Also, a design defined on the point set X_2 will be denoted by $\bar{D}=(X_2,\bar{\mathcal{B}})$.

1. For $i=0,\ldots,t$, let $D_i=(X_1,\mathcal{B}^{(i)})$ be the complete $i-(v_1,i,1)$ design. For $i=t+1,\ldots,k$, let $D_i=(X_1,\mathcal{B}^{(i)})$ be a simple $t-(v_1,i,\lambda_t^{(i)})$ design.

2. Similarly, for $i=0,\ldots,t$, let $\bar{D}_i=(X_2,\bar{\mathcal{B}}^{(i)})$ be the complete $i-(v_2,i,1)$ design. And for $i=t+1,\ldots,k$, let $\bar{D}_i=(X_2,\bar{\mathcal{B}}^{(i)})$ be a simple $t-(v_2,i,\bar{\lambda}_t^{(i)})$ design.

3. Two degenerate cases for designs occur when either $v=k=t=0$ or $v=k$. The first case $v=k=t=0$ gives an "empty" design, denoted by $\emptyset$, however we use the convention that the number of blocks of the empty design is 1 (i.e. the unique block is the empty block). The second case $v=k$ gives a degenerate k-design having just 1 block consisting of all v points. Thus, in these two extreme cases the number of blocks of the designs is always 1.

4. We denote by $T_{(s,t-s)}$ a t-subset T of X with $|T\cap X_1|=s$ and hence $|T\cap X_2|=t-s$, for $s=0,\ldots,t$. It is clear that any t-subset of X is a $T_{(s,t-s)}$ set for some $s\in\{0,\ldots,t\}$.

5. Let X be a finite set and let $u\in\{0,1\}$. The notation $X\times[u]$ has the following meaning. $X\times[0]$ is the empty set $\emptyset$, and $X\times[1]=X$.

We now describe our construction. Consider $(k+1)$ pairs of simple designs $(D_i, \bar{D}_{k-i})$ for $i = 0, \ldots, k$, where $D_i = (X_1, \mathcal{B}^{(i)})$ is a simple $t-(v_1, i, \lambda_t^{(i)})$ design and $\bar{D}_{k-i} = (X_2, \bar{\mathcal{B}}^{(k-i)})$ a simple $t-(v_2, k-i, \bar{\lambda}_t^{(k-i)})$ design, as defined above. For each pair $(D_i, \bar{D}_{k-i})$ define

$$\mathcal{B}_{(i,k-i)} := \{B = B_i \cup \bar{B}_{k-i} \ / \ B_i \in \mathcal{B}^{(i)}, \bar{B}_{k-i} \in \bar{\mathcal{B}}^{(k-i)}\}.$$

Thus, $\mathcal{B}_{(i,k-i)}$ is a collection of k-subsets of X obtained by taking the union of blocks of D_i and $\bar{D}_{k-i}$. Note that the sets $\mathcal{B}_{(i,k-i)}$ and $\mathcal{B}_{(j,k-j)}$ are pairwise disjoint for $i \neq j$ and $i, j = 0, \ldots, k$.

Define

$$\mathcal{B} := \mathcal{B}_{(0,k)} \times [u_0] \cup \mathcal{B}_{(1,k-1)} \times [u_1] \cup \cdots \cup \mathcal{B}_{(k-1,1)} \times [u_{k-1}] \cup \mathcal{B}_{(k,0)} \times [u_k],$$

where $u_i \in \{0, 1\}$, for $i = 0, \ldots, k$.

It should be noted that the notation $\mathcal{B}_{(i,k-i)} \times [u_i]$, as defined in [5.] above, indicates that either we have an empty set $\emptyset$ (when $u_i = 0$) or the set $\mathcal{B}_{(i,k-i)}$ itself (when $u_i = 1$). The empty set case implies that the pair $(D_i, \bar{D}_{k-i})$ is not used and the other case shows the use of $(D_i, \bar{D}_{k-i})$. Thus u_i's are considered as variables.

