INNOVATIVE ECONOMY
NATIONAL COHESION STRATEGY

FNP Foundation for Polish Science

EUROPEAN REGIONAL DEVELOPMENT FUND

The existence of solutions to the compressible Navier-Stokes system via time discretization

Ewelina Zatorska

Institute of Applied Mathematics and Mechanics,
University of Warsaw,
ul.Banacha 2, 02-097 Warszawa, Poland
e.zatorska@mimuw.edu.pl

Abstract

We investigate the time discretization of two compressible Navier-Stokes system with the slip boundary conditions

$$\frac{1}{\triangle t}\left(\varrho^k-\varrho^{k-1}\right)+\mathrm{div}(\varrho^k v^k)=0,$$
$$\frac{1}{\triangle t}\left(\varrho^k v^k-\varrho^{k-1}v^{k-1}\right)+\mathrm{div}(\varrho^k v^k\otimes v^k)-\mu\Delta v^k-(\mu+\nu)\nabla\mathrm{div}v^k+\nabla\pi(\varrho^k)=0,$$
$$v^k\cdot n=0,\quad \text{at } \partial\Omega$$
$$n\cdot\mathbb{T}(v^k,\pi)\cdot\tau+fv^k\cdot\tau=0,\quad \text{at } \partial\Omega$$

$\Omega\subset\mathbb{R}^2$ *is open, bounded with smooth boundary.*
The internal pressure is given by the constitutive equation

$$\pi(\varrho^k)=(\varrho^k)^\gamma,\ \gamma>1.$$

1. Main result

The goal is to show that for $\triangle t = const.$ and in the case when (ϱ^{k-1},v^{k-1}) are given functions satisfying conditions

$$\varrho^{k-1}\geq 0 \text{ a.e. in } \Omega,\quad \varrho^{k-1}\in L_\gamma(\Omega),$$
$$\varrho^{k-1}v^{k-1}\in L_{2\gamma/(\gamma+1)}(\Omega),\quad \varrho^{k-1}(v^{k-1})^2\in L_1(\Omega).$$

the solutions of our system exist in the sense of the following definition

Definition 1 *The pair of functions* $(\varrho^k,v^k)\in L_\gamma(\Omega)\times W_2^1(\Omega)$, $v^k\cdot n=0$ *at* $\partial\Omega$ *is called a weak solution provided*

$$\int_\Omega \varrho^k v^k\cdot\nabla\varphi\,dx=\frac{1}{\triangle t}\int_\Omega(\varrho^k-\varrho^{k-1})\varphi\,dx,\quad \forall\varphi\in C^\infty(\overline{\Omega}),$$

$$\frac{1}{\triangle t}\int_\Omega(\varrho^k v^k-\varrho^{k-1}v^{k-1})\varphi\,dx-\int_\Omega\varrho^k v^k\otimes v^k:\nabla\varphi\,dx+2\mu\int_\Omega\mathbf{D}(v^k):\mathbf{D}(\varphi)\,dx$$
$$+\nu\int_\Omega \mathrm{div}v^k\mathrm{div}\varphi\,dx-\int_\Omega\pi(\varrho^k)\mathrm{div}\varphi\,dx+\int_{\partial\Omega}f(v^k\cdot\tau)(\varphi\cdot\tau)\,dS=0,$$
$$\forall\varphi\in C^\infty(\overline{\Omega});\ \varphi\cdot n=0 \text{ at } \partial\Omega.$$

The existence has been proved in [6]. This result can be summarised as follows:

Theorem 1 *Let* $\Omega\in C^2$ *be a bounded domain,* $\triangle t=const.$, $\mu>0$, $2\mu+3\nu>0$, $\gamma>1$, $f\geq 0$. *Let* $(\varrho^{k-1},v^{k-1})\in L_\gamma(\Omega)\times W_2^1(\Omega)$ *be given functions). Then there exists a weak solution such that*

$$\varrho^k\in L_\infty(\Omega)\quad \text{and } \varrho^k\geq 0,$$
$$v^k\in W_p^1(\Omega)\quad \forall p<\infty,$$
$$\int_\Omega\varrho^k dx=\int_\Omega\varrho^{k-1}dx,$$

moreover $\|\varrho^k\|_\infty\leq(\triangle t)^{\frac{-3\gamma}{2(\gamma-1)^2}}$.

The proof is based on introducing the new type of approximative system being a modification of the one used for the first time by P. Mucha and M. Pokorný [3] for the steady case.

