

## CONSTRUCTION OF 3-DESIGNS USING $(1, \sigma)$ -RESOLUTION

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ABSTRACT. The paper deals with recursive constructions for simple 3-designs based on other 3-designs having  $(1, \sigma)$ -resolution. The concept of  $(1, \sigma)$ -resolution may be viewed as a generalization of the parallelism for designs. We show the constructions and their applications to produce many previously unknown infinite families of simple 3-designs. We also include a discussion of  $(1, \sigma)$ -resolvability of the constructed designs.

### 1. INTRODUCTION

In our previous papers [16, 17] we have presented several recursive constructions for simple 3-designs. In [16], among others, generalizations of the well-known doubling construction of Steiner quadruple systems for 3-designs are introduced. In [17] more general recursive constructions of simple 3-designs are described, whereby ingredient designs may have repeated blocks. The methods in these papers are based on the existence of 3-designs having a parallelism, i.e. the blocks of the design can be partitioned into classes of mutually disjoint blocks such that every point is in exactly one block of each class. Designs with parallelism have shown to be useful for constructing designs in the literature [13], [7], [10], [12], [9], [11], [15], [16, 17].

The concept of  $(1, \sigma)$ -resolvability for  $t - (v, k, \lambda)$  designs may be viewed as a generalization of that of parallelism. For the latter means that the design is  $(1, 1)$ -resolvable. It should be mentioned that if a  $t - (v, k, \lambda)$  design has a parallelism we necessarily have  $k|v$ ; this condition does no longer hold for  $(1, \sigma)$ -resolvability in general. Thus, the natural question is that whether or not the methods in our previous papers [16, 17] can be extended to  $(1, \sigma)$ -resolvable 3-designs. We show that this is in fact the case. Our aim in this paper is to present this generalization. The result provides a general method for constructing simple 3-designs which largely extends the use of complete designs as ingredients for the construction. We show the strength of the method by giving some simple applications to construct a number of families of simple 3-designs, which, to our knowledge, were not previously known to exist. We also include a discussion of  $(1, \sigma)$ -resolvability of the constructed designs.

For notation and general definitions of  $t$ -designs we refer to [3, 8].

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## 2. CONSTRUCTIONS OF 3-DESIGNS USING $(1, \sigma)$ -RESOLUTION

In this section we present recursive constructions of simple 3-designs using  $(1, \sigma)$ -resolution of their ingredients.

2.1. PRELIMINARIES. We begin with a few definitions and set up necessary conditions for the ingredients used in the constructions.

**Definition 1.** A  $t - (v, k, \lambda)$ -design  $(X, \mathcal{B})$  is said to be  $(s, \sigma)$ -resolvable for a given  $s \in \{1, \dots, t\}$ , if its block set  $\mathcal{B}$  can be partitioned into  $w$  classes  $\pi_1, \dots, \pi_w$  such that  $(X, \pi_i)$  is a  $s - (v, k, \sigma)$  design for all  $i = 1, \dots, w$ . Each  $\pi_i$  is called a resolution class.

It is worth noting that the concept of resolvability (i.e.  $(1, 1)$ -resolvability) for BIBD introduced by Bose in 1942 [6] was generalized by Shrikhande and Raghavarao to  $\sigma$ -resolvability (i.e.  $(1, \sigma)$ -resolvability) for BIBD in 1964 [14]. A definition of  $s$ -resolvability (i.e.  $(s, \sigma)$ -resolvability) for  $t$ -designs with  $t \geq 3$  and  $1 \leq s \leq t$  may be found in [1], for example.

**Remark 1.** If  $(X, \mathcal{B})$  is the complete  $t - (v, k, \binom{v-t}{k-t})$  design, then a  $(t, \sigma)$ -resolution of  $(X, \mathcal{B})$  is a large set of  $t - (v, k, \sigma)$  designs.

It should be remarked that each  $t - (v, k, \lambda)$  design always has a *trivial*  $(s, \lambda_s)$ -resolution consisting of a single class, i.e.  $w = 1$ , for all  $1 \leq s \leq t$ . Throughout the paper when we speak of  $(s, \sigma)$ -resolution we mean that  $w \geq 2$ . Note that  $w = \lambda \binom{v}{t} \binom{k}{s} / \sigma \binom{v}{s} \binom{k}{t}$ .

**Definition 2.** Let  $D$  be a  $t - (v, k, \lambda)$  design admitting a  $(s, \sigma)$ -resolution with  $\pi_1, \dots, \pi_w$  as resolution classes. Define a distance between any two classes  $\pi_i$  and  $\pi_j$  by  $d(\pi_i, \pi_j) = \min\{|i - j|, w - |i - j|\}$ .

For the constructions in this paper we employ designs having a  $(1, \sigma)$ -resolution. We now describe the detailed assumption and notation used throughout the paper.

Let  $\{k_1, \dots, k_n, k_{n+1}, \dots, k_{2n}\}$  and  $k$  be integers with  $2 \leq k_1 < \dots < k_n \leq k/2$  such that  $k_i + k_{n+i} = k$  for  $i = 1, \dots, n$ .

Assume that there exist  $3 - (v, k_i, \lambda^{(i)})$  designs  $D_i = (X, \mathcal{B}_i)$  having a  $(1, \sigma^{(i)})$ -resolution such that  $w_i = w_{n+i}$  for all  $i = 1, \dots, n$ , where  $w_j$  denotes the number of classes in a  $(1, \sigma^{(j)})$ -resolution of  $D_j$ , i.e.  $D_i$  and  $D_{n+i}$  have the same number of resolution classes.

It is also assumed that

1. For each pair  $(D_i, D_{n+i})$ ,  $1 \leq i \leq n$ , either  $D_i$  or  $D_{n+i}$  has to be simple.
2. If a  $D_j$ ,  $j \in \{i, n+i\}$ , is not simple, then  $D_j$  is a union of  $a_j$  copies of a simple  $3 - (v, k_j, \alpha^{(j)})$  design  $C_j$ , wherein  $C_j$  admits a  $(1, \sigma^{(j)})$ -resolution. Thus,  $\lambda^{(j)} = a_j \alpha^{(j)}$ .

Note that the trivial  $2 - (v, 2, 1)$  design will be considered as a  $3 - (v, 2, \lambda)$  design with  $\lambda = 0$ .

Further we need to specify the way of setting up  $(1, \sigma^{(j)})$ -resolution classes for  $D_j$ , when  $D_j$  is the union of  $a_j$  copies  $C_j$ .

Let  $P^{(j)} = \{\pi_1^{(j)}, \dots, \pi_{t_j}^{(j)}\}$  be a  $(1, \sigma^{(j)})$ -resolution of the simple design  $C_j$ . The corresponding  $(1, \sigma^{(j)})$ -resolution of  $D_j$  is chosen to be the “concatenation” of  $a_j$

sets  $P^{(j)}$ . This means that the  $w_j = a_j t_j$  resolution classes of  $D_j$  are arranged in the following way

$$\pi_1^{(j)}, \dots, \pi_{t_j}^{(j)}, \pi_1^{(j)}, \dots, \pi_{t_j}^{(j)}, \dots, \pi_1^{(j)}, \dots, \pi_{t_j}^{(j)}.$$

Finally, we also assume that there exists a  $3 - (v, k, \Lambda)$  design  $D = (X, \mathcal{B})$ , when it is needed in our construction.

