Asymptotic behavior of global solutions to a model of cell invasion

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1. Introduction

THE cell invasion into the surrounded extracellular matrix is a process that plays an important role in various biological phenomena like wound healing, tumor invasion or morphogenesis. During this poster we will focus on the tumor invasion. Tumor cells on contact with the extracellular matrix produce proteolytic enzymes which degrade the extracellular matrix creating a space where the cells can move following the gradient of the matrix. Such a movement is called *haptotaxis*

Our objective is to study a version of a model which underlies the models in [1, 2] which involves three variables, $u(x,t)$ the density of invasive cells, $v(x,t)$ the density of the extracellular matrix and $m(x,t)$ the concentration of degradative enzymes, each of them are considered in the space x and time t. More precisely the model is the following:

$$\begin{cases} u_t = \underbrace{\rho\Delta u}_{Diffusion} - \underbrace{\nabla\cdot(u\chi(v)\nabla v)}_{Haptotaxis} + \underbrace{\mu u(1-u-v)}_{Proliferation} & \text{in } \Omega\times(0,T), \\ v_t = -\underbrace{\gamma m v}_{Degradation} & \text{in } \Omega\times(0,T), \\ m_t = \underbrace{\Delta m}_{Diffusion} - \underbrace{\beta m}_{Decay} + \underbrace{\alpha u g(v)}_{Production} & \text{in } \Omega\times(0,T), \\ \rho\dfrac{\partial u}{\partial n} - u\chi(v)\dfrac{\partial v}{\partial n} = \dfrac{\partial m}{\partial n} = 0 & \text{on } \partial\Omega\times(0,T), \\ (u,v,m)(x,0) = (u_0,v_0,m_0)(x) & \text{in } \Omega. \end{cases} \tag{1}$$

Here $\Omega\subset\mathbb{R}^N$ is a bounded regular domain, g, $\chi\in C^2([0,+\infty))$ with $g(s)$, $\chi(s)\geq 0$ for all $s\geq 0$, $\mu\geq 0$ and α,β,γ,ρ positive constants.

A similar model with an additional chemotactic term in the first equation was considered in [3, 4] where it was shown the global existence and uniqueness of a classical solution in the three dimensional case when the logistic growth rate is rate whereas in the one or two dimensional case this hypothesis was removed, see also [5] for the case of nonlinear diffusion of the cells.

The poster is organized as follows. In the next section we introduce the results concerning the local and global solvability of (1) and the continuity of the solutions respect to the initial data. In the last section we will show all the stationary solutions of (1) and we will study the nonlinear stability of the system.

2. Global existence and continuity respect to the initial data

OUR first step is to show the existence and uniqueness of local in time solutions as well as the continuity respect to the initial data. To this aim we will need to introduce some notation.

Let $p\in(1,\infty)$ and let us define the operator

$$A_p u := -\Delta u + \beta u$$

with domain

$$D(A_p) := \left\{ u\in W^{2,p}(\Omega) \ : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}.$$

Since $Re\,\sigma(A_p)\geq\beta>0$, where $\sigma(A_p)$ stands for the spectrum of A_p, we can introduce the fractional powers A_p^θ for all $\theta>0$. Let us set

$$X_p^\theta := D(A_p^\theta).$$

For simplicity we will state the Theorem concerning local existence and the continuity respect to the initial data just in the 3-Dimensional case.

Theorem 2.1 *Let $p\in(3,6)$ and $\theta\in\left(\frac{1}{2}+\frac{3}{2p},1\right)$. Given the non-negative initial data*

$$\mathbf{u_0} = (u_0,v_0,m_0)\in W^{1,2}(\Omega)\times W^{1,\infty}(\Omega)\times X_p^\theta = \mathbf{Y}\,,$$

there exists $\tau(\|\mathbf{u_0}\|_{\mathbf{Y}})>0$ such that the problem (1) has a unique non-negative solution

$$\begin{aligned} u &\in C([0,\tau];W^{1,2}(\Omega))\cap C^1((0,\tau);W^{1,\infty}(\Omega))\,, \\ v &\in C([0,\tau];W^{1,\infty}(\Omega))\cap C^1((0,\tau);W^{1,\infty}(\Omega))\,, \\ m &\in C([0,\tau];X_p^\theta)\cap C^1((0,\tau);X_p^\theta)\cap C((0,\tau);W^{2,p}(\Omega))\,. \end{aligned}$$

Moreover, the solution depends continuously on the initial data.

In order to show that $T_{max}=+\infty$, where T_{max} denote the maximal existence time, we introduce the change of variable $q=ue^{-\int_0^v\chi(s)ds}$ and we apply the Gagliardo-Nirenberg inequality and the parabolic regularity.