2. Approximation

Introduce the notation:

$$\alpha=\tfrac{1}{\triangle t},\quad h=\varrho^{k-1},\quad \varrho=\varrho^k,\quad v=v^k,\quad g=v^{k-1}.$$

The new scheme of approximation is the following:

$$\alpha\left(\varrho-hK(\varrho)\right)+\mathrm{div}(K(\varrho)\varrho v)-\epsilon\Delta\varrho=0,$$
$$\alpha\left(\varrho v-hg\right)+\mathrm{div}(K(\varrho)\varrho v\otimes v)-\mu\Delta v-(\mu+\nu)\nabla\mathrm{div}v+\nabla P(\varrho)+\epsilon\nabla v\nabla\varrho=0$$
$$\tfrac{\partial\varrho}{\partial n}=0,\quad \text{at}\quad \partial\Omega$$
$$v\cdot n=0,\quad \text{at}\quad \partial\Omega$$
$$n\cdot T(v,P(\varrho))\cdot\tau+fv\cdot\tau=0,\quad \text{at}\quad \partial\Omega$$

where

$$P(\varrho)=\gamma\int_0^\varrho s^{\gamma-1}K(s)ds,$$
$$K(\varrho)=\begin{cases}1 & \varrho\leq m_1,\\ 0 & \varrho\geq m_2,\\ \in(0,1) & \varrho\in(m_1,m_2),\end{cases}$$
$$K(\cdot)\in C^1(\mathbb{R})\quad K'(\varrho)<0 \text{ in } (m_1,m_2),$$

where m_2-m_1 is constant.
The existence of a regular solution is provided due to the following theorem

Theorem 2 *Let* $\Omega\in C^2$ *be a bounded domain. Let* ϵ, α *be positive constants. Let* $h\in L_\infty$, $h\geq 0$, $hg\in L_{2\gamma/(\gamma+1)}$, $hg^2\in L_1$. *Then there exists a regular solution* (ϱ,v), $\varrho\in W_p^2,\ v\in W_p^2$ *for all* $p<\infty$.
Moreover

$$0\leq\varrho\leq m_2\quad in\ \Omega,$$
$$\int_\Omega\varrho dx\leq\int_\Omega hdx.$$

Sketch of the proof In the first step we define for $p\in[1,\infty]$:

$$M_p=\left\{w\in W_p^1; w\cdot n=0 \text{ at } \partial\Omega\right\}.$$

Then the following proposition holds true

Proposition 1 *Let assumptions of Theorem 2 be satisfied. Then the operator* $S:M_\infty\to W_p^2$, *where*

$$S(v)=\varrho,$$
$$\alpha\varrho+\mathrm{div}(K(\varrho)\varrho v)-\epsilon\Delta\varrho=\alpha hK(\varrho),$$
$$\tfrac{\partial\varrho}{\partial n}=0,\quad at\quad \partial\Omega$$

is well defined for any $p<\infty$. *Moreover*

- $\varrho=S(v)$ *satisfies*
$$\int_\Omega\varrho dx\leq\int_\Omega hdx.$$
- *If* $h\geq 0$ *then* $\varrho\geq 0$ *a.e. in* Ω.
- *If* $\|v\|_{1,\infty}\leq L,\ L>0$ *then*
$$\|\varrho\|_{2,p}\leq C(\epsilon,p,\Omega)(1+L)\|h\|_p,\quad 1<p<\infty.$$

In the next step we consider the Lamé operator

$$\mathcal{T}:\ M_\infty\to M_\infty$$

defined as follows: $w=\mathcal{T}(v)$ is a solution to the problem

$$-\mu\Delta w-(\mu+\nu)\nabla\mathrm{div}w=\alpha hg-\alpha\varrho v-\mathrm{div}(K(\varrho)\varrho v\otimes v)-\nabla P(\varrho)-\epsilon\nabla v\nabla\varrho,$$
$$w\cdot n=0\quad \text{at,}\quad \partial\Omega$$
$$n\cdot(2\mu D(w)+\nu\mathrm{div}wI)\cdot\tau+fv\cdot\tau=0,\quad \text{at}\quad \partial\Omega.$$

Employing the Leray-Schauder fixed point theorem we finish the proof of existence, provided we have the information from the energy estimate.