**Notation:**

- $\pi_1^{(\ell)}, \dots, \pi_{w_\ell}^{(\ell)}$  denote the  $w_\ell$  classes in a  $(1, \sigma^{(\ell)})$ -resolution of  $D_\ell$  for  $\ell = 1, \dots, 2n$ . Recall that  $w_h = w_{n+h}$  for  $h = 1, \dots, n$ .
- The distance defined on the resolution classes of  $D_\ell$  is then  $d^{(\ell)}(\pi_i^{(\ell)}, \pi_j^{(\ell)}) = \min\{|i - j|, w_\ell - |i - j|\}$ .
- $b^{(j)} = \sigma^{(j)}v/k$  denotes the number of blocks in each class of a  $(1, \sigma^{(j)})$ -resolution of  $D_j$ .
- $u_j := \sigma^{(j)}$  denotes the number of blocks containing a point in each class of a  $(1, \sigma^{(j)})$ -resolution of  $D_j$ .
- $\lambda_2^{(j)} = \lambda^{(j)}(v - 2)/(k_j - 2)$  denotes the number of blocks of  $D_j$  containing two points.

2.2. CONSTRUCTION I. In this section we describe the first construction by using the set-up above for the case  $k_n \neq k/2$ .

Let  $\tilde{D}_i = (\tilde{X}, \tilde{\mathcal{B}}_i)$  be a copy of  $D_i$  defined on the point set  $\tilde{X}$  such that  $X \cap \tilde{X} = \emptyset$ . Also let  $\tilde{D} = (\tilde{X}, \tilde{\mathcal{B}})$  be a copy of  $D$ .

Define blocks on the point set  $X \cup \tilde{X}$  as follows:

- I. blocks of  $D$  and blocks of  $\tilde{D}$ ;
- II. blocks of the form  $A \cup \tilde{B}$  for any  $A \in \pi_i^{(h)}$  and  $\tilde{B} \in \tilde{\pi}_j^{(n+h)}$  with  $\varepsilon_h \leq d^{(h)}(\pi_i^{(h)}, \pi_j^{(h)}) \leq s_h, \varepsilon_h = 0, 1, \text{ for } h = 1, \dots, n$ ;
- III. blocks of the form  $\tilde{A} \cup B$  for any  $\tilde{A} \in \tilde{\pi}_i^{(h)}$  and  $B \in \pi_j^{(n+h)}$  with  $\varepsilon_h \leq d^{(h)}(\pi_i^{(h)}, \pi_j^{(h)}) \leq s_h, \varepsilon_h = 0, 1, \text{ for } h = 1, \dots, n$ .

Here, and in the sequel, the non-negative integers  $s_h, h = 1, \dots, n$ , denote the parameters that have to be determined, for which the defined blocks of types I, II and III form a 3-design. Thus,  $s_h$ , should not be confused with  $s$  in  $(s, \sigma)$ -resolution as defined above.

**Notation:** Define  $z_h = (2s_h + 1 - \varepsilon_h)$  if  $s_h < \frac{w}{2}$ , and  $z_h = (2s_h - \varepsilon_h)$  if  $s_h = \frac{w}{2}$ , for  $h = 1, \dots, n$ .

Any 3 points  $a, b, c \in X$ , resp.  $\tilde{a}, \tilde{b}, \tilde{c} \in \tilde{X}$  are contained in

- $\Lambda$  blocks of type I,
- $z_h \lambda^{(h)} b^{(n+h)}$  blocks of type II for  $h = 1, \dots, n$ ,
- $z_h \lambda^{(n+h)} b^{(h)}$  blocks of type III for  $h = 1, \dots, n$ .

Thus  $a, b, c$  appear together in

$$\Lambda + \sum_{h=1}^n z_h \lambda^{(h)} b^{(n+h)} + z_h \lambda^{(n+h)} b^{(h)}$$

blocks. Set

$$\Delta = \sum_{h=1}^n z_h \lambda^{(h)} b^{(n+h)} + z_h \lambda^{(n+h)} b^{(h)}.$$

Now consider 3 points  $a, b, \tilde{c}$ , where  $a, b \in X$  and  $\tilde{c} \in \tilde{X}$ . Because of the symmetry the number of blocks containing 3 points  $a, b, \tilde{c}$ , is equal to the number of blocks containing  $\tilde{a}, \tilde{b}, c$ . For each  $h = 1, \dots, n$ , any two points  $a$  and  $b$  are contained in  $\lambda_2^{(h)}$  blocks of  $D_h$  and in  $\lambda_2^{(n+h)}$  blocks of  $D_{n+h}$ ; further, the point  $\tilde{c}$  is in  $u_h$  (resp.  $u_{n+h}$ ) blocks of each resolution class of  $\tilde{D}_h$  (resp.  $\tilde{D}_{n+h}$ ).

So  $a, b, \tilde{c}$  appear in

- $z_h \lambda_2^{(h)} u_{n+h}$  blocks of type II for  $h = 1, \dots, n$ ,
- $z_h \lambda_2^{(n+h)} u_h$  blocks of type III for  $h = 1, \dots, n$ .

Thus  $a, b, \tilde{c}$  are contained together in

$$\Theta := \sum_{h=1}^n z_h \lambda_2^{(h)} u_{n+h} + z_h \lambda_2^{(n+h)} u_h$$

blocks.

Therefore the blocks defined in I, II and III will form a 3-design if

$$\Lambda + \Delta = \Theta,$$

or

$$\Lambda = \Theta - \Delta.$$

Note that  $\Lambda = \Theta - \Delta \geq 0$ . The case  $\Lambda = \Theta - \Delta = 0$  implies that  $D$  and  $\tilde{D}$  are not needed in the construction. In both cases either  $\Theta - \Delta > 0$  or  $\Theta - \Delta = 0$  the constructed blocks form a simple  $3 - (2v, k, \Theta)$  design with

$$\Theta = \sum_{h=1}^n \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h)\} z_h,$$

where  $1 \leq z_h \leq w_h$  if both  $D_h$  and  $D_{n+h}$  are simple and  $1 \leq z_h \leq t_j$  if  $D_j$  is non-simple,  $j \in \{h, n+h\}$ .

What remains to be verified is the simplicity of the resulting design when either  $D_h$  or  $D_{n+h}$  is non-simple. Evidently, if both  $D_h$  and  $D_{n+h}$  are simple for all  $1 \leq h \leq n$ , then the constructed design is simple.

To start with we observe that two blocks constructed from two pairs  $(D_i, D_{n+i})$  and  $(D_j, D_{n+j})$ ,  $i \neq j$ , are always distinct. Further any two blocks of different types are also distinct. Thus, we need to consider two blocks of the same type, in particular, of type II or type III constructed from a pair  $(D_j, D_{n+j})$ . W.l.o.g. we may assume that  $D_j$  is a union of  $a_j$  copies of a simple  $3 - (v, k_j, \alpha^{(j)})$  design  $C_j$  and  $D_{n+j}$  is simple.

The following argument is the same for blocks of types II and III. So let  $E = A_1 \cup \tilde{B}_1$  and  $F = A_2 \cup \tilde{B}_2$  be two blocks of type II of the resulting design, where  $A_1 \in \pi_{i_1}^{(j)}$ ,  $\tilde{B}_1 \in \tilde{\pi}_{h_1}^{(n+j)}$ ,  $A_2 \in \pi_{i_2}^{(j)}$  and  $\tilde{B}_2 \in \tilde{\pi}_{h_2}^{(n+j)}$ . Suppose  $E = F$ . Then  $\tilde{B}_1 = \tilde{B}_2$ , and hence  $h_1 = h_2$ , since  $\tilde{D}_{n+j}$  is simple. Consequently,  $A_1 = A_2$ , so we have

1. either  $i_1 = i_2$ ,

2. or  $i_1 \neq i_2$ .