We examine the necessary conditions for which $(X, \mathcal{B})$ forms a simple t-design. Consider the block set $\mathcal{B}_{(i,k-i)}$. We see that each t-subset $T_{(s,t-s)}$ of X is contained in

$$\lambda_s^{(i)} . \bar{\lambda}_{t-s}^{(k-i)}$$

blocks of $\mathcal{B}_{(i,k-i)}$, for $s = 0, \ldots, t$. It is clear because any s-set of X_1 is contained in $\lambda_s^{(i)}$ blocks of D_i and any $(t-s)$-set of X_2 is contained in $\bar{\lambda}_{t-s}^{(k-i)}$ blocks of $\bar{D}_{k-i}$. Note that $\lambda_s^{(i)} . \bar{\lambda}_{t-s}^{(k-i)}$ could be equal to 0; this is the case when $i < s$ or $k - i < t - s$. Define

$$\Lambda_{s,t-s}^{(i,k-i)} := \lambda_s^{(i)} . \bar{\lambda}_{t-s}^{(k-i)}.$$

It follows that for a given t-set $T_{(s,t-s)}$ of X the number of blocks in $\mathcal{B}$ containing $T_{(s,t-s)}$ is equal to

$$\begin{aligned} L_{s,t-s} &:= u_0.\Lambda_{s,t-s}^{(0,k)} + u_1.\Lambda_{s,t-s}^{(1,k-1)} + \cdots + u_k.\Lambda_{s,t-s}^{(k,0)} \\ &= \sum_{i=0}^{k} u_i.\Lambda_{s,t-s}^{(i,k-i)} \\ &= \sum_{i=0}^{k} u_i.\lambda_s^{(i)}.\bar{\lambda}_{t-s}^{(k-i)}, \end{aligned}$$

Since any t-set T of X is of form $T_{s,t-s}$ for some $s \in \{0, \ldots, t\}$, so if

$$L_{0,t} = L_{1,t} = L_{2,t-2} = \cdots = L_{t,0} := \Lambda,$$

where Λ is a positive integer, then $(X, \mathcal{B})$ forms a simple t-design with parameters $t - (v, k, \Lambda)$.

We record the result of the construction discussed above in the following theorem.

Theorem 1.1 *Let v, k, t be integers with $v > k > t \geq 2$. Let X be a v-set and let $X = X_1 \cup X_2$ be a partition of X with $|X_1| = v_1$ and $|X_2| = v_2$. Let $D_i = (X_1, \mathcal{B}^{(i)})$ be the complete $i-(v_1, i, 1)$ design for $i = 0, \ldots, t$ and let $D_i = (X_1, \mathcal{B}^{(i)})$ be a simple $t-(v_1, i, \lambda_t^{(i)})$ design for $i = t+1, \ldots, k$. Similarly, let $\bar{D}_i = (X_2, \bar{\mathcal{B}}^{(i)})$ be the complete $i-(v_2, i, 1)$ design for $i = 0, \ldots, t$, and let $\bar{D}_i = (X_2, \bar{\mathcal{B}}^{(i)})$ be a simple $t-(v_2, i, \bar{\lambda}_t^{(i)})$ design for $i = t+1, \ldots, k$. Define*

$$\mathcal{B} = \mathcal{B}_{(0,k)} \times [u_0] \cup \mathcal{B}_{(1,k-1)} \times [u_1] \cup \cdots \cup \mathcal{B}_{(k-1,1)} \times [u_{k-1}] \cup \mathcal{B}_{(k,0)} \times [u_k],$$

where

$$\mathcal{B}_{(i,k-i)} = \{B = B_i \cup \bar{B}_{k-i} \ / \ B_i \in \mathcal{B}^{(i)}, \bar{B}_{k-i} \in \bar{\mathcal{B}}^{(k-i)}\}.$$

Assume that

$$L_{0,t} = L_{1,t-1} = L_{2,t-2} = \cdots = L_{t,0} := \Lambda, \tag{1}$$

for a positive integer Λ, where

$$L_{s,t-s} \;=\; \sum_{i=0}^{k} u_i.\lambda_s^{(i)}.\bar{\lambda}_{t-s}^{(k-i)}, \tag{2}$$

$s = 0, \ldots, t$, and $u_i \in \{0, 1\}$, for $i = 0, \ldots, k$. Then $(X, \mathcal{B})$ is a simple $t-(v, k, \Lambda)$ design.

Two remarks should be included. Firstly, Eq.(1) always has at least one solution giving rise to the complete $t-(v, k, \binom{v-t}{k-t})$ design. In other words, if each ingredient design is a complete design with its corresponding parameters, then we obtain the complete design as a result. Secondly, we mainly focus on simple designs, so we have formulated Theorem 1.1 accordingly. But, the construction by no means restricts to simple $t-$designs. It works for both simple and non-simple designs. In fact, the construction only uses the "balance property" which depends on the indices $\lambda_t^{(i)}$, and not on any "structural property" of the ingredient designs. Thus, if any of the ingredient designs is non-simple, then so is the resulting design constructed from a solution of Eq.(1).