3. Stationary solutions and asymptotic behavior

STATIONARY solutions are candidate to be the limit of the solutions of (1) as time goes to infinity. Therefore, it will be helpful to provide some information about the stationary problem. In particular, by the simplicity of the v-equation, we are able to identify all the stationary solutions. However through this section we require an additional condition on g

$$g(v)\neq 0 \text{ if } v\neq 0.$$

Theorem 3.1 *If $(u^*,v^*,m^*)\in C^1(\overline{\Omega})\times W^{1,\infty}(\Omega)\times C^1(\overline{\Omega})$ are non-negative solutions to the stationary problem associated to (1), then are given by:*

$$\begin{aligned} (u^*,v^*,m^*) &= (0,\widetilde{v},0)\,, \\ (u^*,v^*,m^*) &= (k,0,\tfrac{k\alpha}{\beta}g(0))\,, \end{aligned}$$

where $k=0$ or $k=1$ if $\mu>0$, $k\geq 0$ is an arbitrary constant if $\mu=0$ and $\widetilde{v}\in W^{1,\infty}(\Omega)$ is an arbitrary non-negative function.

Now, we deal with the long time behavior. The next Lemma asserts, under some conditions, the component u is separated from zero independently of time.

Lemma 3.2 *Let $\mu\geq 0$, v_0 positive and if $\mu>0$ additionally we assume $v_0<1$. If there exists a constant $a>0$ such that $u_0\geq a$, then every global solution satisfies*

$$u(x,t)\geq\min\{1,a\}e^{\int_0^{v_0}\chi(s)ds},$$

for all $x\in\Omega$, $t>0$.

Since the next results depend on the behavior of g around zero, we distinguish between $g(0)\neq 0$ and $g(0)=0$.

• $g(0)\neq 0$ then by the hypotheses on g there exists a constant $C>0$ such that $g(s)>C$ for all $s\in[0,sup_\Omega\{v_0\}]$. Therefore, by the strong maximum principle and the previous Lemma we get

Lemma 3.3 *Under the conditions of Lemma 3.2 there exist $\tau,\delta>0$ such that*

$$m(x,t)\geq\delta>0,\quad\forall t\geq\tau.$$

The previous lemmas and some computations involving the components v,m allow us to conclude the strong decay of v to zero. In particular we have that for all $t\geq\tau>0$

$$\int_\Omega|\nabla v|^2\leq Ce^{-kt},$$

for all $k\in(0,\delta)$.

Theorem 3.4 *Let $p>2$, $\beta<1$, $\mu\geq 0$, $\tau>0$, $t>\tau$, $\mathbf{u_0}\in\mathbf{Y}$, $m_0\geq 0$, $v_0>0$, $u_0>0$ and $v_0<1$ if $\mu>0$. Then there exist $C,\theta,\delta>0$ such that the unique solution to (1) satisfies*

$$\|m(t)-(\alpha/\beta)u_\mu\|_{X_p^\beta}\leq Ce^{-\theta t},\quad\|v(t)\|_{W^{1,\infty}}\leq Ce^{-\delta t},\quad\|u(t)-u_\mu\|_{W^{1,\infty}}\leq Ce^{-\theta t},\ t\geq\tau>0,$$

where

$$u_\mu := \begin{cases} \frac{1}{|\Omega|}\int_\Omega u & \text{if } \mu=0, \\ 1 & \text{if } \mu>0. \end{cases}$$

• $g(0)=0$ In this case we use a different argument and we will not be able , as in the previous case, to determine the rate of convergence of the solution to the constant stationary solution. Thus, let us define

$$y(t)=\|u(t)-u_\mu\|_2.$$

Our objective is to prove that $\lim_{t\to+\infty}y(t)=0$. Notice that the inequality

$$\int_0^{+\infty}y(s)ds<C$$

is not sufficient to claim that $y(t)$ goes to zero as time goes to infinity. We need an additional condition in order to have a control of the oscillatory behavior of $y(t)$. For example, the condition

$$\lim_{t\to+\infty}|y_t(s)|ds=0.$$

Using the previous idea we can prove the following:

Theorem 3.5 *Let $C>0$ and $g(v)\geq Cv$. Under the hypotheses of Theorem 3.4, we deduce:*

$$\lim_{t\to+\infty}\|u(t)-u_\mu\|_2=0,\quad\lim_{t\to+\infty}\|v(t)\|_\infty,\quad\lim_{t\to+\infty}\|m(t)\|_2=0.$$

References

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