In the first case,  $E$  and  $F$  are the same block. In the second case,  $E$  and  $F$  are repeated blocks; this can happen only if  $|i_2 - i_1|$  is a multiple of  $t_j$ , i.e.  $t_j \mid |i_2 - i_1|$ , this is because the resolution classes of  $D_j$  are chosen to be the concatenation of  $a_j$  copies of a given set  $P^{(j)}$  of resolution classes of  $C_j$ . Now, as  $\varepsilon_j \leq d^{(j)}(\pi_{i_1}^{(j)}, \pi_{h_1}^{(j)}) \leq s_j$  and  $\varepsilon_j \leq d^{(j)}(\pi_{i_2}^{(j)}, \pi_{h_2}^{(j)}) = d^{(j)}(\pi_{i_2}^{(j)}, \pi_{h_1}^{(j)}) \leq s_j$ , it follows that  $z_j > t_j$ . Therefore, the second case will not occur if  $z_j \leq t_j$ .

Hence, if  $z_j \leq t_j$  for all non-simple  $D_j$ 's, the resulting design remains simple.

With the notation above, we summarize Construction I in the following theorem.

**Theorem 1.** *Let  $\{k_1, \dots, k_n, k_{n+1}, \dots, k_{2n}\}$  and  $k$  be integers with  $2 \leq k_1 < \dots < k_n < k/2$  and  $k_i + k_{n+i} = k$  for  $i = 1, \dots, n$ . Assume that there exist  $3 - (v, k_i, \lambda^{(i)})$  designs  $D_i = (X, \mathcal{B}_i)$  admitting a  $(1, \sigma^{(i)})$ -resolution such that  $w_i = w_{n+i}$ , where  $w_j$  is the number of resolution classes of  $D_j$ . Assume further that at least one design from each pair  $(D_i, D_{n+i})$ ,  $1 \leq i \leq n$ , is simple and if a  $D_j$ ,  $j \in \{i, n+i\}$ , is not simple, then  $D_j$  is a union of  $a_j$  copies of a simple  $3 - (v, k_j, \alpha^{(j)})$  design  $C_j$  admitting a  $(1, \sigma^{(j)})$ -resolution, i.e.  $\lambda^{(j)} = a_j \alpha^{(j)}$ . Let  $t_j$  denote the number of resolution classes of  $C_j$ . Let*

$$\Theta := \sum_{h=1}^n \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h)\} z_h,$$

$$\Delta := \sum_{h=1}^n \{(\lambda^{(h)} b^{(n+h)} + \lambda^{(n+h)} b^{(h)})\} z_h.$$

(i) Assume that

$$(1) \quad 0 = \Theta - \Delta,$$

with  $1 \leq z_h \leq w_h$  if both  $D_h$  and  $D_{n+h}$  are simple and  $1 \leq z_h \leq t_j$  if  $D_j$  is non-simple,  $j \in \{h, n+h\}$ . Then there exists a simple  $3 - (2v, k, \Theta)$  design  $\mathcal{D}$ .

(ii) Assume that

$$(2) \quad 0 < \Theta - \Delta,$$

with  $1 \leq z_h \leq w_h$  if both  $D_h$  and  $D_{n+h}$  are simple and  $1 \leq z_h \leq t_j$  if  $D_j$  is non-simple,  $j \in \{h, n+h\}$ ; further assume that there is a  $3 - (v, k, \Lambda)$  design with  $\Lambda = \Theta - \Delta$ . Then there exists a simple  $3 - (2v, k, \Theta)$  design  $\mathcal{D}$ .

### 2.3. CONSTRUCTION II. In this section we consider the case $k_n = k/2$ .

We observe that the resulting designs in Construction I would have repeated blocks if  $k_n = k/2$  and the block sets of  $D_n$  and  $D_{2n}$  are not disjoint. To deal with the case  $k_n = k/2$  the blocks constructed from the pair  $(D_n, D_{2n})$  need to be modified.

Suppose now  $2 \leq k_1 < \dots < k_n = k/2$ . Take  $D_n = D_{2n}$  and assume that  $D_n$  is simple. Now define the blocks on the point set  $X \cup \tilde{X}$  as follows:

I. blocks of  $D$  and blocks of  $\tilde{D}$ ;

II. blocks of the form  $A \cup \tilde{B}$  for any  $A \in \pi_i^{(h)}$  and  $\tilde{B} \in \tilde{\pi}_j^{(n+h)}$  with  $\varepsilon_h \leq d^{(h)}(\pi_i^{(h)}, \pi_j^{(h)}) \leq s_h$ ,  $\varepsilon_h = 0, 1$ , for  $h = 1, \dots, n-1$ ;

- III. blocks of the form  $\tilde{A} \cup B$  for any  $\tilde{A} \in \tilde{\pi}_i^{(h)}$  and  $B \in \pi_j^{(n+h)}$  with  $\varepsilon_h \leq d^{(h)}(\pi_i^{(h)}, \pi_j^{(h)}) \leq s_h$ ,  $\varepsilon_h = 0, 1$ , for  $h = 1, \dots, n-1$ ;
- IV. blocks of the form  $A \cup \tilde{B}$  for any  $A \in \pi_i^{(n)}$  and  $\tilde{B} \in \tilde{\pi}_j^{(2n)}$  with  $\varepsilon_n \leq d^{(n)}(\pi_i^{(n)}, \pi_j^{(n)}) \leq s_n$ ,  $\varepsilon_n = 0, 1$ .

Construction II differs from Construction I only in blocks of type IV. Observe that any three points  $a, b, c \in X$  (resp.  $\tilde{a}, \tilde{b}, \tilde{c} \in \tilde{X}$ ) are contained in  $z_n \lambda^{(n)} b^{(n)}$  blocks of type IV; any three points  $a, b, \tilde{c}$  with  $a, b \in X$  and  $\tilde{c} \in \tilde{X}$  (resp.  $\tilde{a}, \tilde{b}, c$ ) are contained in  $z_n \lambda_2^{(n)} u_n$  blocks of type IV. All other countings as well as the proof of simplicity of the resulting design remain unchanged as shown in Construction I.

We obtain the following theorem for the case  $k_n = k/2$ .

**Theorem 2.** *Let  $\{k_1, \dots, k_n, k_{n+1}, \dots, k_{2n}\}$  and  $k$  be integers with  $2 \leq k_1 < \dots < k_n = k/2$  and  $k_i + k_{n+i} = k$  for  $i = 1, \dots, n$ . Assume that there exist  $3 - (v, k_i, \lambda^{(i)})$  designs  $D_i = (X, \mathcal{B}_i)$  admitting a  $(1, \sigma^{(i)})$ -resolution such that  $w_i = w_{n+i}$ , where  $w_j$  is the number of resolution classes of  $D_j$ . Assume further that at least one design from each pair  $(D_i, D_{n+i})$ ,  $1 \leq i \leq n$ , is simple and if a  $D_j$ ,  $j \in \{i, n+i\}$ , is not simple, then  $D_j$  is a union of  $a_j$  copies of a simple  $3 - (v, k_j, \alpha^{(j)})$  design  $C_j$  admitting a  $(1, \sigma^{(j)})$ -resolution, i.e.  $\lambda^{(j)} = a_j \alpha^{(j)}$ . Let  $t_j$  denote the number of resolution classes of  $C_j$ . Let*

$$\Theta^* := \lambda_2^{(n)} u_n z_n + \sum_{h=1}^{n-1} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h)\} z_h,$$

$$\Delta^* := \lambda^{(n)} b^{(n)} z_n + \sum_{h=1}^{n-1} \{(\lambda^{(h)} b^{(n+h)} + \lambda^{(n+h)} b^{(h)})\} z_h.$$

(i) Assume that

$$(3) \quad 0 = \Theta^* - \Delta^*,$$

with  $1 \leq z_h \leq w_h$  if both  $D_h$  and  $D_{n+h}$  are simple and  $1 \leq z_h \leq t_j$  if  $D_j$  is non-simple,  $j \in \{h, n+h\}$ . Then there exists a simple  $3 - (2v, k, \Theta^*)$  design  $\mathcal{D}$ .