2 Applications

In this section we illustrate the construction in Theorem 1.1 through a number of examples which also prove the strength of the method. In fact, for some given parameters with $t = 4, 5, 6$, we have constructed a large number of new simple designs.

In the following we will employ the notation from Chapter 4 : t-Designs with $t \geq 3$ of the Handbook of Combinatorial Designs. The parameter set $t-(v, k, \lambda)$ of a design will be written as $t-(v, k, m\lambda_{\text{min}})$. Since the supplement of a simple $t-(v, k, \lambda)$ design is a $t-(v, k, \lambda_{\text{max}} - \lambda)$ design, we usually consider simple $t-(v, k, \lambda)$ designs with $\lambda \leq \lambda_{\text{max}}/2$. Thus, the upper limit of m of a constructed design will be LIM $= \lfloor \lambda_{\text{max}}/(2\lambda_{\text{min}}) \rfloor$. But, it should be remarked that, when an ingredient design with index λ is used, then λ can take on all possible values, i.e. $\lambda_{\text{min}} \leq \lambda \leq \lambda_{\text{max}}$.

2.1 Simple $5-(36,k,\Lambda)$ designs

A detailed example will illustrate the construction.

2.1.1 Simple $5-(36,10,\Lambda)$ designs

Let $X = X_1 \cup X_2$ be a partition of the point set X with $|X| = 36$ into two subsets X_1 and X_2 with $|X_1| = |X_2| = 18$. For $i = 0,1,2,3,4,5$ let $D_i = (X_1, \mathcal{B}^{(i)})$ be the complete $i-(18,i,1)$ designs. For $i = 6,7,8,9,10$ let $D_i = (X_1, \mathcal{B}^{(i)})$ be a simple $5-(18,i,\lambda_5^{(i)})$ design. These designs have the following parameters.

- $5-(18,6,\lambda_5^{(6)}) = 5-(18,6,m)$, $m = 1,2,\ldots,13$.
- $5-(18,7,\lambda_5^{(7)}) = 5-(18,7,m6)$, $m = 1,2,\ldots,13$
- $5-(18,8,\lambda_5^{(8)}) = 5-(18,8,m2)$, $m = 1,2,\ldots,143$
- $5-(18,9,\lambda_5^{(9)}) = 5-(18,9,m5)$, $m = 1,2,\ldots,143$
- $5-(18,10,\lambda_5^{(10)}) = 5-(18,10,m9)$, $m = 1,2,\ldots,143$ (the complement of a $5-(18,8,m2)$).

Correspondingly, let $\bar{D}_i = (X_2, \bar{\mathcal{B}}^{(i)})$ be simple designs defined on X_2. We first compute $L_{0,5}$, $L_{1,4}$, $L_{2,3}$. We have

$$L_{s,5-s} = \sum_{i=0}^{10} u_i.\lambda_s^{(i)}.\bar{\lambda}_{5-s}^{(10-i)}, \tag{3}$$

$s = 0,\ldots,5$, and $u_i \in \{0,1\}$ for $i = 0,\ldots,10$.

Since $\bar{\lambda}_5^{(4)} = \bar{\lambda}_5^{(3)} = \bar{\lambda}_5^{(2)} = \bar{\lambda}_5^{(1)} = \bar{\lambda}_5^{(0)} = 0$ and $\bar{\lambda}_5^{(5)} = 1$, we have

$$\begin{aligned} L_{0,5} &= u_0\lambda_0^{(0)}\bar{\lambda}_5^{(10)} + u_1\lambda_0^{(1)}\bar{\lambda}_5^{(9)} + u_2\lambda_0^{(2)}\bar{\lambda}_5^{(8)} + u_3\lambda_0^{(3)}\bar{\lambda}_5^{(7)} + u_4\lambda_0^{(4)}\bar{\lambda}_5^{(6)} + u_5\lambda_0^{(5)}\bar{\lambda}_5^{(5)} \\ &= u_0\bar{\lambda}_5^{(10)} + u_1 18\bar{\lambda}_5^{(9)} + u_2 153\bar{\lambda}_5^{(8)} + u_3 816\bar{\lambda}_5^{(7)} + u_4 3060\bar{\lambda}_5^{(6)} + u_5 8568. \end{aligned}$$