3. Passage to the limit

The energy estimate together with estimate of higher power of ϱ (using the Bogovskii operator) give rise to:

$$\|\varrho_\epsilon\|_\infty\leq m_2,\quad \|v_\epsilon\|_{1,2}\leq C\alpha,$$
$$\|P(\varrho_\epsilon)\|_2\leq C(\alpha),$$
$$\|v_\epsilon\|_{1,q}+\epsilon^{1/2}\|\nabla\varrho_\epsilon\|_2\leq C(m_2,\alpha,q)\qquad \text{for } 1\leq q<\infty,$$
$$\epsilon\|\nabla v_\epsilon\nabla\varrho_\epsilon\|_q\leq C(m_2,\alpha,q)\qquad \text{for } 1\leq q<2.$$

Therefore, passing with $\epsilon\to 0$ we have

$$\alpha\left(\varrho-\overline{hK(\varrho)}\right)+\mathrm{div}(\overline{K(\varrho)\varrho v})=0,$$
$$\alpha\left(\varrho v-hg\right)+\mathrm{div}(\overline{K(\varrho)\varrho v}\otimes v)-\mu\Delta v-(\mu+\nu)\nabla\mathrm{div}v+\nabla\overline{P(\varrho)}=0,$$
$$v\cdot n=0,\quad at\quad \partial\Omega$$
$$n\cdot\mathbb{T}(v,\overline{P(\varrho)})\cdot\tau+fv\cdot\tau=0,\quad at\quad \partial\Omega.$$

Thus, we should answer the following questions:

1. Is $K(\varrho)=1$?
2. Is $\overline{P(\varrho)}=\varrho^\gamma$?

Ad.1. We introduce the Helmholtz decomposition of the velocity vector field:

$$v=\nabla\phi+\nabla^\perp A,$$

where the divergence-free part $\nabla^\perp A=\left(-\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_1}\right)A$ and the gradient part ϕ are given by:

$$\begin{cases}\Delta A=\mathrm{rot}v & \text{in } \Omega\\ \nabla^\perp A\cdot n=0 & \text{at } \partial\Omega\end{cases},\quad \begin{cases}\Delta\phi=\mathrm{div}v & \text{in } \Omega,\\ \frac{\partial\phi}{\partial n}=0 & \text{at } \partial\Omega\end{cases}.$$

Inserting this decomposition to both: the approximative and the limit system one gets

$$\nabla G_\epsilon=-\frac{1}{\triangle t}(\varrho_\epsilon v_\epsilon-hg)-\mathrm{div}(K(\varrho_\epsilon)\varrho_\epsilon v_\epsilon\otimes v_\epsilon)+\mu\Delta\nabla^\perp A_\epsilon-\epsilon\nabla\varrho_\epsilon\nabla v_\epsilon,$$
$$\nabla G=-\frac{1}{\triangle t}(\varrho v-hg)-\mathrm{div}(\overline{K(\varrho)\varrho v}\otimes v)+\mu\Delta\nabla^\perp A.$$

Where $G,\ G_\epsilon$ are defined as:

$$G_\epsilon=-(2\mu+\nu)\Delta\phi_\epsilon+P(\varrho_\epsilon),\qquad G=-(2\mu+\nu)\Delta\phi+\overline{P(\varrho)}.$$

Step 1. We show that

$$\|G\|_2\leq C(\alpha),\quad \text{and}\quad \|\nabla G\|_q\leq C(\alpha,m_2),\quad \text{for } q>2.$$

Where we take advantage of the following property:

$$\|\Delta\nabla^\perp A\|_q\leq\|\nabla\omega\|_q\leq\alpha\|hg\|_q+\alpha\|\varrho v\|_q+\|\mathrm{div}(\overline{K(\varrho)\varrho v}\otimes v)\|_q+C\|v\cdot\tau\|_{1-1/q,q,\partial\Omega},$$

where ω is a weak solution to the following problem:

$$-\mu\Delta\omega\ =\ -\alpha\mathrm{rot}(hg-\varrho v)-\mathrm{rotdiv}(\overline{K(\varrho)\varrho v}\otimes v)\quad \text{in } \Omega,$$
$$\omega\ =\ \left(2\chi-\frac{f}{\mu}\right)v\cdot\tau\quad \text{at } \partial\Omega.$$

Step 2. We prove that

$$\nabla(G_\epsilon-G)\rightharpoonup 0\quad in\ L_2(\Omega).$$

We need the observation $\|\Delta\nabla^\perp(A_\epsilon^k-A^k)\|_{-1,2}\leq\|\nabla(\omega_\epsilon-\omega)\|_{-1,2}$, and

$$\Delta\nabla^\perp(A_\epsilon-A)=B_\epsilon^1+B_\epsilon^2,$$

where

$$B_\epsilon^1\to 0\quad \text{in } W_2^{-1}(\Omega),\quad B_\epsilon^2\rightharpoonup 0\quad \text{in } L_2(\Omega).$$

Step 3. Next we show a crucial information about the convergence of the density:

Lemma 1 *Let* $\kappa>0$ *and let* m *satisfy*

$$\|G\|_\infty^{1/\gamma}<m<m_1\quad and\quad \frac{m^{\gamma+1}}{m_2}-\|G\|_\infty-2\alpha(2\mu+\nu)\geq\kappa>0$$

then we have

$$\lim_{\epsilon_n\to 0^+}|\{x\in\Omega:\varrho_{\epsilon_n}(x)>m\}|=0.$$

Thus $\int_\Omega\varrho_\epsilon K(\varrho_\epsilon)\phi dx\to\int_\Omega\varrho\phi dx$, $\forall\phi\in C^\infty(\Omega)$, hence $K(\varrho)=1$ a.e. in Ω.