(ii) Assume that

$$(4) \quad 0 < \Theta^* - \Delta^*,$$

with  $1 \leq z_h \leq w_h$  if both  $D_h$  and  $D_{n+h}$  are simple and  $1 \leq z_h \leq t_j$  if  $D_j$  is non-simple,  $j \in \{h, n+h\}$ ; further assume that there is a  $3 - (v, k, \Lambda)$  design with  $\Lambda = \Theta^* - \Delta^*$ . Then there exists a simple  $3 - (2v, k, \Theta^*)$  design  $\mathcal{D}$ .

### 3. APPLICATIONS

In this section we show applications of Constructions I and II for some small values of  $n$ . It turns out that we can construct many new infinite families of simple 3-designs by merely using complete designs as ingredients. For these applications we implicitly use the following result and observation.

- **Baranyai's Theorem** [2]. The trivial  $k - (v, k, 1)$  design is  $(1,1)$ -resolvable (i.e. having a parallelism) if and only if  $k|v$ .

- **Block orbits.** If  $\gcd(v, k) = 1$ , then the  $k - (v, k, 1)$  design is  $(1, k)$ -resolvable. The resolution classes are the block orbits of a fixed point free automorphism of order  $v$ .

### 3.1. APPLICATIONS OF CONSTRUCTION I.

3.1.1.  $n = 1$ . We consider the most simple case of Construction I, namely the case with  $n = 1$ ,  $k_1 = 2$  and  $k_2 = 3$ .

Let  $v > 5$  be an integer such that  $v \equiv 0 \pmod{2}$  and  $\gcd(v, 3) = 1$ .

- $D_1$  is the union of  $a_1 = (v-2)/6$  copies of the complete  $2 - (v, 2, 1)$  design  $C_1$ . By Baranyai's Theorem  $C_1$  is  $(1,1)$ -resolvable, and the number of resolution (parallel) classes of  $C_1$  is  $t_1 = (v-1)$ . For  $D_1$  we have  $\lambda^{(1)} = 0$ ,  $\lambda_2^{(1)} = (v-2)/6$ ,  $u_1 = 1$ ,  $b^{(1)} = v/2$  and  $w_1 = a_1 t_1$ .
- $D_2$  is the complete  $3 - (v, 3, 1)$  design. Recall by the observation above that  $D_2$  admits a  $(1, 3)$ -resolution, which is derived from the block orbits of a fixed point-free automorphism of order  $v$  on the point set. For  $D_2$  we have  $\lambda^{(2)} = 1$ ,  $\lambda_2^{(2)} = v - 2$ ,  $u_2 = 3$ ,  $b^{(2)} = v$  and  $w_2 = (v-1)(v-2)/6$ .
- $D$  is the complete  $3 - (v, 5, \Lambda) = 3 - (v, 5, \binom{v-3}{2})$  design.

With the notation of Theorem 1 we can check that

$$\Lambda = \Theta - \Delta$$

if  $z_1 = (v-4)/2$ , where

$$\Theta = \{\lambda_2^{(1)} u_2 + \lambda_2^{(2)} u_1\} z_1 = 3(v-2)z_1/2,$$

$$\Delta = \{\lambda^{(1)} b^{(2)} + \lambda^{(2)} b^{(1)}\} z_1 = v z_1/2,$$

$$\Lambda = \binom{v-3}{2}.$$

The constructed design then has parameters  $3 - (2v, 5, \frac{3}{4}(v-2)(v-4))$ . Since  $a_1 = (v-2)/6$ , we have that  $v \equiv 2 \pmod{6}$ . Thus we have shown the following.

**Theorem 3.** *There is a simple*

$$3 - (2v, 5, \frac{3}{4}(v-2)(v-4))$$

*design for any integer  $v \equiv 2 \pmod{6}$ .*

We can construct another family of 3-designs with moderate value for  $\Theta$ . Let  $v = 2^f + 1$  with odd  $f$ .

- $D_1$  is the union of  $a_1 = 2^f - 1$  copies of the complete  $2 - (2^f + 1, 2, 1)$  design  $C_1$ . So,  $D_1$  is  $(1,2)$ -resolvable with  $\lambda^{(1)} = 0$ ,  $\lambda_2^{(1)} = a_1 = 2^f - 1$ ,  $u_1 = 2$ ,  $b^{(1)} = 2^f + 1$  and  $w_1 = a_1 t_1$  with  $t_1 = 2^{f-1}$ .
- $D_2$  is the complete  $3 - (2^f + 1, 3, 1)$  design. Since  $f$  is odd, we have  $2^f + 1 \equiv 0 \pmod{3}$ . So,  $D_2$  is  $(1,1)$ -resolvable. For  $D_2$  we have  $\lambda^{(2)} = 1$ ,  $\lambda_2^{(2)} = 2^f - 1$ ,  $u_2 = 1$ ,  $b^{(2)} = (2^f + 1)/3$  and  $w_2 = 2^{f-1}(2^f - 1)$ .
- $D$  is a  $3 - (2^f + 1, 5, 10(2^f - 2))$  design, which is obtained from the  $4 - (2^f + 1, 5, 20)$  design [5] with  $\gcd(f, 6) = 1$ . Thus  $\Lambda = 10(2^f - 2)$ .

Now

$$\begin{aligned}\Theta &= \{\lambda_2^{(1)}u_2 + \lambda_2^{(2)}u_1\}z_1 = 3(2^f - 1)z_1, \\ \Delta &= \{\lambda^{(1)}b^{(2)} + \lambda^{(2)}b^{(1)}\}z_1 = (2^f + 1)z_1, \\ \Lambda &= 10(2^f - 2).\end{aligned}$$

Hence

$$\Lambda = \Theta - \Delta$$

if  $z_1 = 5$ . The constructed design has parameters  $3 - (2(2^f + 1), 5, 15(2^f - 1))$ . We have the following.

**Theorem 4.** *There is a simple  $3 - (2(2^f + 1), 5, 15(2^f - 1))$  design for  $\gcd(f, 6) = 1$ .*

3.1.2.  $n = 2$ . We construct a family of simple 3-designs with  $k = 7$  by using Construction I with  $n = 2$ .

Let  $v$  be an integer such that  $v \equiv 0 \pmod{4}$ ,  $\gcd(v, 3) = 1$  and  $\gcd(v, 5) = 1$ .