Since $\bar{\lambda}_4^{(3)} = \bar{\lambda}_4^{(2)} = \bar{\lambda}_4^{(1)} = \bar{\lambda}_4^{(0)} = 0$ and $\lambda_1^{(0)} = 0$, we have

$$\begin{aligned} L_{1,4} &= u_1\lambda_1^{(1)}\bar{\lambda}_4^{(9)} + u_2\lambda_1^{(2)}\bar{\lambda}_4^{(8)} + u_3\lambda_1^{(3)}\bar{\lambda}_4^{(7)} + u_4\lambda_1^{(4)}\bar{\lambda}_4^{(6)} + u_5\lambda_1^{(5)}\bar{\lambda}_4^{(5)} + u_6\lambda_1^{(6)}\bar{\lambda}_4^{(4)} \\ &= u_1\frac{14}{5}\bar{\lambda}_5^{(9)} + u_2\frac{17\times 7}{2}\bar{\lambda}_5^{(8)} + u_3\frac{136\times 14}{3}\bar{\lambda}_5^{(7)} + u_4 680\times 7\bar{\lambda}_5^{(6)} + \\ &\quad u_5 2380\times 14 + u_6 476\lambda_5^{(6)}. \end{aligned}$$

Further, since $\bar{\lambda}_3^{(2)} = \bar{\lambda}_3^{(1)} = \bar{\lambda}_3^{(0)} = \lambda_2^{(0)} = \lambda_2^{(1)} = 0$, we have

$$\begin{aligned} L_{2,3} &= u_2\lambda_2^{(2)}\bar{\lambda}_3^{(8)} + u_3\lambda_2^{(3)}\bar{\lambda}_3^{(7)} + u_4\lambda_2^{(4)}\bar{\lambda}_3^{(6)} + u_5\lambda_2^{(5)}\bar{\lambda}_3^{(5)} + u_6\lambda_2^{(6)}\bar{\lambda}_3^{(4)} + u_7\lambda_2^{(7)}\bar{\lambda}_3^{(3)} \\ &= u_2\frac{21}{2}\bar{\lambda}_5^{(8)} + u_3\frac{16\times 35}{2}\bar{\lambda}_5^{(7)} + u_4 120\times 35\bar{\lambda}_5^{(6)} + u_5 560\times 105 + \\ &\quad u_6 140\times 15\lambda_5^{(6)} + u_7 56\lambda_5^{(7)}. \end{aligned}$$

Similarly, we compute

$$\begin{aligned} L_{3,2} &= u_3\lambda_3^{(3)}\bar{\lambda}_2^{(7)} + u_4\lambda_3^{(4)}\bar{\lambda}_2^{(6)} + u_5\lambda_3^{(5)}\bar{\lambda}_2^{(5)} + u_6\lambda_3^{(6)}\bar{\lambda}_2^{(4)} + u_7\lambda_3^{(7)}\bar{\lambda}_2^{(3)} + u_8\lambda_3^{(8)}\bar{\lambda}_2^{(2)} \\ &= u_3 56\bar{\lambda}_5^{(7)} + u_4 15\times 140\bar{\lambda}_5^{(6)} + u_5 105\times 560 + u_6 35\times 120\lambda_5^{(6)} + \\ &\quad u_7\frac{35\times 16}{2}\lambda_5^{(7)} + u_8\frac{21}{2}\lambda_5^{(8)}. \end{aligned}$$

$$\begin{aligned} L_{4,1} &= u_4\lambda_4^{(4)}\bar{\lambda}_1^{(6)} + u_5\lambda_4^{(5)}\bar{\lambda}_1^{(5)} + u_6\lambda_4^{(6)}\bar{\lambda}_1^{(4)} + u_7\lambda_4^{(7)}\bar{\lambda}_1^{(3)} + u_8\lambda_4^{(8)}\bar{\lambda}_1^{(2)} + u_9\lambda_4^{(9)}\bar{\lambda}_1^{(1)} \\ &= u_4 476\bar{\lambda}_5^{(6)} + u_5 14\times 2380 + u_6 7\times 680\lambda_5^{(6)} + u_7\frac{14\times 136}{3}\lambda_5^{(7)} + \\ &\quad u_8\frac{7\times 17}{2}\lambda_5^{(8)} + u_9\frac{14}{5}\lambda_5^{(9)}. \end{aligned}$$

$$\begin{aligned} L_{5,0} &= u_5\lambda_5^{(5)}\bar{\lambda}_0^{(5)} + u_6\lambda_5^{(6)}\bar{\lambda}_0^{(4)} + u_7\lambda_5^{(7)}\bar{\lambda}_0^{(3)} + u_8\lambda_5^{(8)}\bar{\lambda}_0^{(2)} + u_9\lambda_5^{(9)}\bar{\lambda}_0^{(1)} + u_{10}\lambda_5^{(10)}\bar{\lambda}_0^{(0)} \\ &= u_5 8568 + u_6 3060\lambda_5^{(6)} + u_7 816\lambda_5^{(7)} + u_8 153\lambda_5^{(8)} + u_9 18\lambda_5^{(9)} + u_{10}\lambda_5^{(10)}. \end{aligned}$$