Ad.2. We test the approximate continuity equation by $\ln\frac{m_2}{\varrho_\epsilon+\delta}$ for $\delta>0$.
Then $\delta\to 0^+$ [Fatou Lemma] + Def. G_ϵ + $\epsilon\to 0^+$ yields

$$\int_\Omega\overline{P(\varrho)\varrho}\,dx+\frac{2\mu+\nu}{\Delta t}\int_\Omega\overline{(\varrho-h)\ln\varrho}\,dx\leq\int_\Omega G\varrho^k\,dx.$$

Next, we test the limit continuity equation by $\ln\frac{\delta}{\varrho_n+\delta}$ for $\delta>0$.
Then $n\to\infty$ + $\delta\to 0^+$ [Lebesgue monotone theorem] + Def. G yields

$$\int_\Omega G\varrho\,dx=(2\mu+\nu)\frac{1}{\Delta t}\int_\Omega(\varrho-h)\ln\varrho^k\,dx+\int_\Omega\overline{P(\varrho)}\varrho\,dx$$

where ϱ_n is obtained due to the lemma

Lemma 2 *Let* $\Omega\in C^{0,1}$, $v^k\in W_q^1(\Omega),\ 1<q<\infty,\ \varrho\in L_p(\Omega),\ 1<p<\infty$, $v\varrho\in L_s$, $1/s=(1/p)+(1/q)$. *Then there exists* $\varrho_n\in C^\infty(\overline{\Omega})$ *such that*

$$v\cdot\nabla\varrho_n\to v\cdot\nabla\varrho\quad \text{in } L_s(\Omega)\quad \text{and}\quad \varrho_n\to\varrho\quad \text{in } L_p(\Omega).$$

By comparison of above expressions and the standard arguments for weak limits of convex functions we show

$$\overline{(\varrho^k)^\gamma}=(\varrho^k)^\gamma\quad \text{for a.e. } x\in\Omega\quad \Rightarrow \text{strong convergence of density}.$$

Remark 1 *The density obtained in the above procedure is bounded by some* m *as we could see from Lemma 1, in particular,* m *satisfies*

$$m^\gamma\geq C\left(\alpha+\alpha^{3/2}m+\alpha^3+\alpha^{\frac{6\gamma}{2\gamma+\gamma q-2q}}m^{1+2(1-2/q)\gamma}\right),$$

moreover, for $q\to 2^+$ *and for* $1<\gamma<2$ *one gets*

$$\|\varrho\|_\infty\leq\alpha^{\frac{3\gamma}{2(\gamma-1)^2}}.$$

References

[1] E. Feireisl, A. Novotný: Singular limits in thermodynamics of viscous fluids. Advances in Mathematical Fluid Mechanics. Birkhäuser Verlag, Basel, 2009.

[2] P.-L. Lions: Mathematical Topics in Fluid Mechanics, Vol 2: Compressible Models. Oxford Lecture Series in Mathematics and its Applications, 10. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998.

[3] P.B. Mucha, M. Pokorný: On a new approach to the issue of existence and regularity for the steady compressible Navier-Stokes equations. Nonlinearity 19 (2006), no. 8, 1747–1768.

[4] S. Novo, A. Novotný, M. Pokorný: Some notes to the transport equation and to the Green formula. Rend. Sem. Mat. Univ. Padova 106 (2001), 65–76.

[5] A. Novotný, I. Straškraba: Introduction to the mathematical theory of compressible flow. Oxford Lecture Series in Mathematics and its Applications, 27. Oxford University Press, Oxford, 2004.

[6] E. Zatorska: Analysis of semidiscretization of the compressible Navier-Stokes equations. (preprint)

This work has been supported by the International Ph.D. Projects Programme of Foundation for Polish Science operated within the Innovative Economy Operational Programme 2007-2013 funded by European Regional Development Fund (Ph.D. Programme: Mathematical Methods in Natural Sciences) and partially supported by the Polish Ministry of Science and Higher Education, grant No. N201 547438.