- $D_1$  is the union of  $a_1 = \binom{v-2}{3}/20$  copies of the complete  $2 - (v, 2, 1)$  design  $C_1$ . So,  $D_1$  is  $(1, 1)$ -resolvable. Here we have  $\lambda^{(1)} = 0$ ,  $\lambda_2^{(1)} = a_1$ ,  $u_1 = 1$  and  $b^{(1)} = v/2$  and  $w_1 = a_1 t_1$  with  $t_1 = (v - 1)$ .
- $D_3$  is the complete  $3 - (v, 5, \binom{v-3}{2})$  design, which is  $(1, 5)$ -resolvable. For  $D_3$  we have  $\lambda^{(3)} = \binom{v-3}{2}$ ,  $\lambda_2^{(3)} = \binom{v-2}{3}$ ,  $u_3 = 5$ ,  $b^{(3)} = v$  and  $w_3 = \binom{v-1}{4}/5$ .
- $D_2$  is the union of  $a_2 = (v - 3)$  copies of the complete  $3 - (v, 3, 1)$  design  $C_2$ . So,  $D_2$  is  $(1, 3)$ -resolvable. For  $D_2$  we have  $\lambda^{(2)} = v - 3$ ,  $\lambda_2^{(2)} = (v - 2)(v - 3)$ ,  $u_2 = 3$ ,  $b^{(2)} = v$  and  $w_2 = a_2 t_2$  with  $t_2 = \binom{v-1}{2}/3$ .
- $D_4$  is the complete  $3 - (v, 4, v - 3)$  design, which is  $(1, 1)$ -resolvable. For  $D_4$  we have  $\lambda^{(4)} = v - 3$ ,  $\lambda_2^{(4)} = \binom{v-2}{2}$ ,  $u_4 = 1$ ,  $b^{(4)} = v/4$  and  $w_4 = \binom{v-1}{3}$ .

We have

$$\begin{aligned}\Theta &= (\lambda_2^{(1)}u_3 + \lambda_2^{(3)}u_1)z_1 + (\lambda_2^{(2)}u_4 + \lambda_2^{(4)}u_2)z_2 \\ &= \frac{5}{4} \binom{v-2}{3} z_1 + 5 \binom{v-2}{2} z_2\end{aligned}$$

$$\begin{aligned}\Delta &= (\lambda^{(1)}b^{(3)} + \lambda^{(3)}b^{(1)})z_1 + (\lambda^{(2)}b^{(4)} + \lambda^{(4)}b^{(2)})z_2 \\ &= \frac{1}{4}v(v-3)(v-4)z_1 + \frac{5}{4}v(v-3)z_2\end{aligned}$$

Construction I will yield a simple  $3 - (2v, 7, \Theta)$  design, when there exist values for  $z_1$  and  $z_2$  such that  $\Theta - \Delta = 0$ .

Set

$$\Theta - \Delta := -Az_1 + Bz_2.$$

Then we have

$$A = \frac{1}{24}(v-3)(v-4)(v+10)$$

and

$$B = \frac{5}{4}(v-3)(v-4).$$

It follows that  $\Theta - \Delta = 0$  if we have  $Az_1 = Bz_2$ , which reduces to the equation

$$(v+10)z_1 = 30z_2,$$



where  $z_1 \leq t_1$  and  $z_2 \leq t_2$ , i.e.  $z_1 \leq v - 1$  and  $z_2 \leq (v - 1)(v - 2)/6$ . It is clear that  $z_1 = 30m$  and  $z_2 = (v + 10)m$  for integer  $m \leq (v - 1)/30$  are solutions to the equation. From  $z_1 = 30m$  and  $z_2 = (v + 10)m$  we obtain

$$\Theta = \frac{35}{4}v(v - 2)(v - 3)m.$$

Recall that  $v \equiv 0 \pmod 4$ ,  $v \equiv 1, 2 \pmod 3$ , and  $\gcd(5, v) = 1$ . Moreover, since  $a_1 = \binom{v-2}{3}/20$  must be an integer, we have  $v \equiv 2, 3, 4 \pmod 5$ . Now the congruence system  $v \equiv 0 \pmod 4$ ,  $v \equiv 1, 2 \pmod 3$ ,  $v \equiv 2, 3, 4 \pmod 5$  has  $v \equiv 4, 8, 28, 32, 44, 52 \pmod{60}$  as solutions. Thus we have proven the following.

**Theorem 5.** *There is a simple  $3 - (2v, 7, \frac{35}{4}v(v - 2)(v - 3)m)$  design for any integer  $v \equiv 4, 8, 28, 32, 44, 52 \pmod{60}$  (with  $v \geq 32$ ) and any integer  $m \leq (v - 1)/30$ .*

### 3.2. APPLICATIONS OF CONSTRUCTION II.

#### 3.2.1. $n = 1$ . Here is the first example.

Let  $f > 3$  be an odd integer such that  $\gcd(f, 3) = 1$ .

- $D_1$  is the complete  $3 - (2^f + 1, 3, 1)$  design.  $D_1$  is  $(1, 1)$ -resolvable. For  $D_1$  we have  $\lambda^{(1)} = 1$ ,  $\lambda_2^{(1)} = 2^f - 1$ ,  $u_1 = 1$ ,  $b^{(1)} = (2^f + 1)/3$  and  $w_1 = 2^{f-1}(2^f - 1)$ .
- $D$  is a  $3 - (2^f + 1, 6, \Lambda)$  design, which is obtained from the  $4 - (2^f + 1, 6, \lambda)$  design [4] with  $\gcd(f, 6) = 1$ , where  $\lambda \in \{10, 60, 70, 90, 100, 150, 160\}$ . Thus  $\Lambda = \lambda(2^f - 2)/3$ .

Now from Theorem 2 we have  $\Theta^* = \lambda_2^{(1)}u_1z_1$ ,  $\Delta^* = \lambda^{(1)}b^{(1)}z_1$ . So,  $\Theta^* - \Delta^* = \frac{2}{3}(2^f - 2)z_1$ . Thus  $\Lambda = \Theta^* - \Delta^*$  if  $z_1 = \lambda/2$ . The constructed design has parameters  $3 - (2(2^f + 1), 6, \Theta^*)$  with  $\Theta^* = (2^f - 1)z_1 = (2^f - 1)\lambda/2$ .

We have the following.

**Theorem 6.** *There exists a simple  $3 - (2(2^f + 1), 6, (2^f - 1)m)$  design for  $m \in \{5, 30, 35, 45, 50, 75, 80\}$  and  $\gcd(f, 6) = 1$ .*

We consider another example of general form. Let  $v, k$  be integers with  $v > k \geq 3$  and  $\gcd(v, k) = 1$ .

- $D_1$  is the complete design  $3 - (v, k, \binom{v-3}{k-3})$ . So,  $\lambda^{(1)} = \binom{v-3}{k-3}$ ,  $\lambda_2^{(1)} = \binom{v-2}{k-2}$ ,  $u_1 = k$ ,  $b^{(1)} = v$ , and  $w_1 = \binom{v-1}{k-1}/k$ .
- $D$  is a  $3 - (v, 2k, \Lambda)$  design.

We have  $\Theta^* = \lambda_2^{(1)}u_1z_1$ ,  $\Delta^* = \lambda^{(1)}b^{(1)}z_1$ . Construction II yields a simple  $3 - (2v, 2k, \Theta^*)$  design, when it holds

$$\Theta^* - \Delta^* = (\lambda_2^{(1)}u_1 - \lambda^{(1)}b^{(1)})z_1 = \Lambda,$$

or

$$2\binom{v-3}{k-2}z_1 = \Lambda,$$

with  $z_1 \leq \binom{v-1}{k-1}/k$ . In this case we have

$$\Theta^* = \frac{k(v-2)}{2(v-k)}\Lambda.$$

We record the result obtained above.