Each set of values of $u_i \in \{0,1\}$, $i = 0,\ldots,10$, and $\lambda_5^{(j)}$ and $\bar{\lambda}_5^{(j)}$, $j = 6,\ldots,10$, for which the condition

$$L_{0,5} = L_{1,4} = L_{2,3} = L_{3,2} = L_{4,1} = L_{5,0} := \Lambda \tag{4}$$

is fullfilled for a positive integer Λ will yield a simple $5-(36,10,\Lambda)$ design.

Note that a $5-(36,10,\lambda)$ design will be written as $5-(36,10,m63)$ with $\lambda_{\text{min}} = 63$ and $\lambda_{\text{max}} = \binom{31}{5} = 169911$. So, LIM $= \lfloor 169911/2*63 \rfloor = 1348$. By solving Eq.(1) above, we obtain designs for all $m63 \leq 2697$. Altogether 75 values for m have been found, of which 37 values of $m \leq$ LIM. However, since not all simple $5-(18,i,\lambda_5^{(i)})$ designs are known to exist, for example, $5-(18,6,m)$ designs are known for $m = 4,5,6,7,8,9,13$ only (here $5-(18,6,13)$ is the complete design), we just obtain the following 10 new non-trivial simple $5-(36,10,m63)$ designs for $m = 542, 621, 645, 669, 748, 772, 932, 956, 1304, 1328$. More precisely, Table 1 below shows the details of these 10 solutions.

m	$\lambda_5^{(5)}$	$\lambda_5^{(6)}$	$\lambda_5^{(7)}$	$\lambda_5^{(8)}$	$\lambda_5^{(9)}$	$\lambda_5^{(10)}$
542	0	5	6	60	210	990
621	0	6	0	126	75	135
645	0	6	6	78	275	495
669	0	6	12	30	475	855
748	0	7	6	96	340	0
772	0	7	12	48	540	360
932	0	9	0	192	60	720
956	0	9	6	144	260	1080
1304	1	0	66	112	100	792
1328	1	0	72	64	300	1152

An entry 0 in a column of the table implies that $u_i = 0$, otherwise $u_i = 1$. No values for $\bar{\lambda}_5^{(j)}$ are given in the table, because we have $\lambda_5^{(j)} = \bar{\lambda}_5^{(j)}$, $j = 6, 7, 8, 9, 10$, for all these solutions.

Remark 2.1 In order to simplify the expressions $L_{s,5-s}$ we may introduce the following variables $x_j = u_j\lambda_5^{(j)}$ and $y_j = u_{k-j}\bar{\lambda}_5^{(j)}$ for $j = 6, 7, 8, 9, 10$. More precisely,

$$x_j = \begin{cases} 0 & \text{if } u_j = 0 \\ \lambda_5^{(j)} & \text{if } u_j = 1 \end{cases}$$

and

$$y_j = \begin{cases} 0 & \text{if } u_{k-j} = 0 \\ \bar{\lambda}_5^{(j)} & \text{if } u_{k-j} = 1 \end{cases}$$

Thus $L_{s,5-s}$ have much simpler forms, in which x_j and y_j are allowed to take on the value of zero. For example,

$$\begin{aligned} L_{2,3} &= \frac{21}{2}y_8 + \frac{16 \times 35}{2}y_7 + 120 \times 35y_6 + u_5 560 \times 105 + 140 \times 15x_6 + 56x_7. \\ L_{1,4} &= \frac{14}{5}y_9 + \frac{17 \times 7}{2}y_8 + \frac{136 \times 14}{3}y_7 + 680 \times 7y_6 + u_5 2380 \times 14 + 476x_6. \\ L_{0,5} &= y_{10} + 18y_9 + 153y_8 + 816y_7 + 3060y_6 + u_5 8568. \end{aligned}$$

2.1.2 Simple $5-(36, k, \lambda)$ designs with $11 \le k \le 15$

We give a summary of the results from the construction of Theorem 1.1 for simple $5-(36, k, \lambda)$ designs for $k = 11, \ldots, 15$, for which $v_1 = v_2 = 18$.

When $v_1 = v_2$, we observe that most of the solutions of Eq.(1) have the property that $\lambda_5^{(k)} = \bar{\lambda}_5^{(k)}$, which we call *symmetric property*. Thus, assuming symmetric property for solutions of Eq.(1) appears to be reasonable. On the other hand, it will reduce the search time for solutions enormously. For $k = 12, 13, 14, 15$ we assume the symmetric property, but even so a great number of new designs have been constructed.

