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Computational Mechanics
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Advanced Numerical Methods. Exercise 11.

Exercise 1: (6 Points)
Let the ODE

$$y'(t) = -15\,y(t), \quad y(0) = 1.$$

be given.

1. Plot the exact solution for $t \in [0, 1]$.
2. Apply the explicit Euler scheme with step lengths $h = 1/4$ and $h = 1/8$ by hand or using MATLAB. Do the numerical solutions show the expected behavior? Relate this to the region of absolute stability of the explicit Euler method.
3. From the viewpoint of stability, what would be an appropriate step length h for this problem using the explicit Euler scheme?

Exercise 2: (6 Points)
Consider the decoupled system $y'(t) = Ay(t)$ with

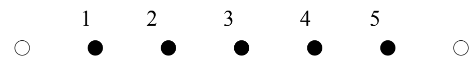
$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

with $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\mathrm{Re}(\lambda_i) < 0, i = 1, 2$. Clearly, each component is a test equation.

1. Apply the explicit Euler scheme *independently* to each component. How would you choose the two different step sizes h_1, h_2 from the viewpoint of absolute stability? (You can draw this in a picture.)
2. If you want to apply the explicit Euler scheme to both equations *simultaneously*, you have to choose a common step size h.
   When does that matter and which one would you choose?
3. How does the picture change if you consider the implicit Euler method? You can argue geometrically.

Exercise 3 (10 Points / Programming (2 weeks time))
Discretize the 1D stationary heat equation

$$u'' = 1, \quad x \in (0,1)$$



with boundary conditions $u(0) = 0$ and $u(1) = 0$ on the open circles with 7 nodes resulting in

$$h = 1/6, \quad x_0 = 0, \quad x_i = x_{i-1} + h, \text{ and } u_i = u(x_i) \quad \text{for } i = 1, \dots, 6.$$

Use the approximation

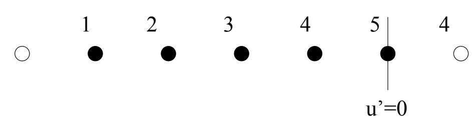
$$u''(x) \approx \frac{1}{h^2}[u(x-h) - 2u(x) + u(x+h)].$$

By substituting x_i for x, x_{i-1} for $x-h$ and x_{i+1} for $x+h$ this leads to a system of linear equations which can be solved in MATLAB.

Sometimes instead of Dirichlet boundary conditions such as $u = 0$ or $u = -1$ on the boundary (fixed temperature) we would like to introduce "isolating" boundary conditions, i.e., $\frac{\partial u}{\partial n} = 0$ on the boundary (in 1D this reduces to $u' = \frac{\partial u}{\partial x} = 0$).

Such boundary conditions are discretized by introducing "ghost points": One layer of points including the values is mirrored at the boundary.

Discretize the 1D equation with $u' = 0$ (resulting in $x_4 = x_6$) on the right side and $u = 0$ on the left side.



Compute and plot both solutions using MATLAB.

Exercise 4: (10 Bonus points)
a) Show that for a sufficiently many times differentiable function u it holds

$$\begin{aligned} u'' &= \tfrac{1}{h^2}[1 \quad -2 \quad 1]u + O(h^2) \\ &:= \tfrac{1}{h^2}[u(x-h) - 2u(x) + u(x+h)] + O(h^2). \end{aligned}$$

b) Using Taylor expansion, show directly that for a sufficiently many times differentiable function u it holds

$$\begin{aligned} \Delta u(x,y) &= \tfrac{\partial^2}{\partial x^2}u(x,y) + \tfrac{\partial^2}{\partial y^2}u(x,y) \\ &= h^{-2}[u(x-h,y) + u(x,y-h) - 4u(x,y) + u(x+h,y) + u(x,y+h)] + O(h^2) \end{aligned}$$

Due Date (theory): 07/12/2012 12:30h
Due Date (programming): 07/19/2012 12:30h