**Theorem 7.** *Let  $v > k \geq 3$  be integers with  $\gcd(v, k) = 1$ . Assume that there exists a simple  $3 - (v, 2k, \Lambda)$  design such that  $m = \Lambda/2 \binom{v-3}{k-2}$  is an integer and  $m \leq \binom{v-1}{k-1}/k$ . Then there exists a simple  $3 - (2v, 2k, \frac{k(v-2)}{2(v-k)}\Lambda)$  design.*

We will illustrate some explicit families for 3-designs from Theorem 7 by taking the  $3 - (v, 2k, \Lambda)$  design  $D$  to be the complete  $3 - (v, 2k, \binom{v-3}{2k-3})$  design.

- **$k = 3$ .**  $D$  is the  $3 - (v, 6, \binom{v-3}{3})$  design. There exists a simple  $3 - (2v, 6, \Theta^*)$  design with  $\Theta^* = \frac{3(v-2)}{2(v-3)} \binom{v-3}{3}$ , if  $m = \binom{v-3}{2k-3}/2 \binom{v-3}{k-2} = (v-4)(v-5)/12$  is an integer. This condition is equivalent to  $v \equiv 1, 2 \pmod{3}$  and  $v \equiv 0, 1 \pmod{4}$ . Hence  $v \equiv 1, 4, 5, 8 \pmod{12}$ .
- **$k = 4$ .**  $D$  is the  $3 - (v, 8, \binom{v-3}{5})$  design. There exists a simple  $3 - (2v, 8, \Theta^*)$  design with  $\Theta^* = \frac{4(v-2)}{2(v-4)} \binom{v-3}{5}$ , if  $m = \binom{v-3}{5}/2 \binom{v-3}{2} = (v-5)(v-6)(v-7)/2 \cdot 3 \cdot 4 \cdot 5$  is an integer. This condition is equivalent to  $v \equiv 1, 3 \pmod{4}$  and  $v \equiv 0, 1, 2 \pmod{5}$ . Hence  $v \equiv 1, 5, 7, 11, 15, 17 \pmod{20}$ .
- **$k = 5$ .**  $D$  is the  $3 - (v, 10, \binom{v-3}{7})$  design. There is a simple  $3 - (2v, 10, \Theta^*)$  design with  $\Theta^* = \frac{5(v-2)}{2(v-4)} \binom{v-3}{7}$ , if  $m = \binom{v-3}{7}/2 \binom{v-3}{2} = (v-6)(v-7)(v-8)(v-9)/16 \cdot 3 \cdot 5 \cdot 7$  is an integer. This condition is equivalent to  $\gcd(v, 5) = 1$ ,  $v \equiv 0, 1, 6, 7 \pmod{8}$  and  $v \equiv 0, 1, 2, 6 \pmod{7}$ .

In summary, we have the following corollary of Theorem 7.

**Corollary 1.** *The following hold.*

- (i) *There is a simple  $3 - (2v, 6, \frac{3(v-2)}{2(v-3)} \binom{v-3}{3})$  design for  $v \equiv 1, 4, 5, 8 \pmod{12}$ .*
- (ii) *There is a simple  $3 - (2v, 8, \frac{4(v-2)}{2(v-4)} \binom{v-3}{5})$  design for  $v \equiv 1, 5, 7, 11, 15, 17 \pmod{20}$ .*
- (iii) *There is a simple  $3 - (2v, 10, \frac{5(v-2)}{2(v-5)} \binom{v-3}{7})$  design for  $v \equiv 0, 1, 2, 6 \pmod{7}$ ,  $v \equiv 0, 1, 6, 7 \pmod{8}$ , and  $\gcd(v, 5) = 1$ .*

3.2.2.  **$n = 2$ .** Let  $v, k$  be integers such that  $v > 2k$ ,  $k \geq 3$ ,  $\gcd(v, 2k) = 1$  and  $\gcd(v, k+1) = 1$ .

- $D_1$  is a union of  $a_1 = \frac{1}{k(2k-1)} \binom{v-2}{2k-2}$  copies of the complete  $2 - (v, 2, 1)$  design  $C_1$ . Since  $\gcd(v, 2) = 1$ ,  $C_1$  is  $(1, 2)$ -resolvable and has  $t_1 = (v-1)/2$  resolution classes. For  $D_1$  we have  $\lambda^{(1)} = 0$ ,  $\lambda_2^{(1)} = \frac{1}{k(2k-1)} \binom{v-2}{2k-2}$ ,  $u_1 = 2$ ,  $b^{(1)} = v$ ,  $w_1 = a_1 t_1$ .
- $D_3$  is the complete  $2 - (v, 2k, \binom{v-3}{2k-3})$  design which is  $(1, 2k)$ -resolvable. For  $D_3$  we have  $\lambda^{(3)} = \binom{v-3}{2k-3}$ ,  $\lambda_2^{(3)} = \binom{v-2}{2k-2}$ ,  $u_3 = 2k$ ,  $b^{(3)} = v$ ,  $w_3 = \frac{1}{2k} \binom{v-1}{2k-1}$ .
- $D_2$  is the complete  $2 - (v, k+1, \binom{v-3}{k-2})$  design which is  $(1, k+1)$ -resolvable. For  $D_2$  we have  $\lambda^{(2)} = \binom{v-3}{k-2}$ ,  $\lambda_2^{(2)} = \binom{v-2}{k-1}$ ,  $u_2 = k+1$ ,  $b^{(2)} = v$ ,  $w_2 = \frac{1}{k+1} \binom{v-1}{k}$ .

We have

$$\begin{aligned} \Theta^* &= (\lambda_2^{(1)}u_3 + \lambda_2^{(3)}u_1)z_1 + \lambda_2^{(2)}u_2z_2 \\ &= \frac{4k}{2k-1} \binom{v-2}{2k-2} z_1 + (k+1) \binom{v-2}{k-1} z_2, \end{aligned}$$

$$\begin{aligned} \Delta^* &= (\lambda^{(1)}b^{(3)} + \lambda^{(3)}b^{(1)})z_1 + \lambda^{(2)}b^{(2)}z_2 \\ &= v \binom{v-3}{2k-3} z_1 + v \binom{v-3}{k-2} z_2 \end{aligned}$$

We then obtain a simple  $3 - (2v, 2(k+1), \Theta^*)$  design, if there exist positive integers  $z_1$  and  $z_2$  with  $z_1 \leq t_1$  and  $z_2 \leq w_2$  for which  $\Theta^* - \Delta^* = 0$ .

Set

$$\Theta^* - \Delta^* := -Az_1 + Bz_2.$$

Then we have

$$\begin{aligned} -A &= \frac{4k}{2k-1} \binom{v-2}{2k-2} - v \binom{v-3}{2k-3} \\ &= - \binom{v-3}{2k-3} \alpha \end{aligned}$$

with  $\alpha = [v(4k^2 - 10k + 2) + 8k]/(2k-2)(2k-1)$ ,

$$\begin{aligned} B &= (k+1) \binom{v-2}{k-1} - v \binom{v-3}{k-2} \\ &= 2 \binom{v-3}{k-2} (v-k-1)/(k-1). \end{aligned}$$

Hence, if  $\Theta^* - \Delta^* = 0$ , we have  $Az_1 = Bz_2$ . In particular, if  $A/B$  is an integer, then for any integer  $1 \leq z_1 \leq t_1$  such that  $z_2 = z_1A/B \leq w_2$ , we obtain a simple  $3 - (2k, 2(k+1), \Theta^*)$  design.

Here we record this result.