- Simple $5-(36, 11, \lambda) = 5-(36, 11, m21)$ designs with LIM $= 17530$. The construction yields 400 values for m with $m \le$ LIM as solutions for Eq.(1). The 73 values for m below

$$\begin{aligned} m = {} & 11832, 8712, 8736, 9404, 9416, 9440, 10084, 10120, 10752, 10889, \\ & 10913, 11432, 11444, 11456, 11545, 12124, 12136, 12225, 12249, \\ & 12261, 12840, 12905, 12929, 12941, 12953, 13496, 14265, 14301, \\ & 10676, 10717, 11356, 11397, 12077, 12101, 12781, 12805, 12894, \\ & 13396, 13485, 13509, 13574, 14076, 14117, 14189, 14254, 14797, \\ & 14821, 15501, 15614, 16205, 16294, 16861, 16909, 13426, 13450, \\ & 14130, 14154, 14834, 14858, 15466, 15538, 16146, 16170, 16271, \\ & 16850, 16874, 16951, 15390, 16070, 16803, 16875, 17483, 17507. \end{aligned}$$

show the constructed simple $5-(36,11,m21)$ designs. Of which 72 values of m yield new designs, except one, $m = 13485$, which has been known already.

- The results for $k = 12, 13, 14, 15$ are recorded in the following Table 2.

Parameters	LIM	# solutions of Eq.(1)	# constructed designs
$5-(36,12,m15)$	87652	3261	240
$5-(36,13,m585)$	6742	2427	359
$5-(36,14,m65)$	155077	26609	1926
$5-(36,15,m143)$	155077	48852	4452

In Table 2 the figures in column "# solutions of Eq.(1)" are the number of solutions of Eq.(1) having the symmetric property, whereas those in column "# constructed designs" are the number of constructed simple designs with parameters in the first column for $m \leq$ LIM. The constructed 5-designs are derived from solutions of Eq.(1) and from known simple 5-designs on 18 points as given in [8].

Remark 2.2 We have also applied our method to constructing $5-(36,k,\Lambda)$ designs for $k = 16, 17, 18$. In each of these cases we can always construct new designs.

Examples 2.1 We display some new simple 5-designs for $k = 11, 12, 13, 14, 15$ explicitly. All but one design have the symmetric property. The missing values for $\lambda_5^{(i)}$ and $\bar{\lambda}_5^{(i)}$ in the following examples imply that the corresponding designs are not used in the construction. Here are the designs.

- $5-(36,11,11832\times 21)$ with $\lambda_5^{(7)} = 54$, $\lambda_5^{(8)} = 16$, $\lambda_5^{(9)} = 240$, $\lambda_5^{(10)} = 1224$, $\bar{\lambda}_5^{(6)} = 8$, $\bar{\lambda}_5^{(7)} = 12$, $\bar{\lambda}_5^{(8)} = 108$, $\bar{\lambda}_5^{(9)} = 360$. This solution does not have the symmetric property.

  $5-(36,11,8712\times 21)$ with $\lambda_5^{(6)} = 4$, $\lambda_5^{(7)} = 6$, $\lambda_5^{(8)} = 142$, $\lambda_5^{(9)} = 40$, $\lambda_5^{(10)} = 72$, $\lambda_5^{(11)} = 1320$, and $\bar{\lambda}_5^{(i)} = \lambda_5^{(i)}$, $i = 6,7,8,9,10,11$.

- $5-(36,12,15337\times 15)$ with $\lambda_5^{(6)} = 4$, $\lambda_5^{(7)} = 6$, $\lambda_5^{(8)} = 30$, $\lambda_5^{(9)} = 55$, $\lambda_5^{(10)} = 27$, $\lambda_5^{(11)} = 660$, $\lambda_5^{(12)} = 660$, and $\bar{\lambda}_5^{(i)} = \lambda_5^{(i)}$, $i = 6,7,8,9,10,11,12$.

  $5-(36,12,50490\times 15)$ with $\lambda_5^{(7)} = 42$, $\lambda_5^{(8)} = 46$, $\lambda_5^{(9)} = 135$, $\lambda_5^{(10)} = 864$, and $\bar{\lambda}_5^{(i)} = \lambda_5^{(i)}$, $i = 7,8,9,10$.