**Theorem 8.** *Let  $v, k$  be integers such that  $v > 2k$ ,  $k \geq 3$ ,  $\gcd(v, 2k) = 1$  and  $\gcd(v, k+1) = 1$ . Define  $A = \frac{v-3}{2k-3} \frac{v(4k^2-10k+2)+8k}{(2k-2)(2k-1)}$  and  $B = 2 \binom{v-3}{k-2} \frac{v-k-1}{k-1}$ . If  $A/B$  is an integer, then for any integer  $1 \leq z_1 \leq (v-1)/2$  such that  $z_2 = z_1A/B \leq \frac{1}{k+1} \binom{v-1}{k}$ , there exists a simple  $3 - (2v, 2(k+1), \Theta^*)$  design with*

$$\Theta^* = \binom{v-2}{2k-2} \frac{4k}{2k-1} z_1 + \binom{v-2}{k-1} (k+1) z_2.$$

We illustrate two special cases with  $k = 3$  and  $k = 4$  of Theorem 8.

•  **$k = 3$ .**

We then have  $A/B = \frac{(v-5)(v-3)}{3.5}$ . The conditions that  $\gcd(v, 6) = \gcd(v, 4) = 1$  and  $A/B$  is an integer are equivalent to  $v \equiv 2 \pmod{3}$ ,  $v \equiv 1, 3 \pmod{4}$  and  $v \equiv 0, 2 \pmod{5}$ . Thus we have  $v \equiv 5, 17, 35, 47 \pmod{60}$ . Note that  $z_2 = z_1A/B$ . In this case we have a  $3 - (2v, 8, \Theta^*)$  with

$$\begin{aligned} \Theta^* &= \binom{v-2}{4} \frac{12}{5} z_1 + \binom{v-2}{2} 4z_2 \\ &= \frac{7}{30} v(v-2)(v-3)(v-5) z_1, \end{aligned}$$

where  $1 \leq z_1 \leq (v-1)/2$ .

•  **$k = 4$ .**

We obtain  $A/B = (v-6)(v-7)(13v+16)/8 \cdot 3 \cdot 5 \cdot 7$ . The requirement that  $\gcd(v, 2k) = \gcd(v, 8) = 1$ ,  $\gcd(v, k+1) = \gcd(v, 5) = 1$  and  $A/B$  is an integer, reduces to  $v \equiv 7 \pmod{8}$ ,  $v \equiv 1, 2, 3 \pmod{5}$  and  $v \equiv 0, 2, 6 \pmod{7}$ . Hence  $v \equiv 7, 23, 63, 111, 167, 191, 223, 231, 247 \pmod{280}$ . And we have a simple  $3 - (2v, 10, \Theta^*)$  design with

$$\begin{aligned} \Theta^* &= \binom{v-2}{6} \frac{16}{7} z_1 + \binom{v-2}{3} 5 z_2 \\ &= 81v \binom{v-2}{6} z_1 / 7(v-5). \end{aligned}$$

In summary, we have proven the following.

**Corollary 2.** *The following hold.*

- (i) *There is a simple  $3 - (2v, 8, \frac{7}{30}v(v-2)(v-3)(v-5)m)$  design for any positive integers  $v \equiv 5, 17, 35, 47 \pmod{60}$  and  $m \leq (v-1)/2$ .*
- (ii) *There is a simple  $3 - (2v, 10, 81v \binom{v-2}{6} m / 7(v-5))$  design for any positive integers  $v \equiv 7, 23, 63, 111, 167, 191, 223, 231, 247 \pmod{280}$  and  $m \leq (v-1)/2$ .*

**3.3.  $(1, \sigma)$ -RESOLVABILITY OF THE CONSTRUCTED DESIGNS.** In this section, we discuss the question of  $(1, \sigma)$ -resolvability of the designs obtained by Constructions I and II. In particular, we will consider the cases  $\Theta - \Delta = 0$  and  $\Theta^* - \Delta^* = 0$ , i.e. the cases where a  $3 - (v, k, \Lambda)$  design  $D$  is not used in the construction.

We make use of the following observation.

- Let  $(D_h, D_{n+h})$  be a pair of designs in Constructions I or II such that  $k_h \neq k_{n+h}$ . For given  $(i, j)$  the blocks constructed from the resolution classes  $(\pi_i^{(h)}, \tilde{\pi}_j^{(n+h)})$  and  $(\tilde{\pi}_i^{(h)}, \pi_j^{(n+h)})$  will be denoted by  $\mathcal{B}_{h, n+h}^{(i, j)}$ . Thus

$$\mathcal{B}_{h, n+h}^{(i, j)} = \{A \cup \tilde{B}, \tilde{A} \cup B \mid A \in \pi_i^{(h)}, \tilde{A} \in \tilde{\pi}_i^{(h)}, B \in \pi_j^{(n+h)}, \tilde{B} \in \tilde{\pi}_j^{(n+h)}\}.$$

Recall that  $\varepsilon_h \leq d^{(h)}(\pi_i^{(h)}, \pi_j^{(h)}) \leq s_h$ . It follows that each point  $x \in X$  or  $\tilde{x} \in \tilde{X}$  appears in

$$\sigma^{(i)} := u_h b^{(n+h)} + u_{n+h} b^{(h)}$$

blocks of  $\mathcal{B}_{h, n+h}^{(i, j)}$ . Note that  $|\mathcal{B}_{h, n+h}^{(i, j)}| = 2b^{(h)}b^{(n+h)}$ .

- For the blocks of type IV in Construction II we have  $D_n = D_{2n}$  i.e.  $k_n = k_{2n}$ . Let  $\mathcal{B}_{n, n}^{(i, j)}$  denote the set of blocks constructed from resolution classes of  $D_n$  and  $\tilde{D}_n$  corresponding to the pair  $(i, j)$ . Then we have

$$\mathcal{B}_{n, n}^{(i, j)} = \{A \cup \tilde{B} \mid A \in \pi_i^{(n)}, \tilde{B} \in \tilde{\pi}_j^{(n)}\}.$$

We have  $|\mathcal{B}_{n, n}^{(i, j)}| = b^{(n)}b^{(n)}$  and each point  $x \in X$  or  $\tilde{x} \in \tilde{X}$  appears in

$$\sigma^{(n)} := u_n b^{(n)}$$

blocks of  $\mathcal{B}_{n, n}^{(i, j)}$ .

Let  $m_1, \dots, m_n$  be positive integers such that

$$m_1\sigma^{(1)} = \dots = m_n\sigma^{(n)} := \sigma.$$

Observe that the blocks constructed by each pair  $(D_h, D_{n+h})$  is a union of  $z_h w_h$  subsets  $\mathcal{B}_{h, n+h}^{(i,j)}$  of equal size. Now assume that  $m_h | z_h w_h$  for all  $h = 1, \dots, n$ . This is equivalent to say that the blocks constructed by the pair  $(D_h, D_{n+h})$  can be partitioned into  $z_h w_h / m_h$  disjoint  $1 - (2v, k_h + k_{n+h}, \sigma) = 1 - (2v, k, \sigma)$  designs. It is then clear that the constructed design is  $(1, \sigma)$ -resolvable.

In summary, by using the notation above we have the following result.

**Proposition 1.** *Let  $\mathcal{D}$  be a  $3 - (2v, k, \Theta)$  (resp.  $3 - (2v, k, \Theta)^*$ ) design obtained by Construction I (resp. Construction II) for which  $\Theta - \Delta = 0$  (resp.  $\Theta^* - \Delta^* = 0$ ). Assume that there exist positive integers  $m_1, \dots, m_n$  with  $m_h | z_h w_h$ , for  $h = 1, \dots, n$ , such that  $m_1\sigma^{(1)} = \dots = m_n\sigma^{(n)} := \sigma$ . Then the constructed design  $\mathcal{D}$  is  $(1, \sigma)$ -resolvable.*

In the rest of this section we consider the  $(1, \sigma)$ -resolvability of some families of 3-designs constructed above.

- We begin with the simple  $3 - (2v, 7, \frac{35}{4}v(v-2)(v-3)m)$  design  $\mathcal{D}$  in Theorem 5, where  $v \equiv 4, 8, 28, 32, 44, 52 \pmod{60}$  (with  $v \geq 32$ ) and integer  $m \leq (v-1)/30$ . The design  $\mathcal{D}$  is obtained by Construction I with  $n = 2$  and  $\Theta - \Delta = 0$ . From the parameters of the ingredients (see the proof of Theorem 5) we have

$$\begin{aligned} \sigma^{(1)} &= u_1 b^{(3)} + u_3 b^{(1)} = v + 5v/2 = 7v/2, \\ \sigma^{(2)} &= u_2 b^{(4)} + u_4 b^{(2)} = 3v/4 + v = 7v/4. \end{aligned}$$

Choose  $m_1 = 1$  and  $m_2 = 2$ . Then we have  $\sigma = \sigma^{(1)} = 2\sigma^{(2)}$ . Now the condition of Proposition 1 reduces to  $m_2 | z_2 w_2$ , i.e.  $2 | (v+10)mw_2$ , which is always satisfied since  $v$  is even. Hence  $\mathcal{D}$  is  $(1, 7v/2)$ -resolvable.

- Consider the designs in Corollary 2 obtained by Construction II with  $n = 2$  and  $\Theta^* - \Delta^* = 0$ .

- (i) Let  $\mathcal{D}$  be a simple  $3 - (2v, 8, \frac{7}{30}v(v-2)(v-3)(v-5)m)$  design from Corollary 2, where  $v \equiv 5, 17, 35, 47 \pmod{60}$  and  $m \leq (v-1)/2$ . Here we have

$$\begin{aligned} \sigma^{(1)} &= u_1 b^{(3)} + u_3 b^{(1)} = 2v + 6v = 8v, \\ \sigma^{(2)} &= u_2 b^{(2)} = 4v. \end{aligned}$$

Take  $m_1 = 1$  and  $m_2 = 2$ , then  $\sigma = \sigma^{(1)} = 2\sigma^{(2)} = 8v$ . The condition is  $m_2 | z_2 w_2$ , i.e.  $2 | z_2 w_2$ , where  $z_2 = z_1 A/B$  with  $A/B = \frac{(v-5)(v-3)}{3 \cdot 5}$ . Since  $v$  is odd, so  $A/B$  is even. Thus  $2 | z_2 w_2$ . Hence  $\mathcal{D}$  is  $(1, 8v)$ -resolvable.

- (ii) Similarly, let  $\mathcal{D}$  be a simple  $3 - (2v, 10, 81v \binom{v-2}{6} m / 7(v-5))$  design from Corollary 2, with  $v \equiv 7 \pmod{8}$ ,  $v \equiv 1, 2, 3 \pmod{5}$ ,  $v \equiv 0, 2, 6 \pmod{7}$  and  $m \leq (v-1)/2$ . We have

$$\begin{aligned} \sigma^{(1)} &= u_1 b^{(3)} + u_3 b^{(1)} = 2v + 2kv = 10v, \\ \sigma^{(2)} &= u_2 b^{(2)} = (k+1)v = 5v. \end{aligned}$$

Take  $m_1 = 1$  and  $m_2 = 2$ , then  $\sigma = \sigma^{(1)} = 2\sigma^{(2)} = 10v$ . The condition is  $m_2 | z_2 w_2$ , i.e.  $2 | z_2 w_2$ , where  $z_2 = z_1 A/B$  with  $A/B = \frac{(v-6)(v-7)(13v+16)}{8 \cdot 3 \cdot 5 \cdot 7}$ . Thus, if either  $z_1 (= m)$  is even or  $A/B$  is even, then the condition  $2 | z_2 w_2$  is satisfied. Hence the design  $\mathcal{D}$  is  $(1, 10v)$ -resolvable. Note that  $A/B$  being an even integer is equivalent to  $16 | (v-7)$  or  $v \equiv 7 \pmod{16}$ ,  $v \equiv 1, 2, 3 \pmod{5}$  and  $v \equiv 0, 2, 6 \pmod{7}$ .

We have proven the following.

**Proposition 2.** *The following hold.*

- (i) *The  $3-(2v, 7, \frac{35}{4}v(v-2)(v-3)m)$  design  $\mathcal{D}$  in Theorem 5 is  $(1, 7v/2)$ -resolvable for  $v \equiv 4, 8, 28, 32, 44, 52 \pmod{60}$  (with  $v \geq 32$ ) and integer  $m \leq (v-1)/30$ .*
- (ii) *The  $3-(2v, 8, \frac{7}{30}v(v-2)(v-3)(v-5)m)$  design  $\mathcal{D}$  from Corollary 2 is  $(1, 8v)$ -resolvable for  $v \equiv 5, 17, 35, 47 \pmod{60}$  and  $m \leq (v-1)/2$ .*
- (iii) *The  $3-(2v, 10, 81v \binom{v-2}{6} m/7(v-5))$  design  $\mathcal{D}$  from Corollary 2 for  $v \equiv 7, 23, 63, 111, 167, 191, 223, 231, 247 \pmod{280}$  and  $m \leq (v-1)/2$  is  $(1, 10v)$ -resolvable, if either  $m$  even or  $16|(v-7)$ .*

It is an open question whether Constructions I and II provide a  $(2, \sigma)$ -resolvable 3-design.

Finally, we include a table listing the simple 3-designs constructed in the paper.

TABLE 1. Families of simple 3-designs constructed using Theorems 1, 2

No.	Constructed design	Condition	Comment
1	$3-(2v, 5, \frac{3}{4}(v-2)(v-4))$	$v \equiv 2 \pmod{6}$	Thm. 3
2	$3-(2(2^f+1), 5, 15(2^f-1))$	$\gcd(f, 6) = 1$	Thm. 4
3	$3-(2v, 7, \frac{35}{4}v(v-2)(v-3)m)$	$v \equiv 4, 8, 28, 32, 44, 52 \pmod{60}$ $v \geq 32, m \leq (v-1)/30$	Thm. 5
4	$3-(2(2^f+1), 6, (2^f-1)m)$	$m \in \{5, 30, 35, 45, 50, 75, 80\}$ $\gcd(f, 6) = 1$	Thm. 6
5	$3-(2v, 6, \frac{3(v-2)}{2(v-3)} \binom{v-3}{3})$	$v \equiv 1, 4, 5, 8 \pmod{12}$	Cor. 1(i)
6	$3-(2v, 8, \frac{4(v-2)}{2(v-4)} \binom{v-3}{5})$	$v \equiv 1, 5, 7, 11, 15, 17 \pmod{20}$	Cor. 1(ii)
7	$3-(2v, 10, \frac{5(v-2)}{2(v-5)} \binom{v-3}{7})$	$v \equiv 0, 1, 2, 6 \pmod{7},$ $v \equiv 0, 1, 6, 7 \pmod{8},$ $\gcd(v, 5) = 1$	Cor. 1(iii)
8	$3-(2v, 8, \frac{7}{30}v(v-2)(v-3)(v-5)m)$	$v \equiv 5, 17, 35, 47 \pmod{60},$ $m \leq (v-1)/2$	Cor. 2(i)
9	$3-(2v, 10, 81v \binom{v-2}{6} m/7(v-5))$	$v \equiv 7, 23, 63, 111, 167,$ $191, 223, 231, 247 \pmod{280},$ $m \leq (v-1)/2$	Cor. 2(ii)

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