- $5-(36,13,1347\times 585)$ with $\lambda_5^{(6)} = 4$, $\lambda_5^{(7)} = 18$, $\lambda_5^{(8)} = 48$, $\lambda_5^{(9)} = 40$, $\lambda_5^{(10)} = 27$, $\lambda_5^{(11)} = 396$, $\lambda_5^{(12)} = 1716$, $\lambda_5^{(13)} = 1287$, and $\bar{\lambda}_5^{(i)} = \lambda_5^{(i)}$, $i = 6,7,8,9,10,11,12,13$.

  $5-(36,13,2448\times 585)$ with $\lambda_5^{(6)} = 4$, $\lambda_5^{(7)} = 48$, $\lambda_5^{(8)} = 48$, $\lambda_5^{(9)} = 120$, $\lambda_5^{(10)} = 360$, and $\bar{\lambda}_5^{(i)} = \lambda_5^{(i)}$, $i = 6,7,8,9,10$.

- $5-(36,14,20400\times 65)$ with $\bar{\lambda}_5^{(6)}=4$, $\bar{\lambda}_5^{(7)}=30$, $\bar{\lambda}_5^{(9)}=60$, $\bar{\lambda}_5^{(10)}=144$, and $\lambda_5^{(i)}=\bar{\lambda}_5^{(i)}$, $i=6,7,9,10$.

  $5-(36,14,19992\times 65)$ with $\lambda_5^{(6)}=4$, $\lambda_5^{(8)}=98$, $\lambda_5^{(9)}=60$, $\lambda_5^{(12)}=1056$, and $\bar{\lambda}_5^{(i)}=\lambda_5^{(i)}$, $i=6,8,9,12$.

- $5-(36,15,19040\times 143)$ with $\lambda_5^{(6)}=4$, $\lambda_5^{(7)}=6$, $\lambda_5^{(8)}=112$, $\lambda_5^{(9)}=320$, $\lambda_5^{(12)}=528$, and $\bar{\lambda}_5^{(i)}=\lambda_5^{(i)}$, $i=6,7,8,9,12$.

  $5-(36,15,119952\times 143)$ with $\lambda_5^{(7)}=42$, $\lambda_5^{(8)}=280$, $\lambda_5^{(10)}=1152$, $\lambda_5^{(12)}=792$, and $\bar{\lambda}_5^{(i)}=\lambda_5^{(i)}$, $i=7,8,10,12$.

Remark 2.3 It is worth mentioning that there may exist different solutions to Eq.(1) leading to the same value Λ for constructed designs. For instance, the following two distinct solutions (a) and (b) of Eq.(1) for $t=5, v=36, k=13$:

(a) $\lambda_5^{(6)}=4$, $\lambda_5^{(7)}=54$, $\lambda_5^{(8)}=128$, $\lambda_5^{(10)}=729$, $\lambda_5^{(11)}=264$, $\bar{\lambda}_5^{(i)}=\lambda_5^{(i)}$, $i=6,7,8,10,11$,

(b) $\lambda_5^{(6)}=7$, $\lambda_5^{(7)}=42$, $\lambda_5^{(8)}=64$, $\lambda_5^{(9)}=240$, $\lambda_5^{(10)}=288$, $\lambda_5^{(11)}=528$, $\bar{\lambda}_5^{(i)}=\lambda_5^{(i)}$, $i=6,7,8,9,10,11$,

lead to simple designs with the same parameters $5-(36,13,3672\times 585)$. However, they are not isomorphic.

2.2 Simple $4-(35,k,\Lambda)$ designs with $k=8,9,10$

We shall choose $v_1=17$ and $v_2=18$.

2.2.1 $k=8$

There is a unique non-trivial solution for Eq.(1) with $\lambda_4^{(5)}=13$, $\lambda_4^{(7)}=264$, $\lambda_4^{(8)}=320$, $\bar{\lambda}_4^{(5)}=14$, $\bar{\lambda}_4^{(7)}=336$, $\bar{\lambda}_4^{(8)}=448$, which yields a simple $4-(35,8,448\times 35)$ design.

2.2.2 $k=9$

There are in total 700 non-trivial solutions for Eq.(1), of which we can construct 452 simple $4-(35,9,\Lambda)$ designs. Here are two examples.

(a) $\lambda_4^{(6)}=18$, $\lambda_4^{(7)}=38$, $\lambda_4^{(8)}=15$, $\lambda_4^{(9)}=27$, $\bar{\lambda}_4^{(5)}=4$, $\bar{\lambda}_4^{(7)}=84$, $\bar{\lambda}_4^{(8)}=133$, $\bar{\lambda}_4^{(9)}=42$, which yields a simple $4-(35,9,369\times 63)$ design.

(b) $\lambda_4^{(5)}=4$, $\lambda_4^{(7)}=84$, $\lambda_4^{(8)}=50$, $\lambda_4^{(9)}=90$, $\bar{\lambda}_4^{(6)}=28$, $\bar{\lambda}_4^{(8)}=294$, $\bar{\lambda}_4^{(9)}=140$, which yields a simple $4-(35,9,414\times 63)$ design.

2.2.3 $k = 10$

There is a huge number of non-trivial solutions for Eq.(1) in this case. For instance, with the restriction that $\lambda_4^{(5)} = 3$, we already have constructed 43225 simple $4-(35, 10, \Lambda)$ designs (many designs have equal value Λ, but they are not isomorphic). Here is an example.

$\lambda_4^{(5)} = 3$, $\lambda_4^{(6)} = 12$, $\lambda_4^{(7)} = 6$, $\lambda_4^{(8)} = 85$, $\lambda_4^{(9)} = 153$, $\lambda_4^{(10)} = 612$, $\bar{\lambda}_4^{(5)} = 2$, $\bar{\lambda}_4^{(6)} = 11$, $\bar{\lambda}_4^{(7)} = 28$, $\bar{\lambda}_4^{(8)} = 70$, $\bar{\lambda}_4^{(9)} = 238$, $\bar{\lambda}_4^{(10)} = 357$, which yields a simple $4-(35, 10, 3043 \times 21)$ design.

2.3 Some simple $6-(46, k, \Lambda)$ designs with $k = 13, 15$

Some further examples for $6-(46, 13, \Lambda)$ and $6-(46, 15, \Lambda)$ designs are given here. In both cases the ingredient designs are on 23 points, i.e. $v_1 = v_2 = 23$.

- $6-(46, 13, 3515 \times 1560)$ with $\lambda_6^{(7)} = 5$, $\lambda_6^{(8)} = 40$, $\lambda_6^{(9)} = 200$, $\lambda_6^{(10)} = 700$, $\lambda_6^{(11)} = 1820$, $\lambda_6^{(12)} = 3640$, $\lambda_6^{(13)} = 5720$, and $\bar{\lambda}_6^{(i)} = \lambda_6^{(i)}$, $i = 7, 8, 9, 10, 11, 12, 13$.

  $6-(46, 13, 4218 \times 1560)$ with $\lambda_6^{(7)} = 6$, $\lambda_6^{(8)} = 48$, $\lambda_6^{(9)} = 240$, $\lambda_6^{(10)} = 840$, $\lambda_6^{(11)} = 2184$, $\lambda_6^{(12)} = 4368$, $\lambda_6^{(13)} = 6864$, and $\bar{\lambda}_6^{(i)} = \lambda_6^{(i)}$, $i = 7, \ldots, 13$.

- $6-(46, 15, 28120 \times 2860)$ with $\lambda_6^{(7)} = 5$, $\lambda_6^{(8)} = 136$, $\lambda_6^{(9)} = 200$, $\lambda_6^{(10)} = 700$, $\lambda_6^{(11)} = 1820$, $\lambda_6^{(12)} = 3640$, $\lambda_6^{(13)} = 5720$, $\lambda_6^{(14)} = 7150$, $\lambda_6^{(15)} = 7150$, and $\bar{\lambda}_6^{(i)} = \lambda_6^{(i)}$, $i = 7, \ldots, 15$.

Remark 2.4 We note that Eq.(1) could have non-trivial solutions when $t+1 \leq k \leq 2t-1$. For example, when $t = 5$, $k = 8$ and $v_1 = v_2 = 22$, Eq.(1) has a non-trivial solution with $u_2 = u_4 = u_6 = 0$ (other u_i are equal to 1), $\lambda_5^{(7)} = 130$, $\bar{\lambda}_5^{(8)} = 160$ and $\bar{\lambda}_5^{(i)} = \lambda_5^{(i)}$, $i = 7, 8$, leading to a simple $5-(44, 8, 4560)$ design. However, the existence of an ingredient design with parameters $5-(22, 7, \lambda_5^{(7)}) = 5-(22, 7, 130)$ seems to be still undecided.

3 Conclusion

We have presented a new recursive construction for simple $t-$designs based on a composition of smaller ingredient designs. The construction leads to find solutions for the indices of the ingredient designs that satisfy a certain set of equalities. With a small number of examples to demonstrate the strength of the method, we have constructed a large amount of new $t-$designs, which were unknown to date. Clearly the method is very fruitful and powerful. We believe that this method would enable interested researchers to improve the Table of simple $t-$designs in the Handbook of Combinatorial Designs considerably.

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